263-5805-00L
Rigid Body **Dynamics** & **Optimizing Mass** Distribution

Moritz Bächer

Agenda

- Motivation
- Coordinate Frames / Motion of a Point
- Mass Properties (Mass, Center of Mass, Moment of Inertia)
- Linear vs. Angular Motion / Momentum
- Equations of Motion
- Optimizing Mass Distribution

Motivation

[John Hopkins University]

[Wikimedia Commons]

Motivation

["A though gang of Spinning Tops", David Earle, www.davidturnswood.com]
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[Sharon Drummond, http://flic.kr/p/95KRph]
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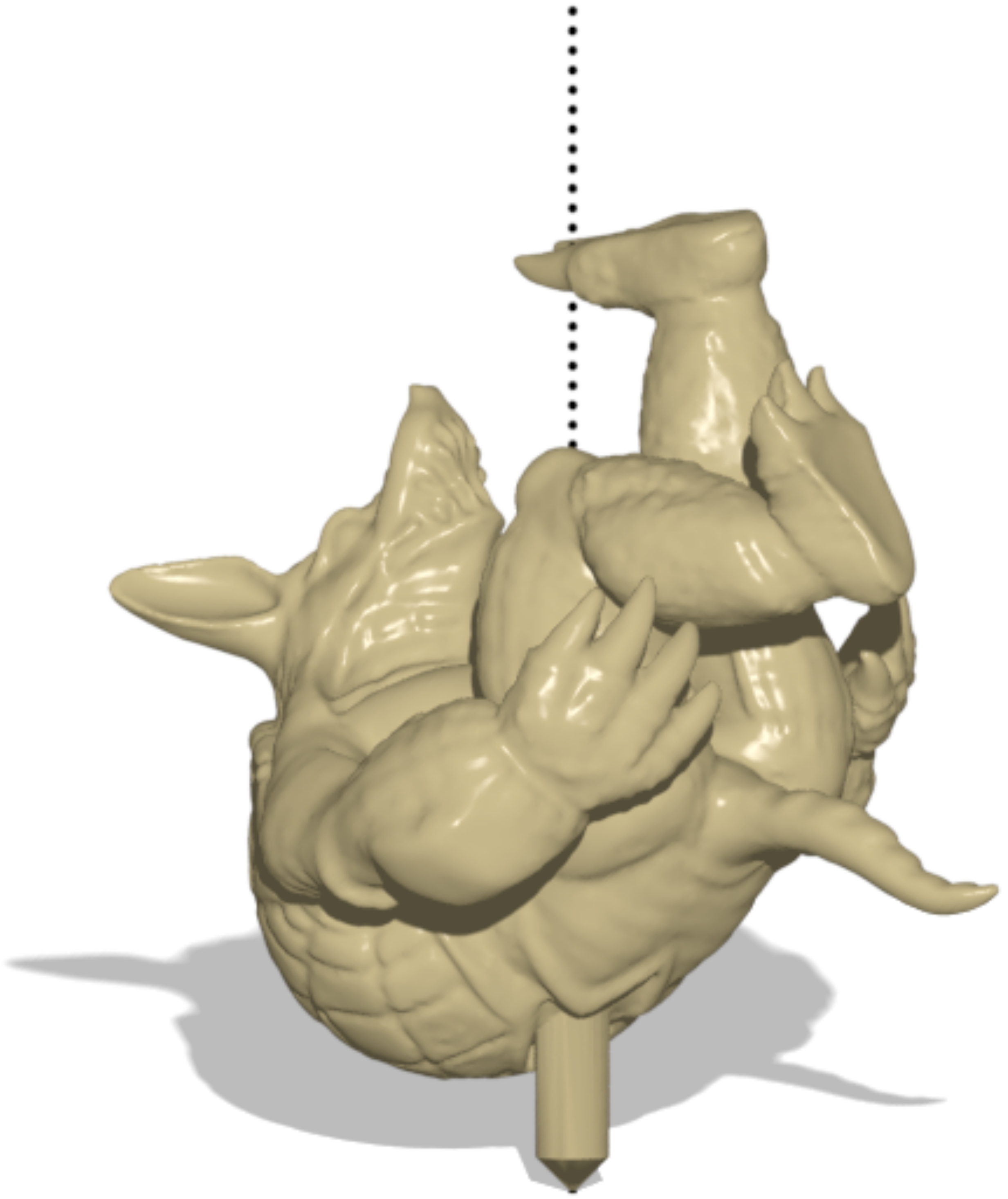
Motivation

[Hager, http://www.hager-drechseln.de/]

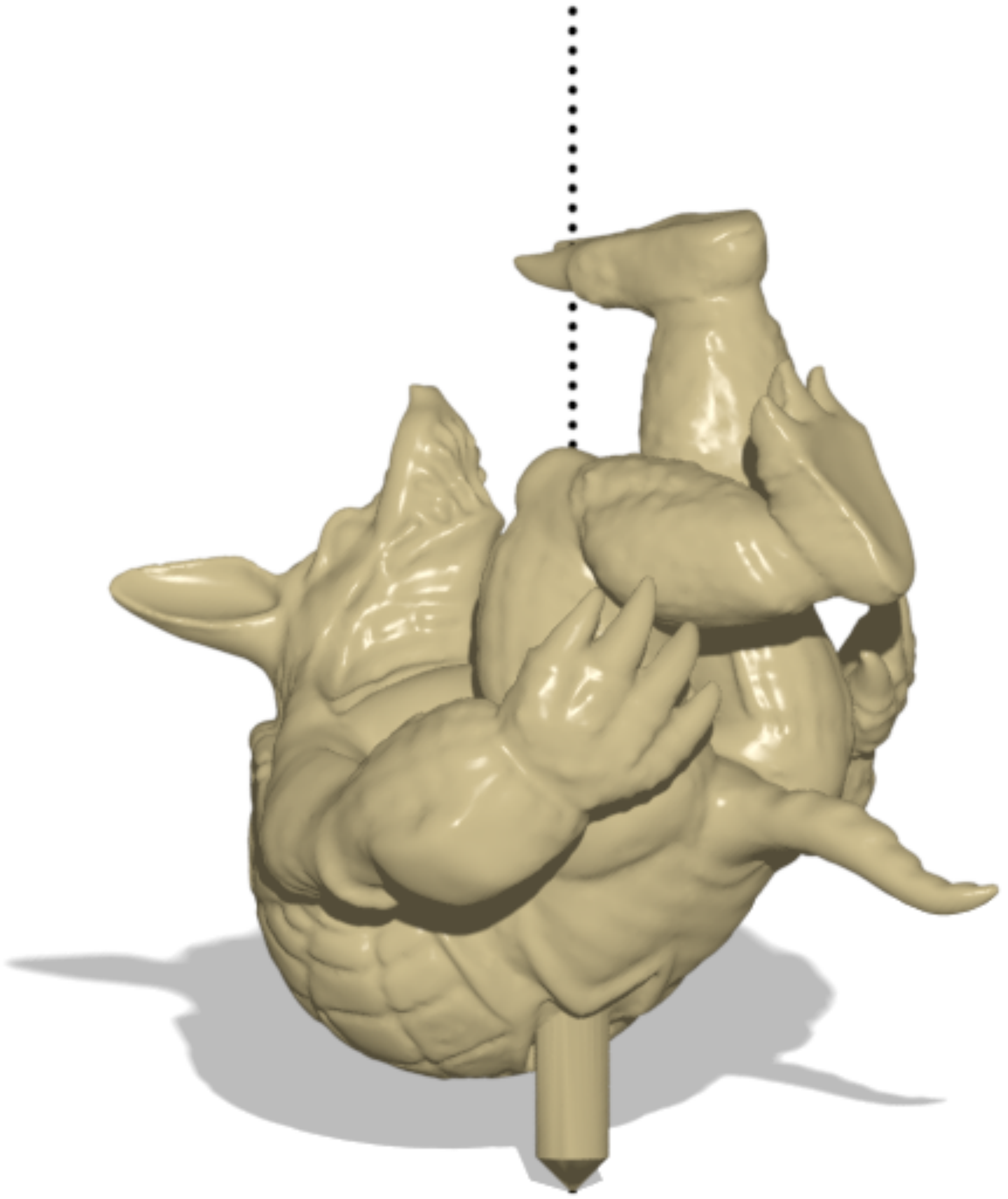
Motivation

[Hager, http://www.hager-drechseln.de/]

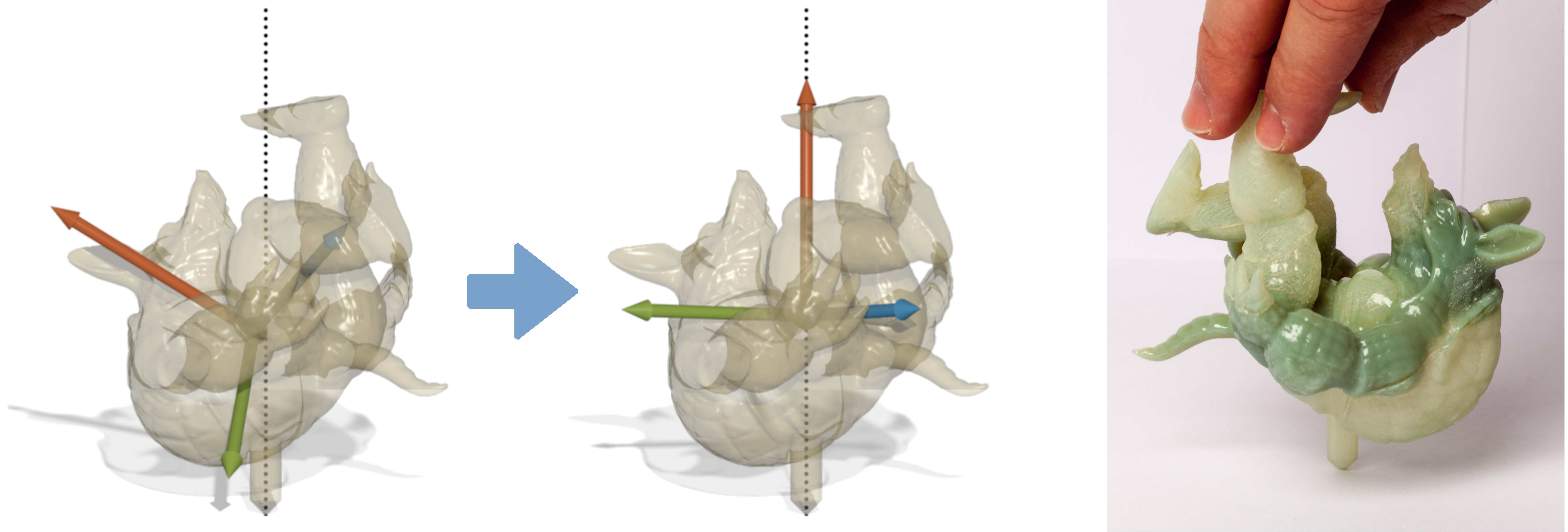
Approach



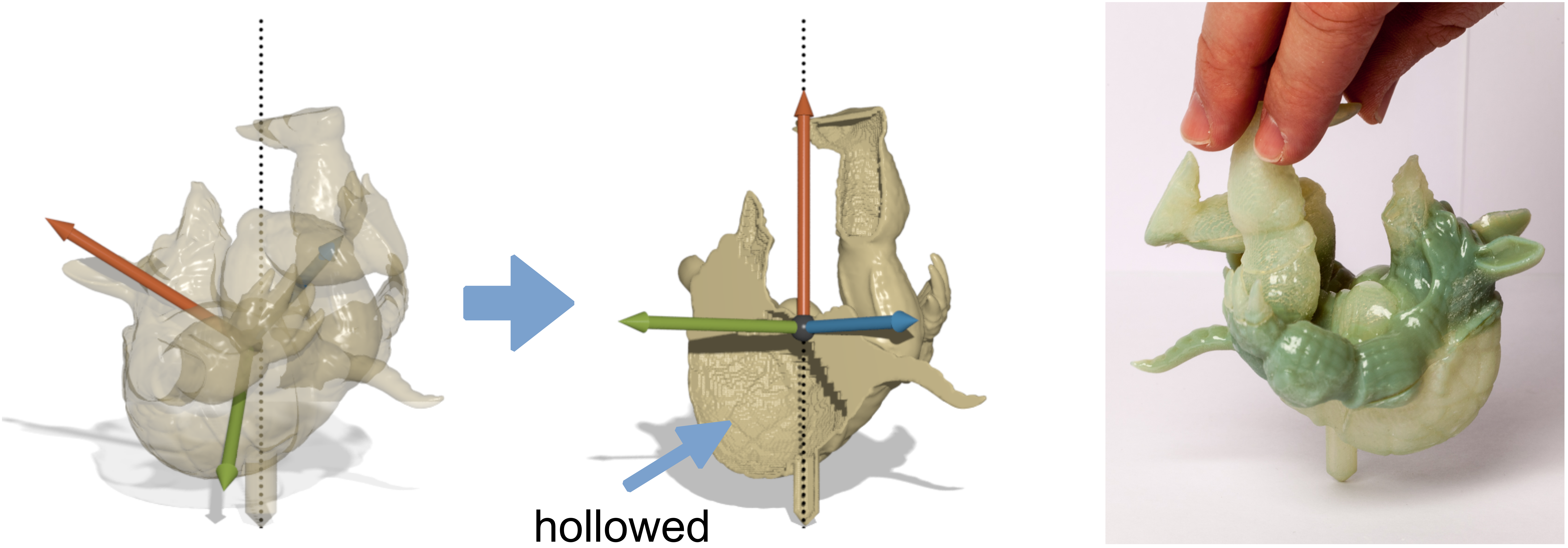
Approach



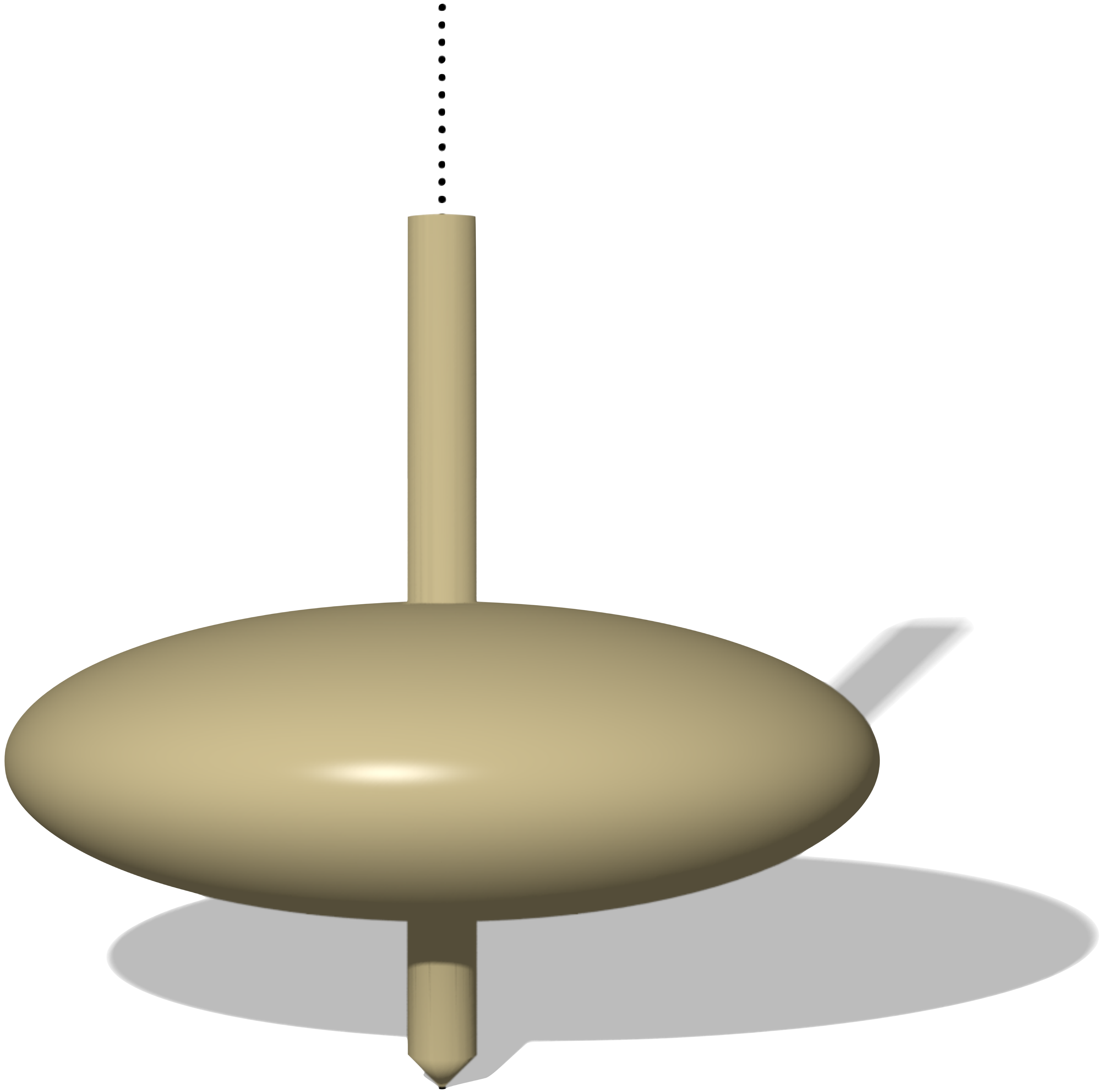
Approach



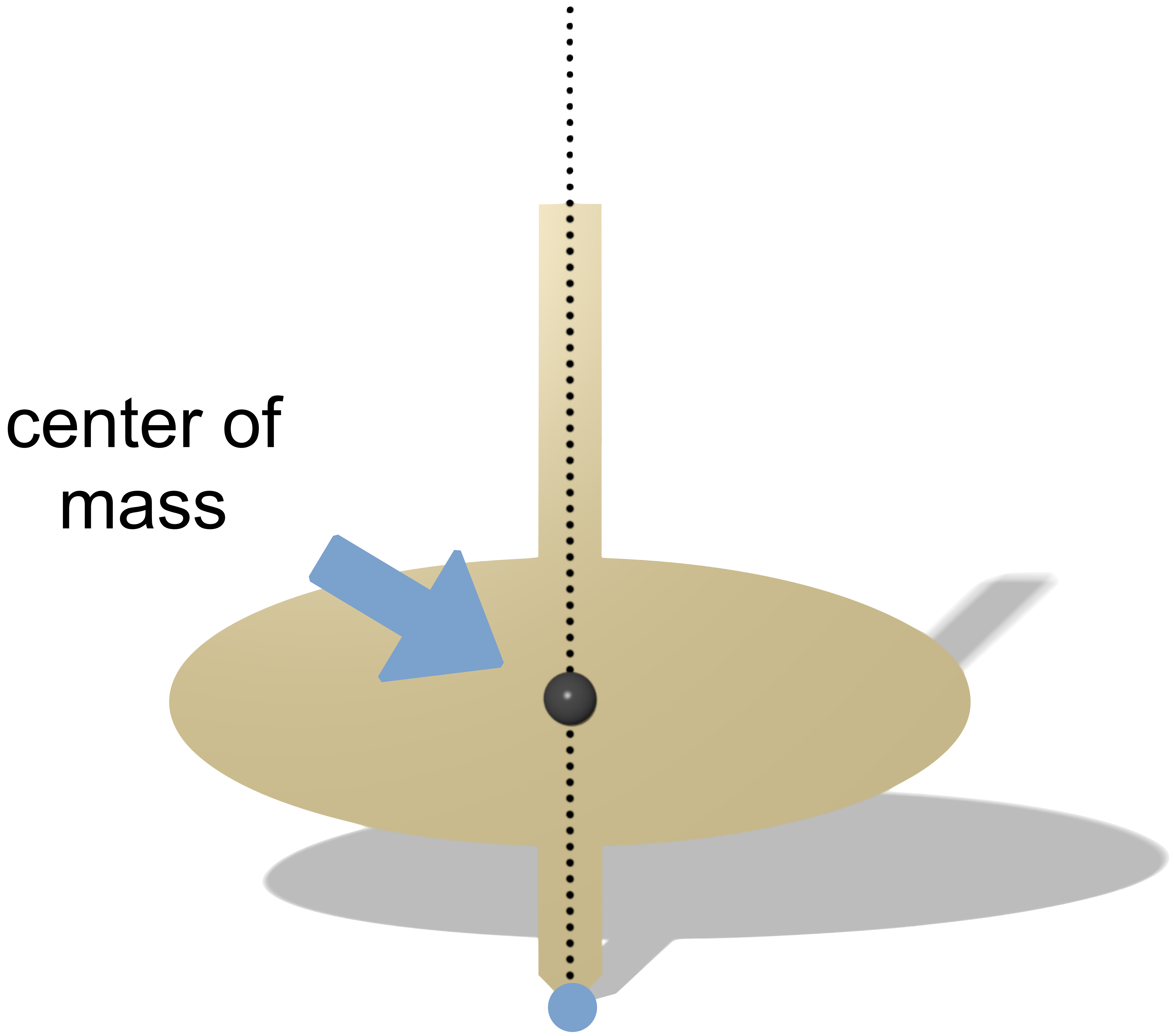
Approach



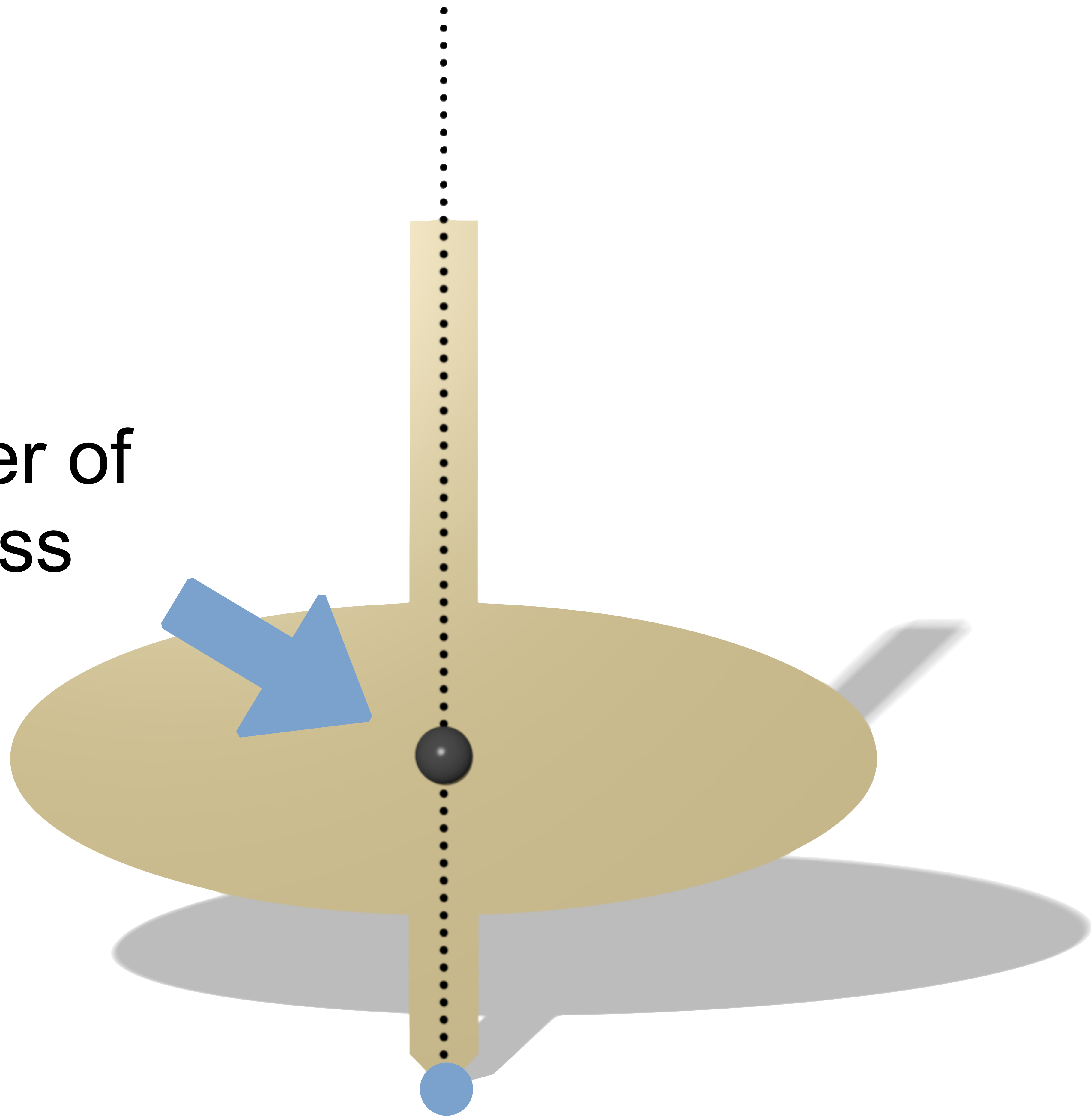
Dynamic Balancing: Challenges

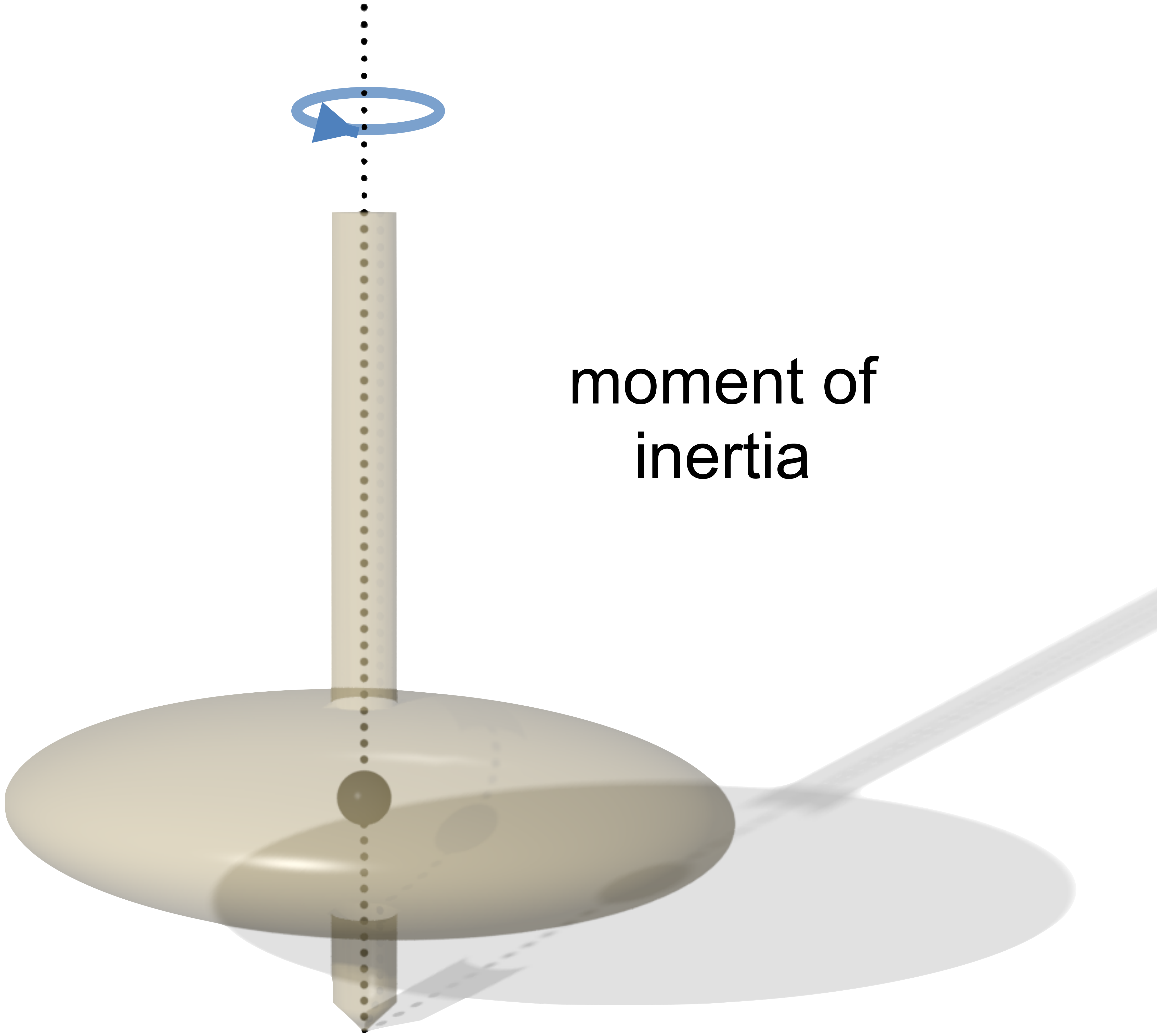


Dynamic Balancing: Challenges

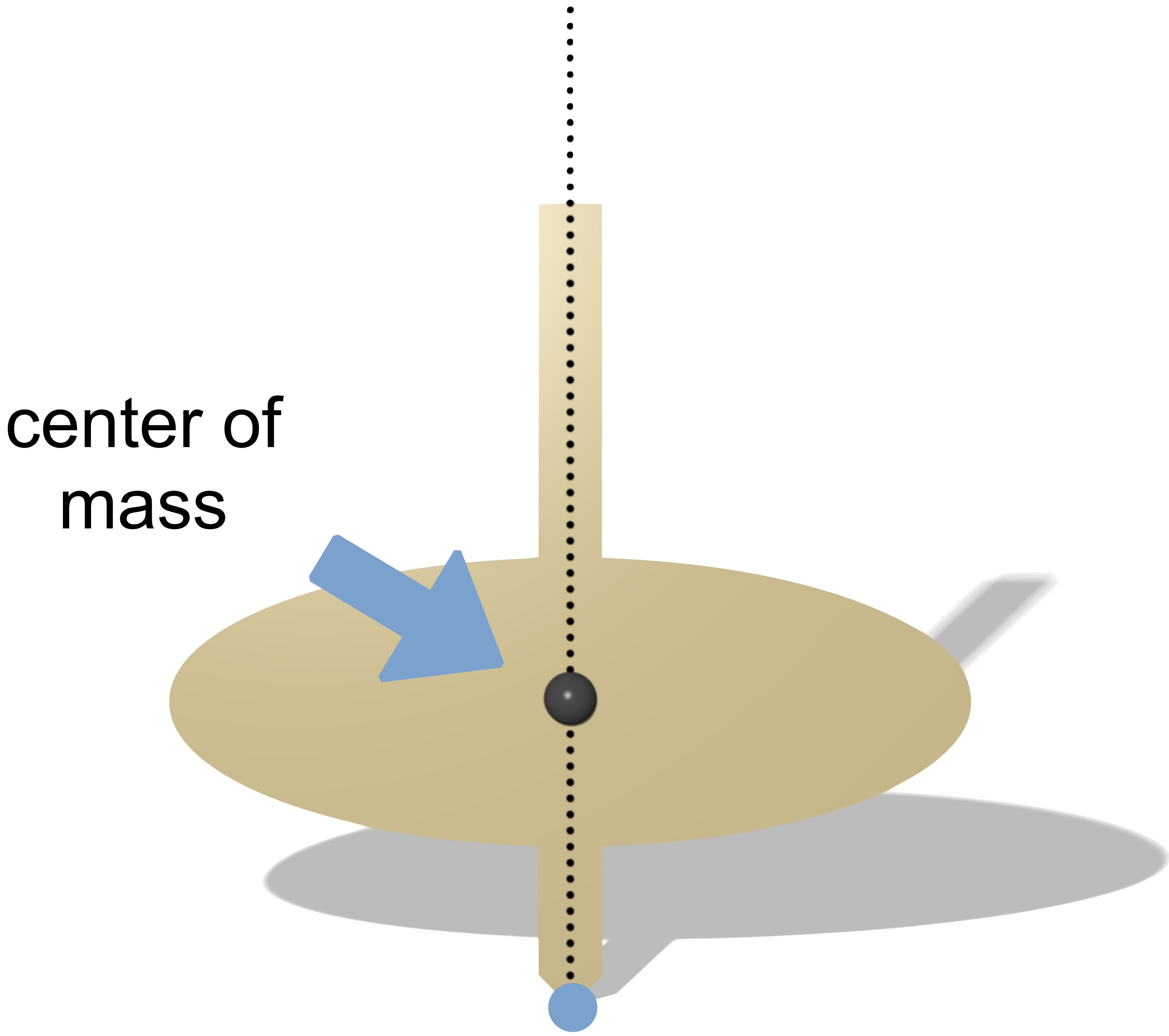


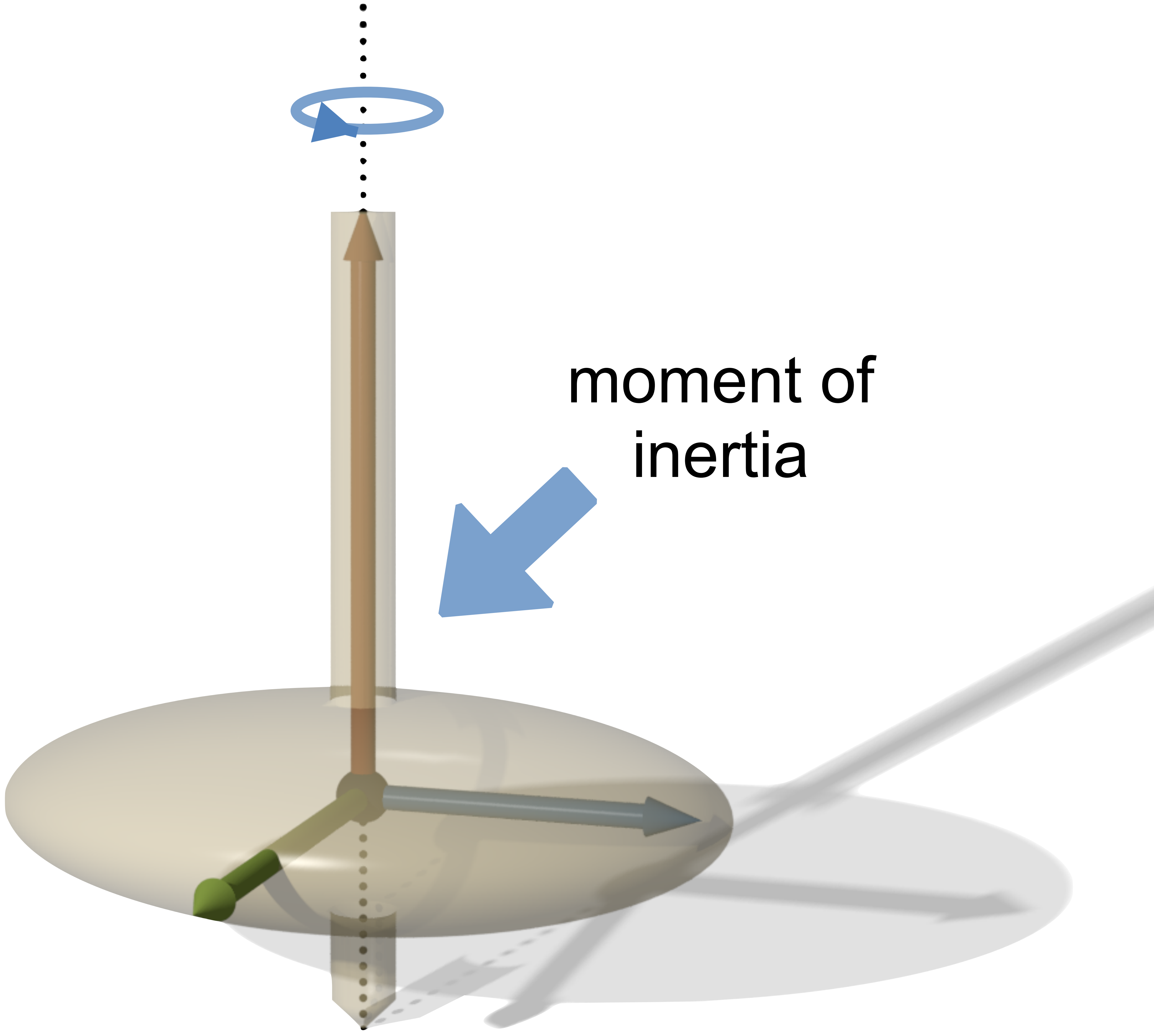
Dynamic Balancing: Challenges





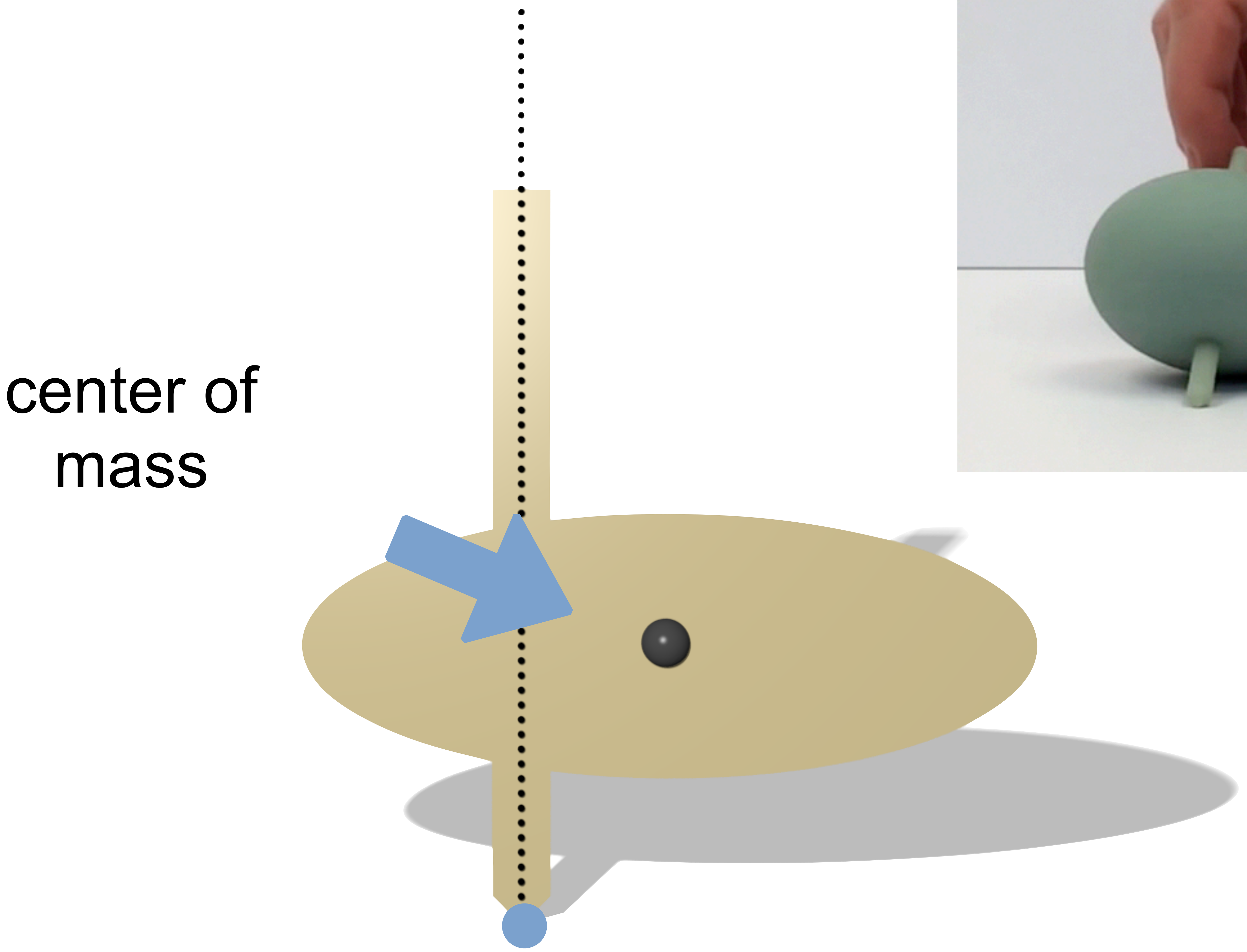
Dynamic Balancing: Challenges

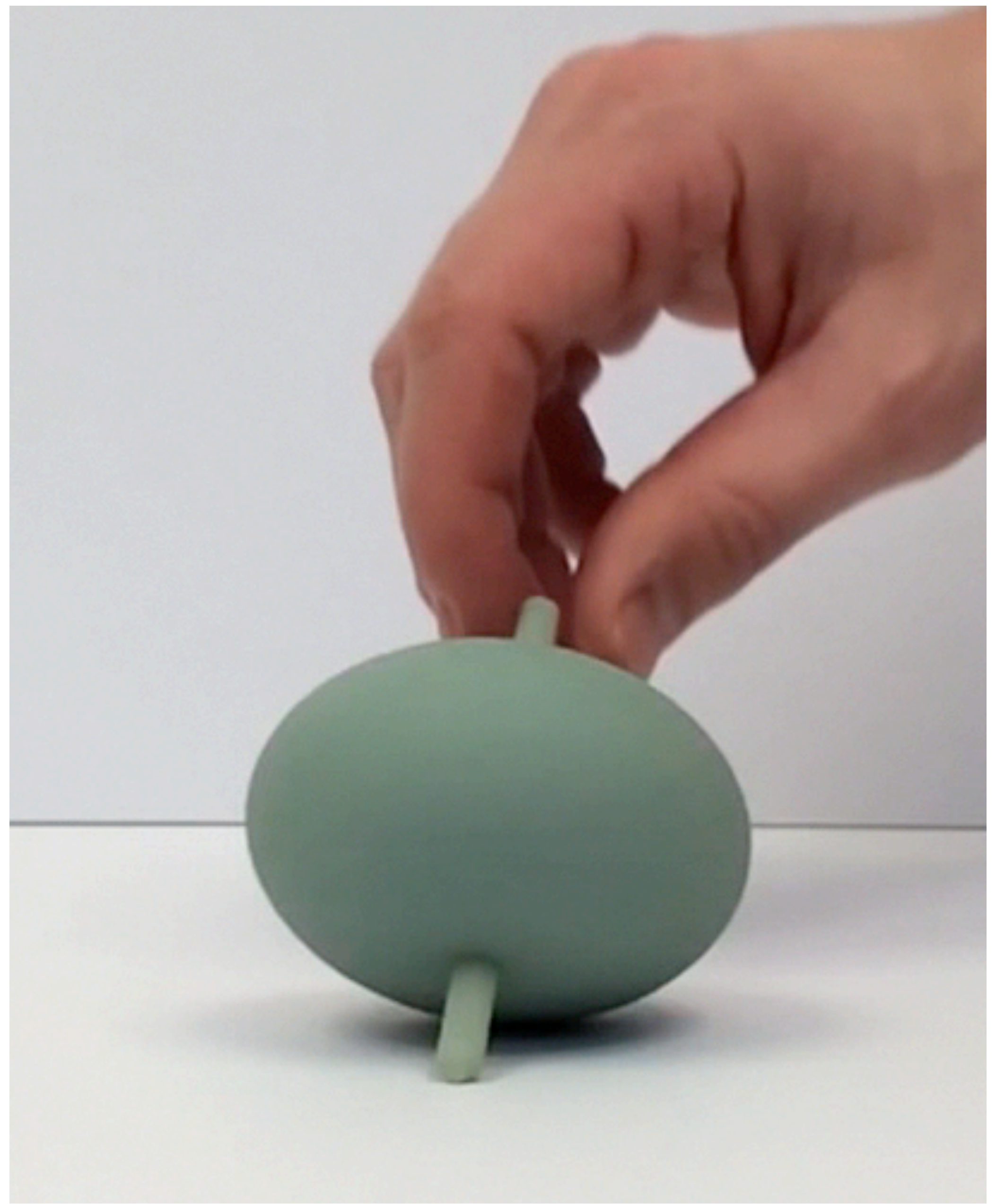


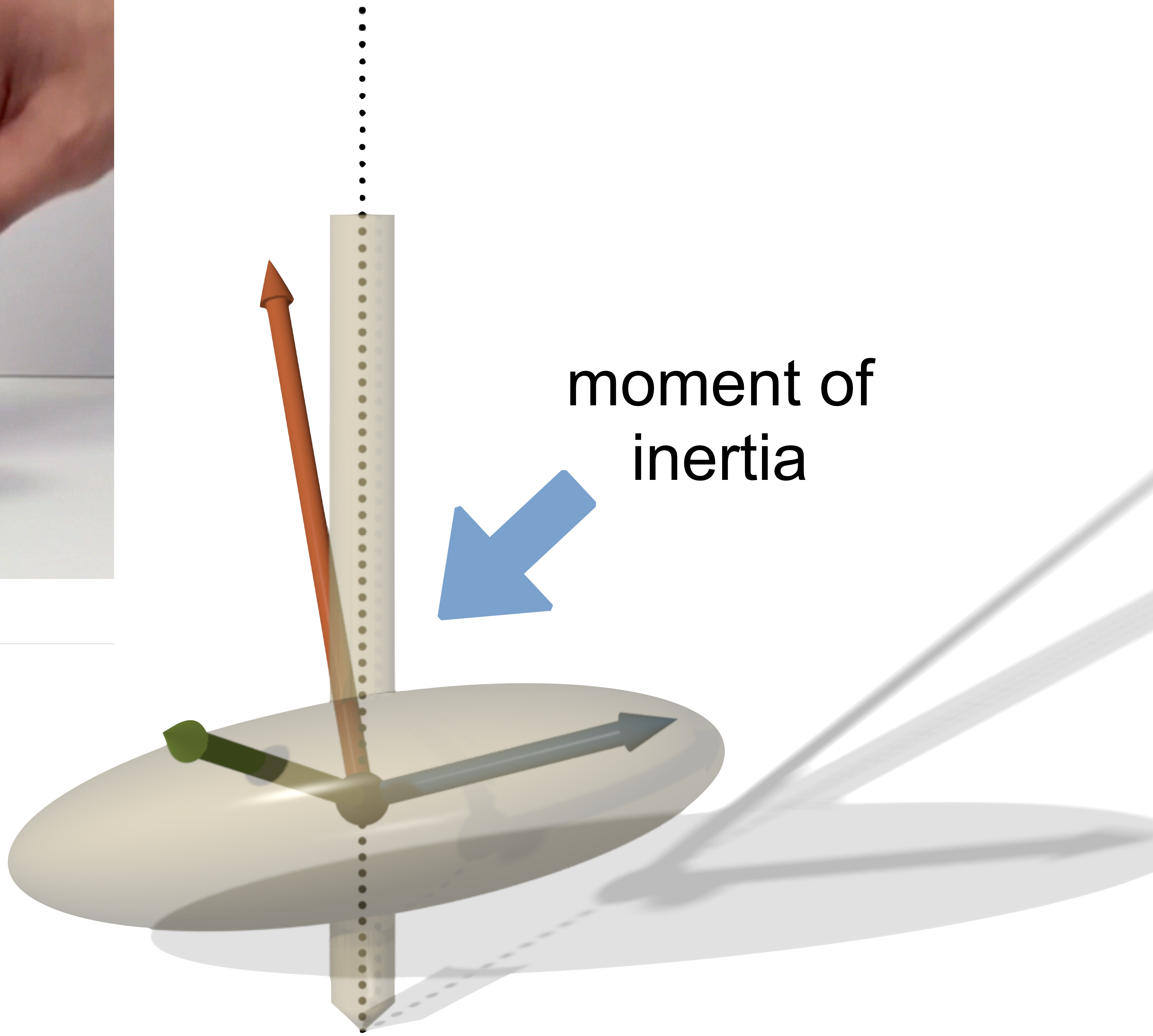


Dynamic Balancing: Challenges

Dynamic Balancing: Challenges







ETH zürich

Disney Research

Agenda

- Motivation
- Coordinate Frames / Motion of a Point
- Mass Properties (Mass, Center of Mass, Moment of Inertia)
- Linear vs. Angular Motion / Momentum
- Equations of Motion
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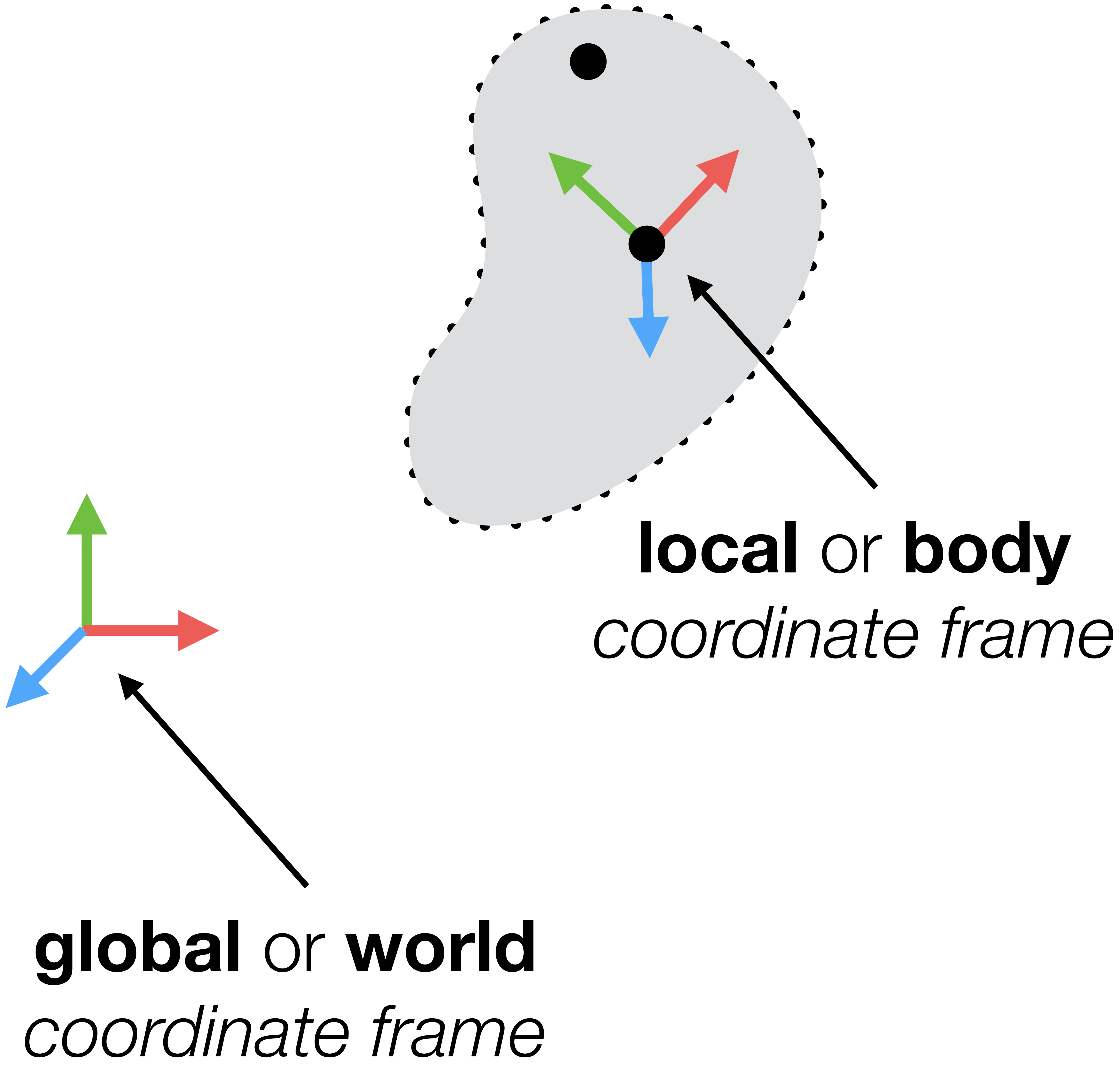
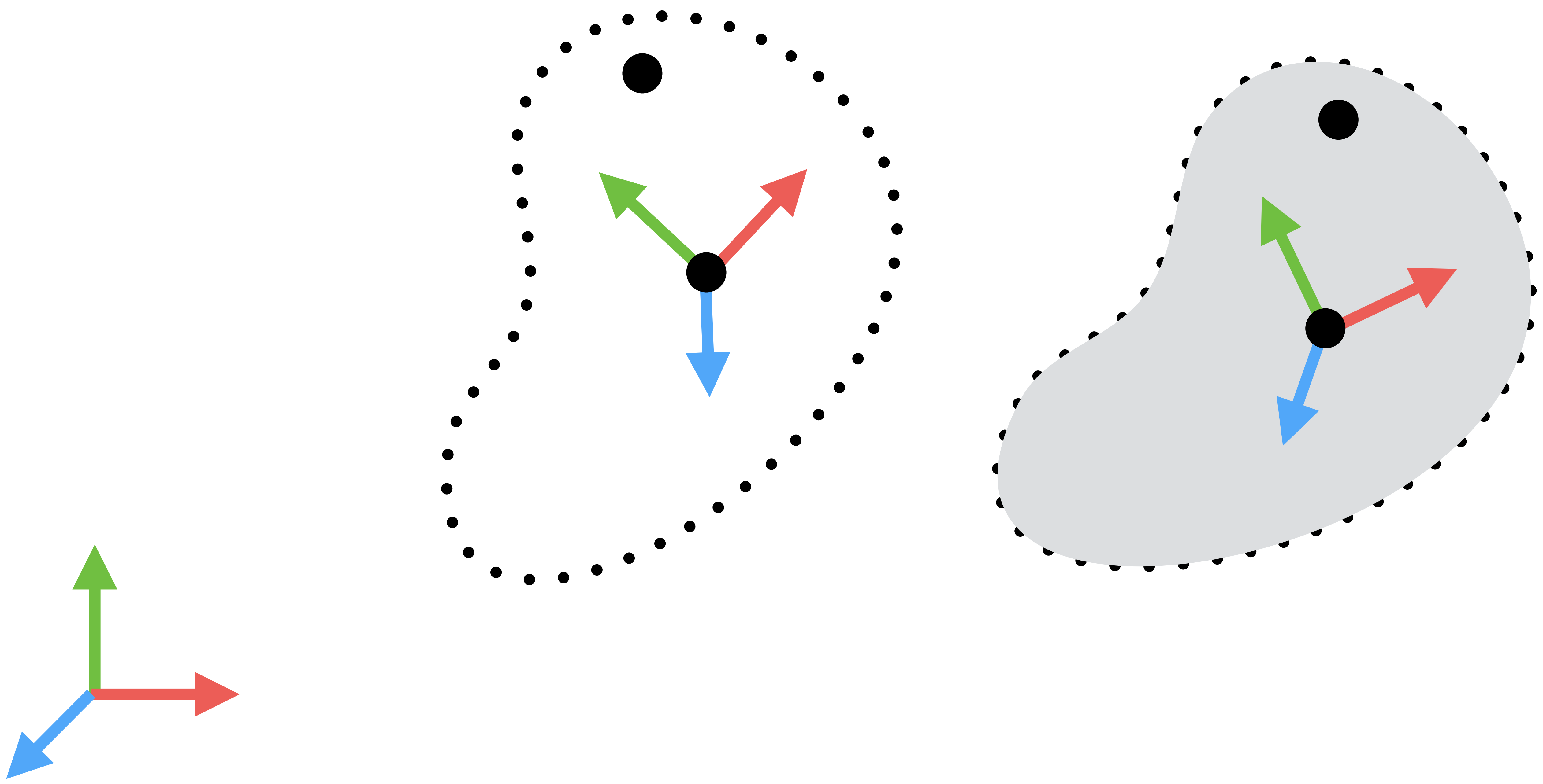
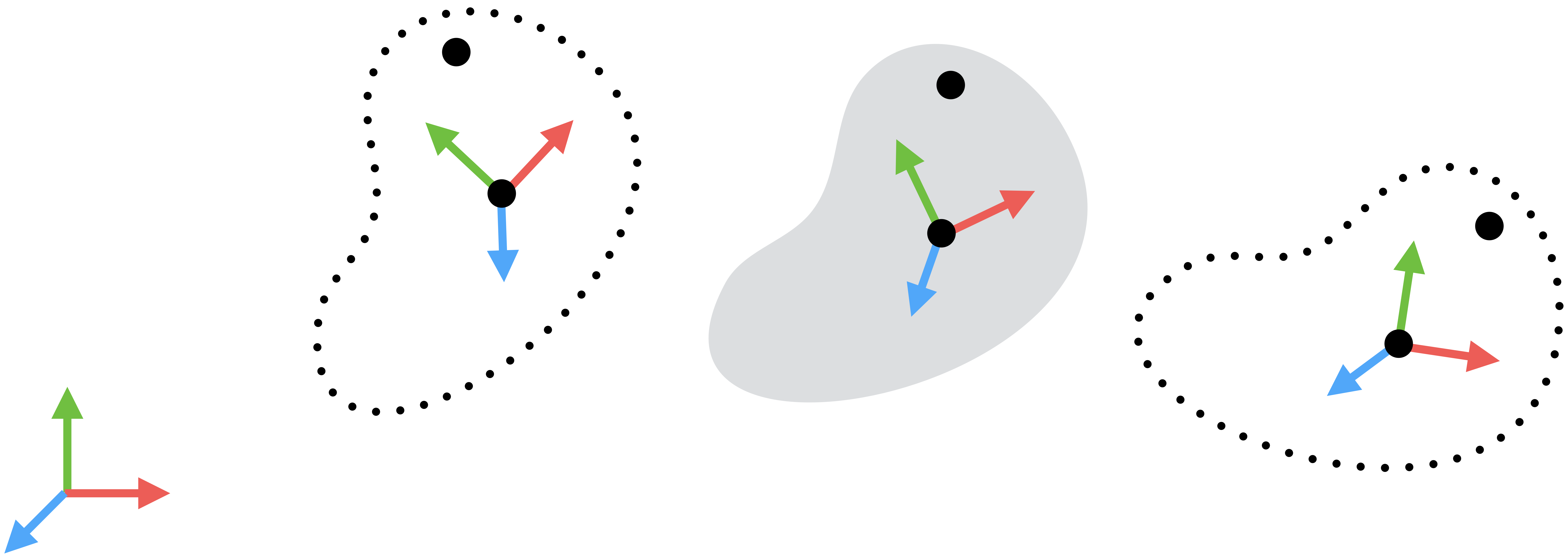
Coordinate Frames



Coordinate Frames

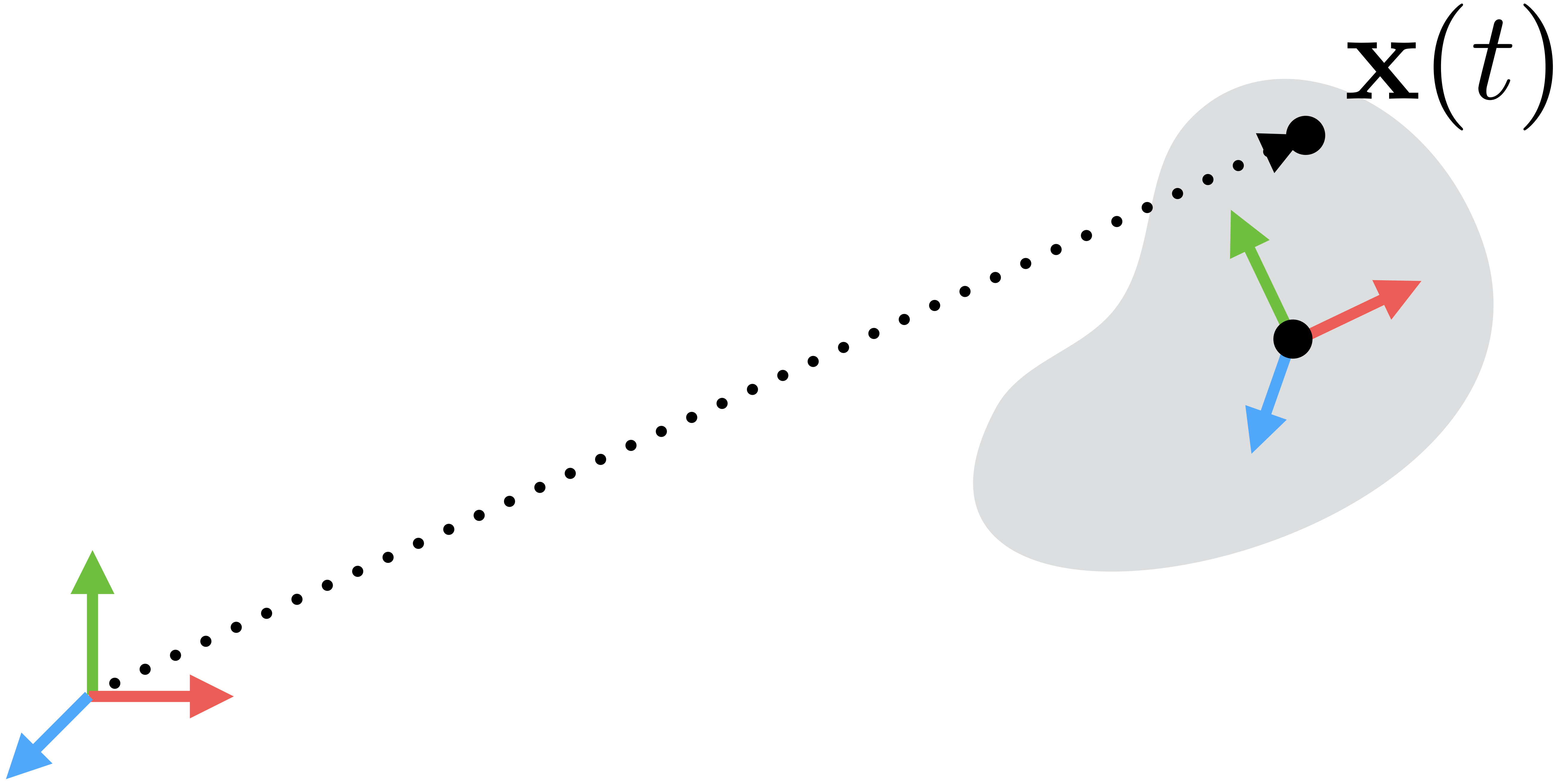


Coordinate Frames

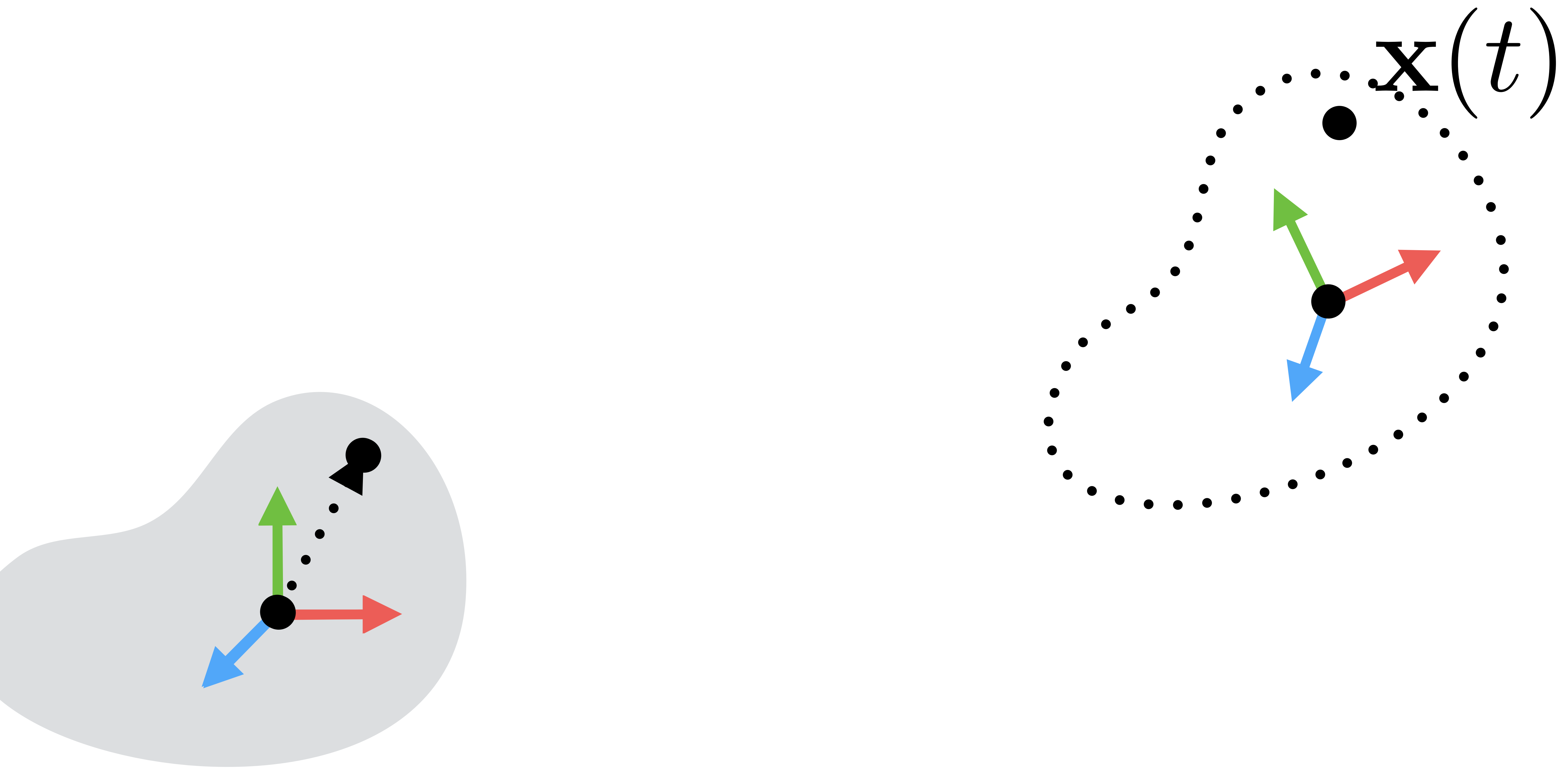


Motion of a Point

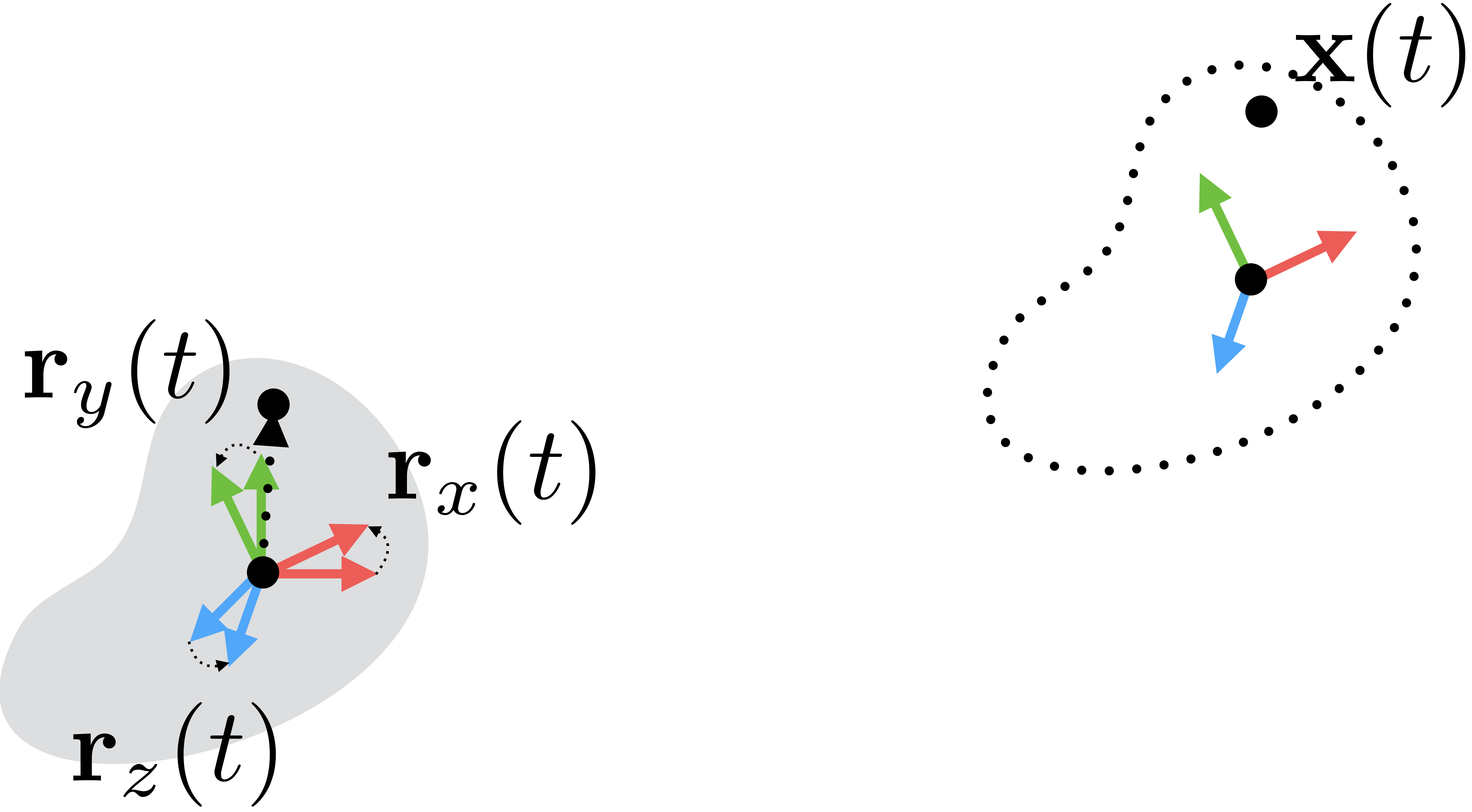
Motion of a Point



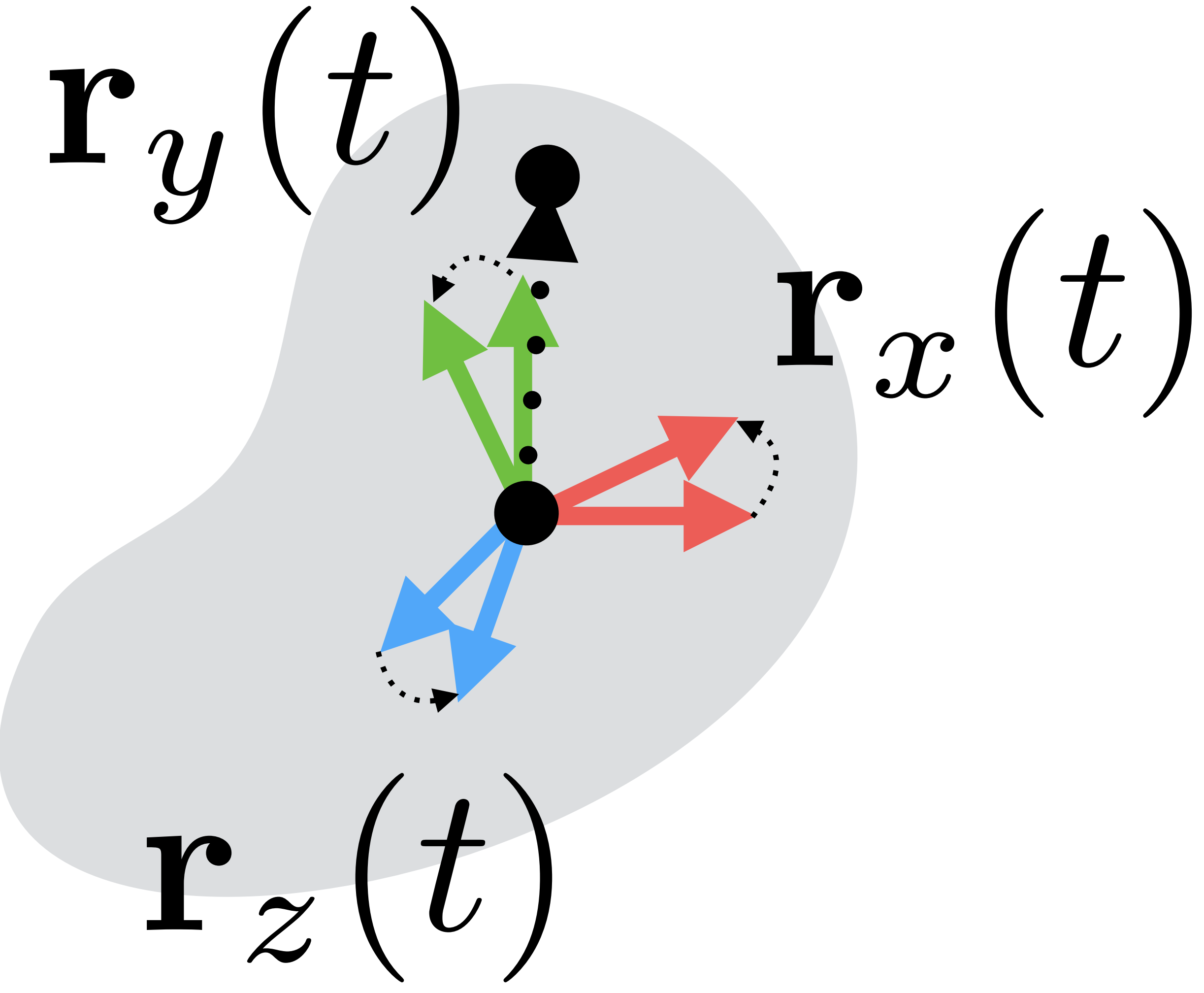
Motion of a Point

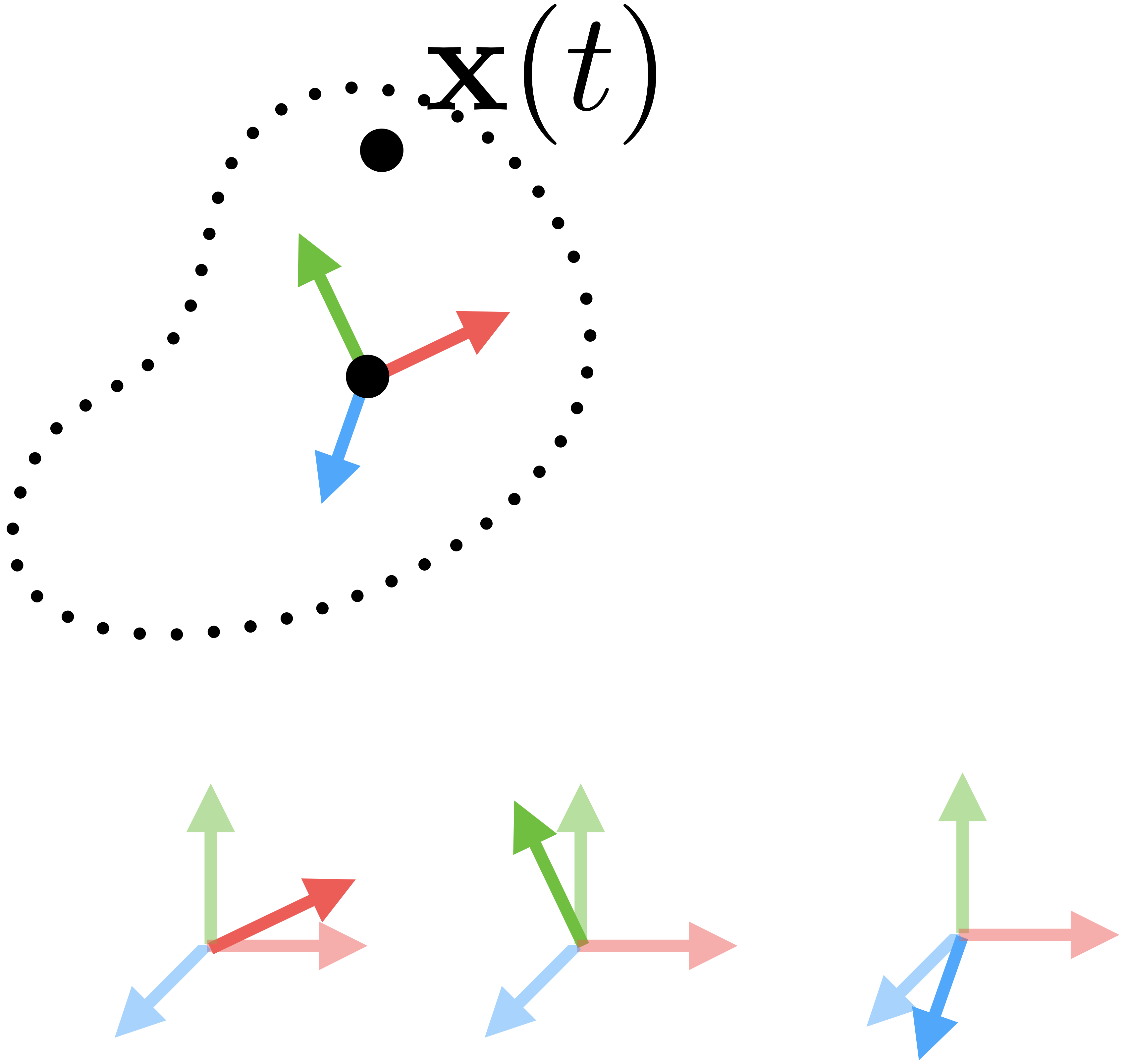


Motion of a Point



Motion of a Point

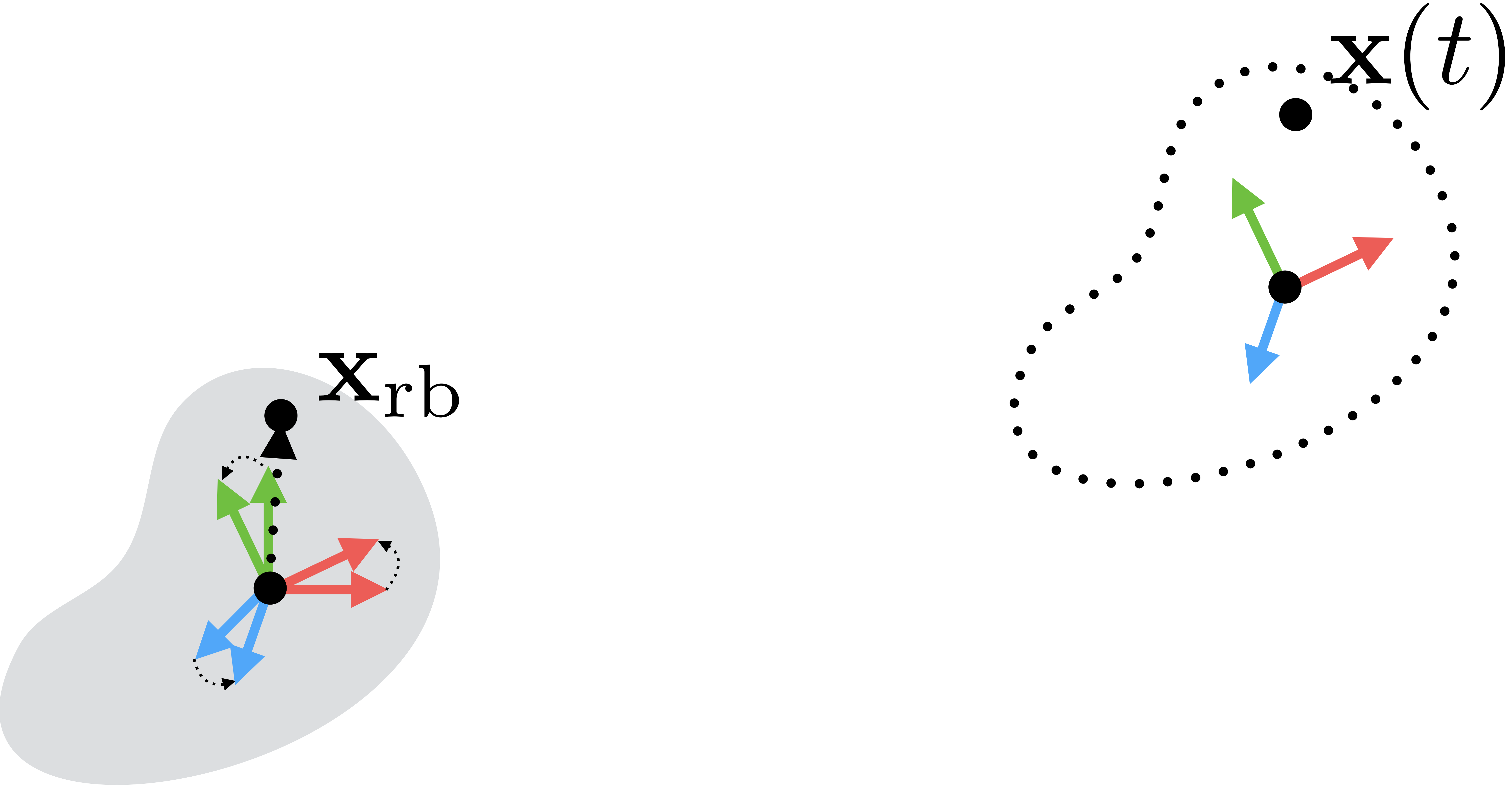




$$\mathbf{R}(t) = [\mathbf{r}_x(t), \mathbf{r}_y(t), \mathbf{r}_z(t)]$$

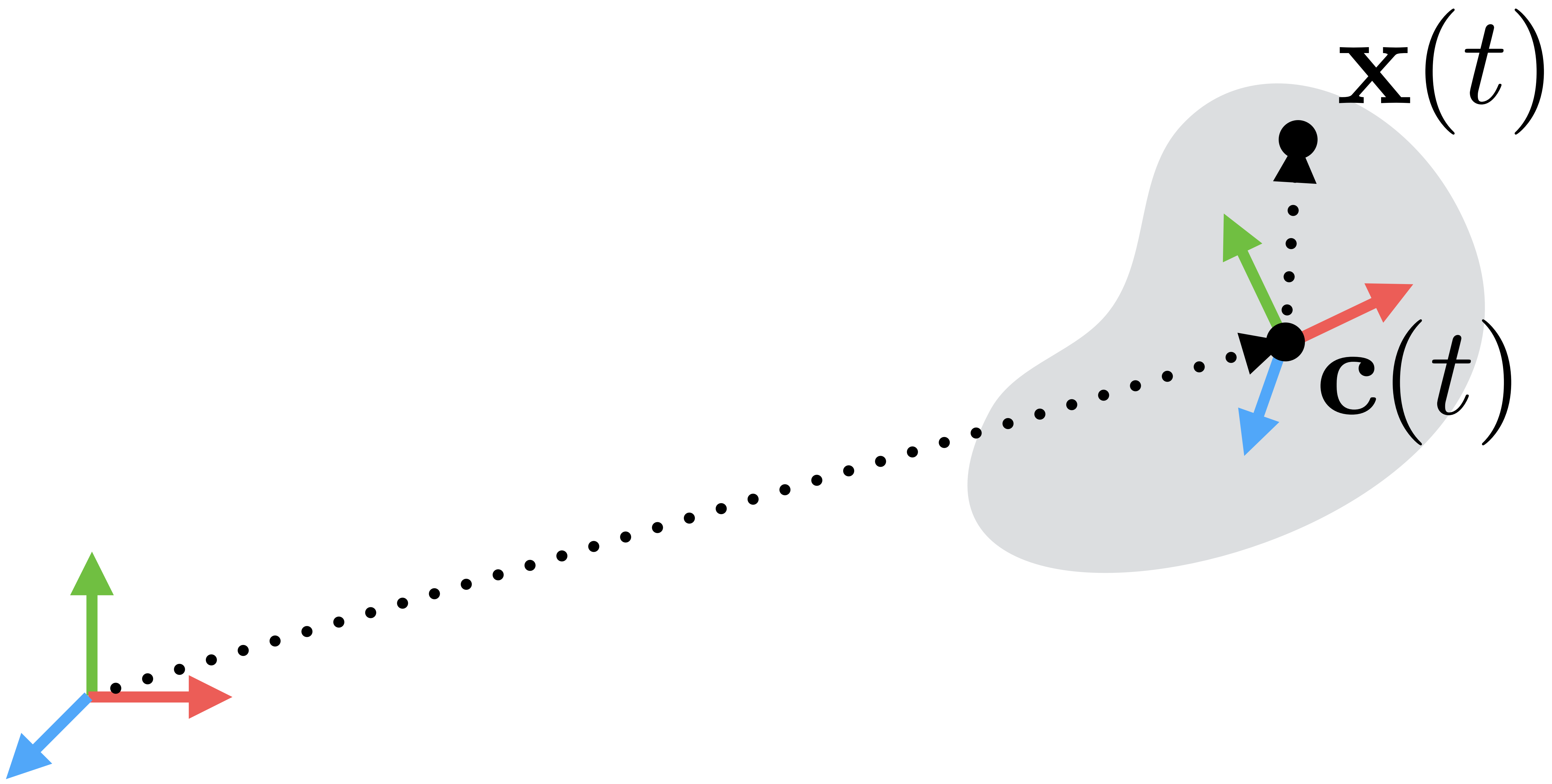
rotation from **body** to **world** coordinates

Motion of a Point



$$\mathbf{R}(t)\mathbf{x}_{\mathrm{rb}}$$

Motion of a Point



$$\mathbf{x}(t) = \mathbf{R}(t)\mathbf{x}_{\mathrm{rb}} + \mathbf{c}(t)$$

position

orientation of body

position of body

Motion of a Point

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{R}}(t)\mathbf{x}_{\mathrm{rb}} + \dot{\mathbf{c}}(t)$$

velocity

1st time derivative of *orientation*

1st time derivative of *position*

Motion of a Point

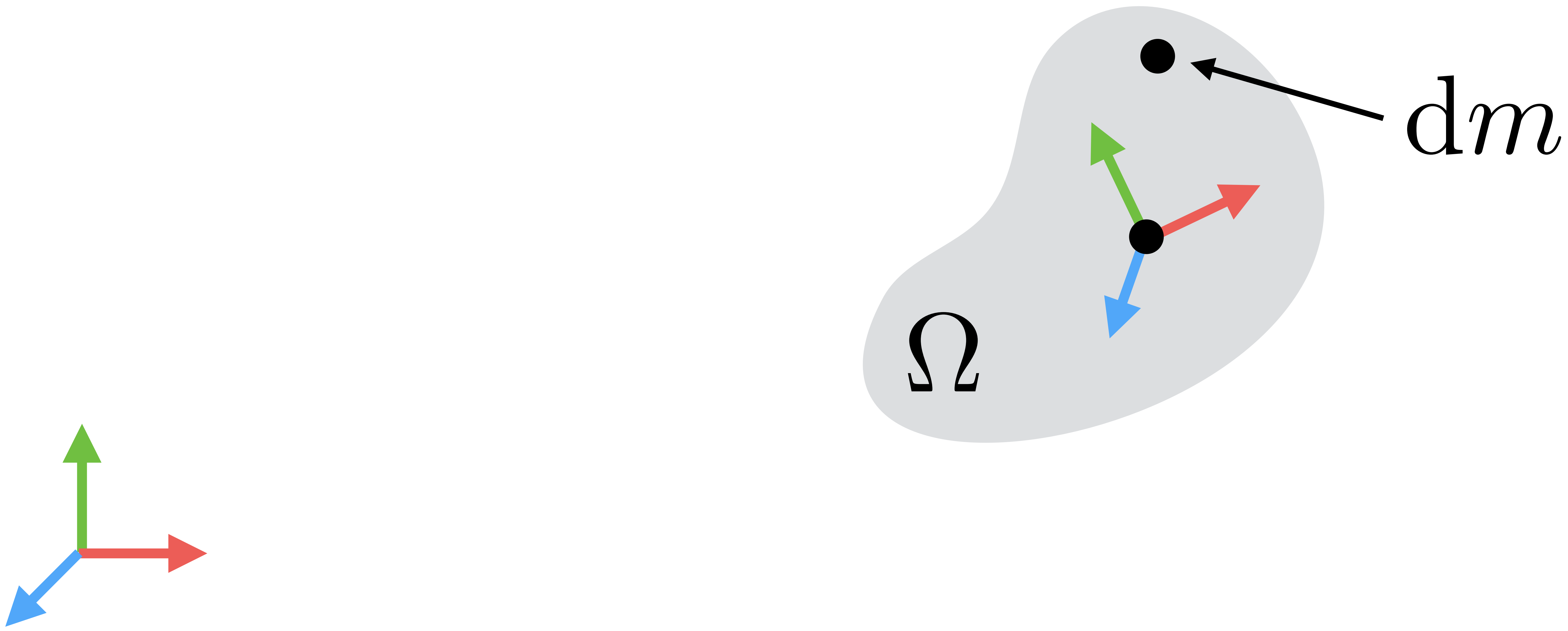
$$\ddot{\mathbf{x}}(t) = \ddot{\mathbf{R}}(t)\mathbf{x}_{\mathrm{rb}} + \ddot{\mathbf{c}}(t)$$

acceleration

2nd time derivative of *orientation*

2nd time derivative of *position*

Mass Properties: Mass



density

$$M = \int_{\Omega} \mathrm{d}m = \int_{\Omega} \rho \, \mathrm{dV}$$

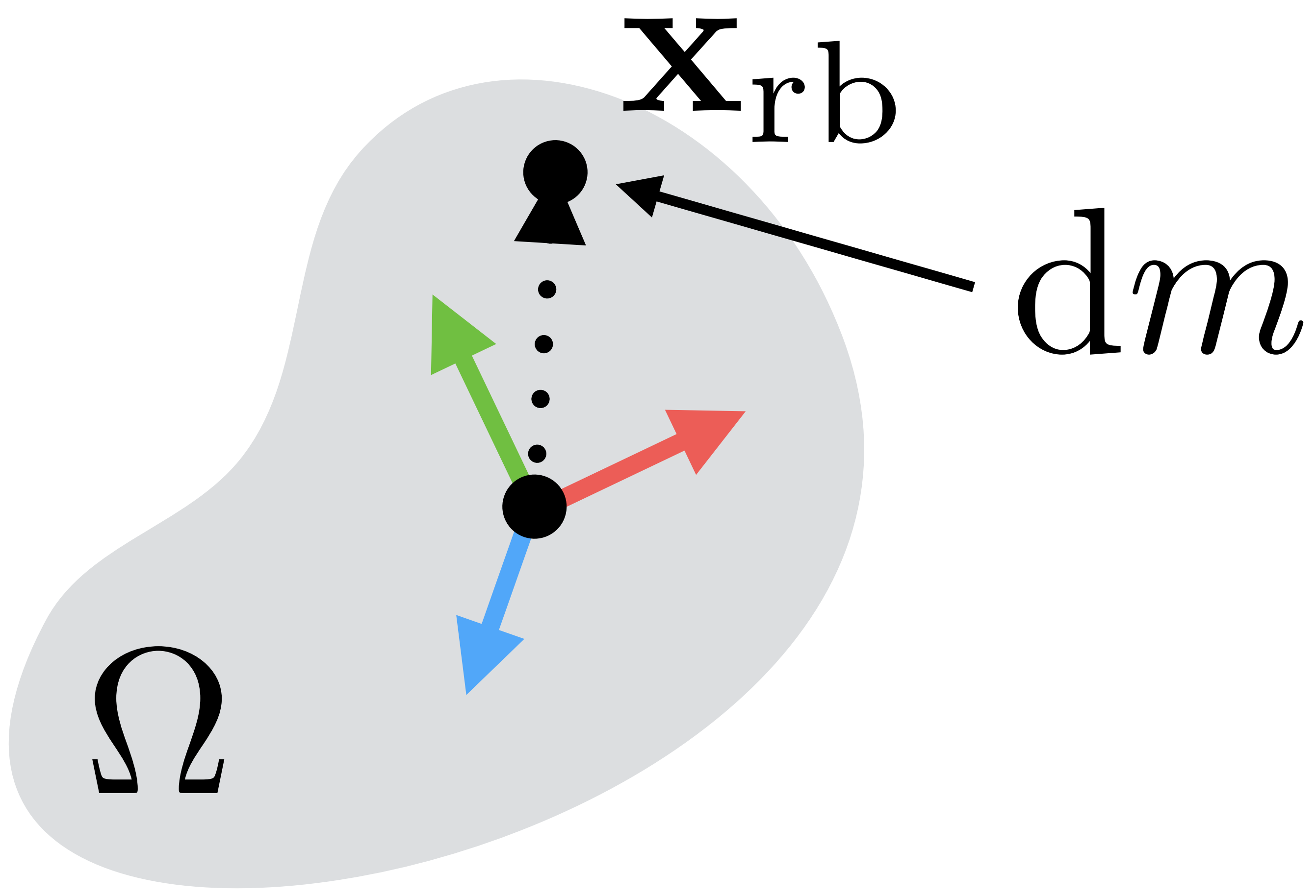
infinitesimal *mass*

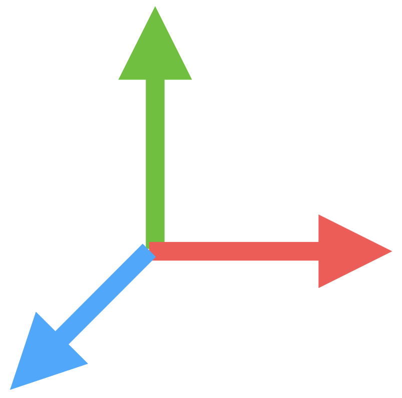
infinitesimal *volume*

Mass Properties: Mass

- independent of coordinate frame
- invariant under translations / rotations

Mass Properties: Center of Mass



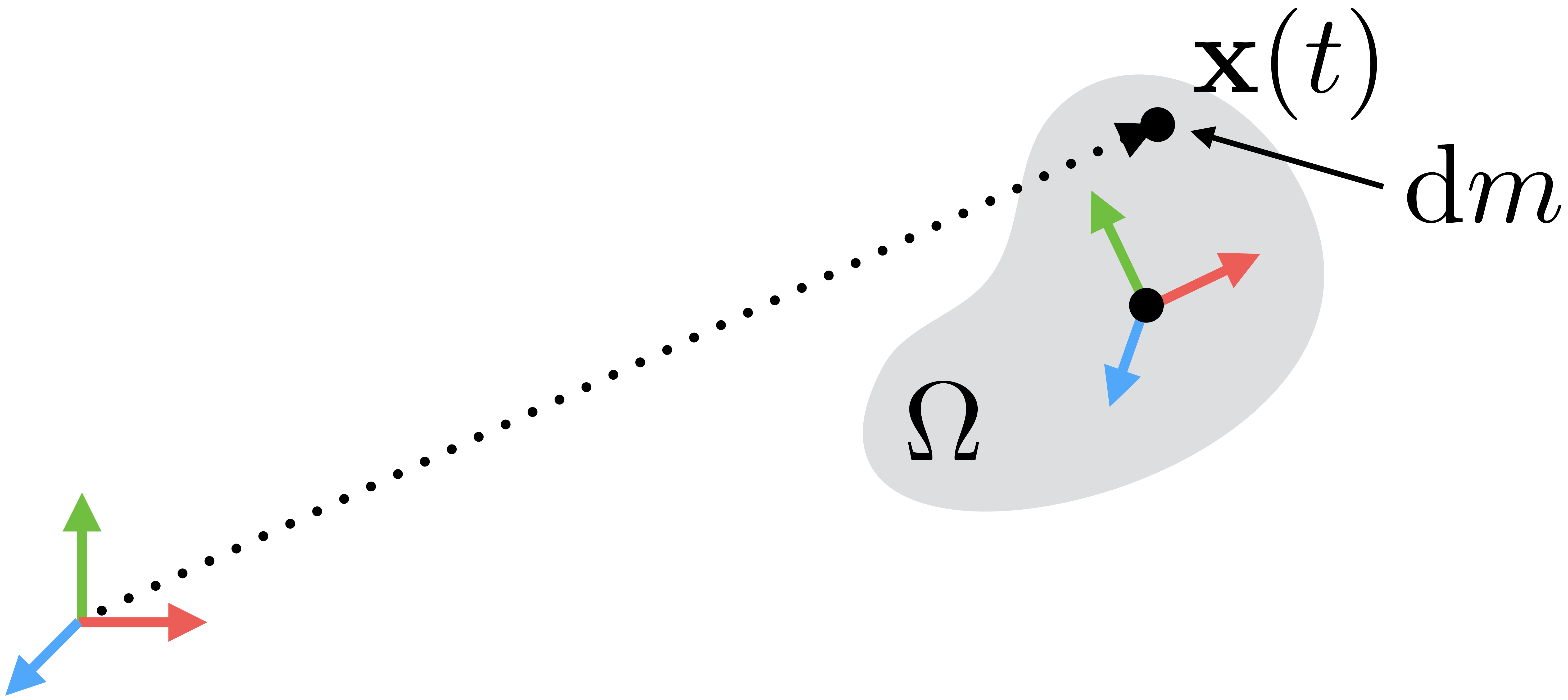


definition

$$\frac{\int_\Omega \mathbf{x}_{\mathrm{rb}}\,\mathrm{d}m}{\int_\Omega \mathrm{d}m} = \frac{\int_\Omega \mathbf{x}_{\mathrm{rb}}\,\rho\,\mathrm{dV}}{M} =: \mathbf{0}$$

mass weighted average

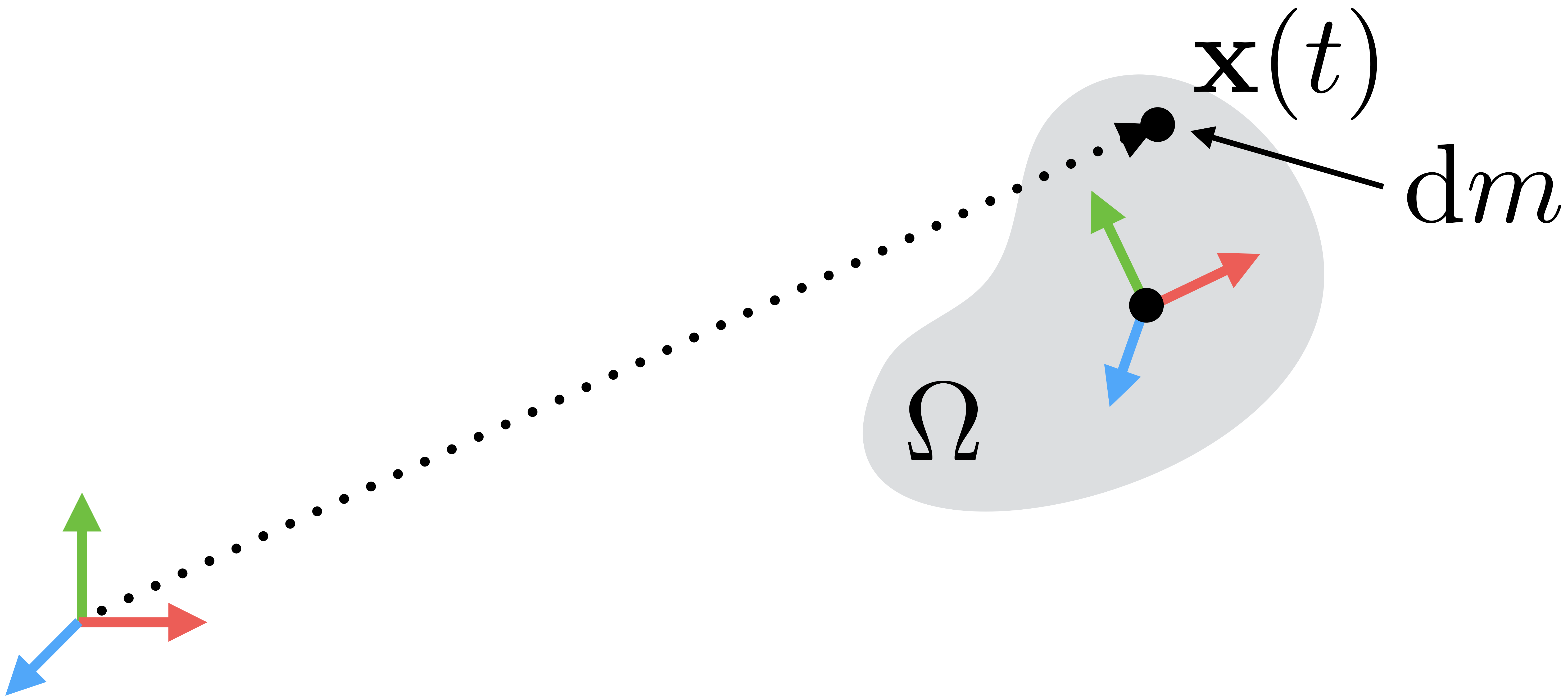
Mass Properties: Center of Mass



$$\frac{\int_\Omega \mathbf{x}(t)\,\mathrm{d}m}{\int_\Omega \mathrm{d}m} = \frac{\int_\Omega \mathbf{R}(t)\mathbf{x}_{\mathrm{rb}} + \mathbf{c}(t)\,\mathrm{d}m}{M}$$

mass weighted average

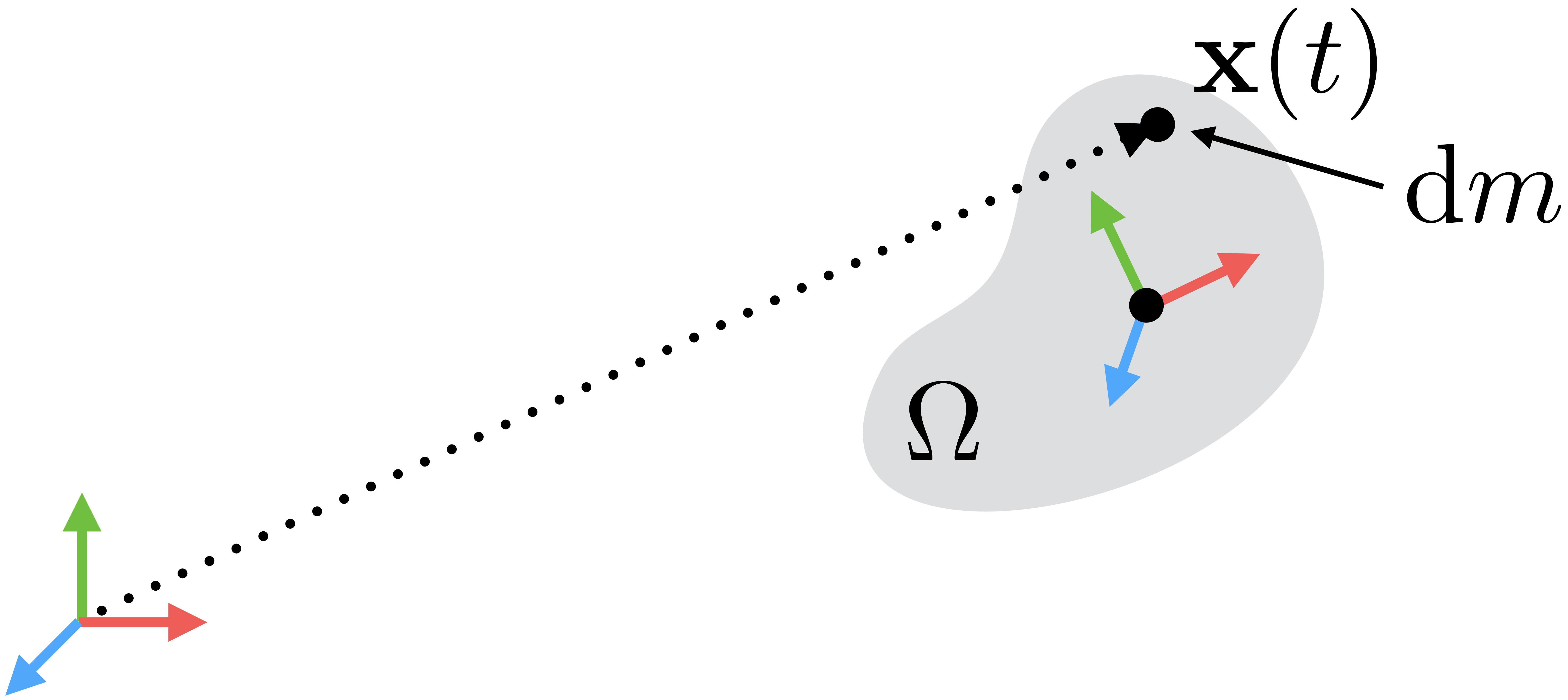
Mass Properties: Center of Mass



$$\frac{\int_\Omega \mathbf{x}(t)\,\mathrm{d}m}{\int_\Omega \mathrm{d}m} = \mathbf{R}(t)\left(\frac{\int_\Omega \mathbf{x}_{\mathrm{rb}}\,\mathrm{d}m}{M}\right) + \mathbf{c}(t)\left(\frac{\int_\Omega \mathrm{d}m}{M}\right)$$

mass weighted
average

Mass Properties: Center of Mass



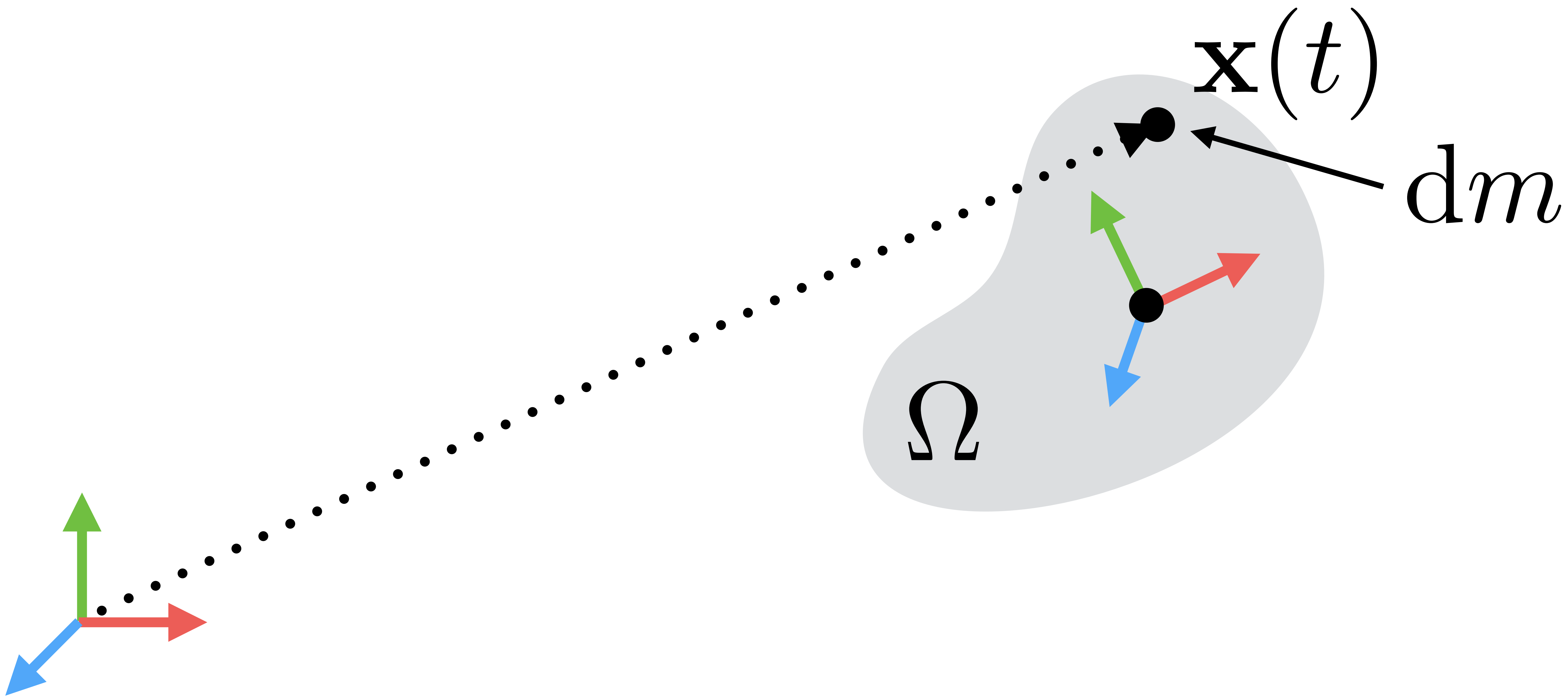
$$\frac{\int_\Omega \mathbf{x}(t)\,\mathrm{d}m}{\int_\Omega \mathrm{d}m} = \mathbf{R}(t)\left(\frac{\int_\Omega \mathbf{x}_{\mathrm{rb}}\,\mathrm{d}m}{M}\right) + \mathbf{c}(t)\left(\frac{\int_\Omega \mathrm{d}m}{M}\right)$$

mass weighted average

zero by *definition*

mass divided by *mass*

Mass Properties: Center of Mass



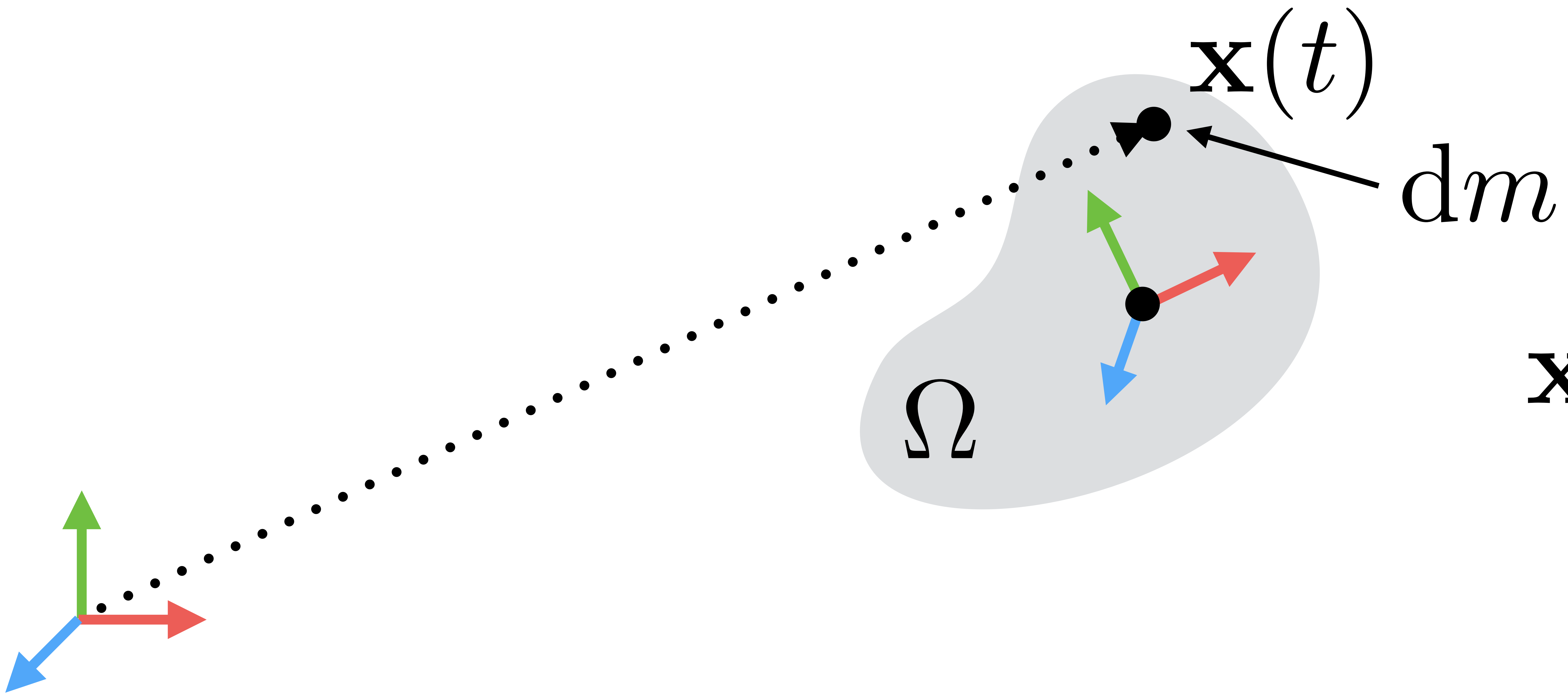
$$\frac{\int_\Omega \mathbf{x}(t)\,\mathrm{d}m}{\int_\Omega \mathrm{d}m} = \mathbf{c}(t) \quad \Rightarrow \quad \frac{\int_\Omega \mathbf{x}(t) - \mathbf{c}(t)\,\mathrm{d}m}{\int_\Omega \mathrm{d}m} = \mathbf{0}$$

mass weighted average

Mass Properties: Center of Mass

- invariant under rotations
- ignore *angular motion* if only interested in *motion* of *center of mass*

Mass Properties: Moment of Inertia



$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\mathbf{I}(t) = \int_\Omega \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} \mathrm{d}m$$

omitting time dependence

Skew-Symmetric Matrix Operator

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b}$$

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

cross product

matrix-vector product

skew-symmetric matrix operator

elements of vector

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Skew-Symmetric Matrix Operator

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b}$$

RULES

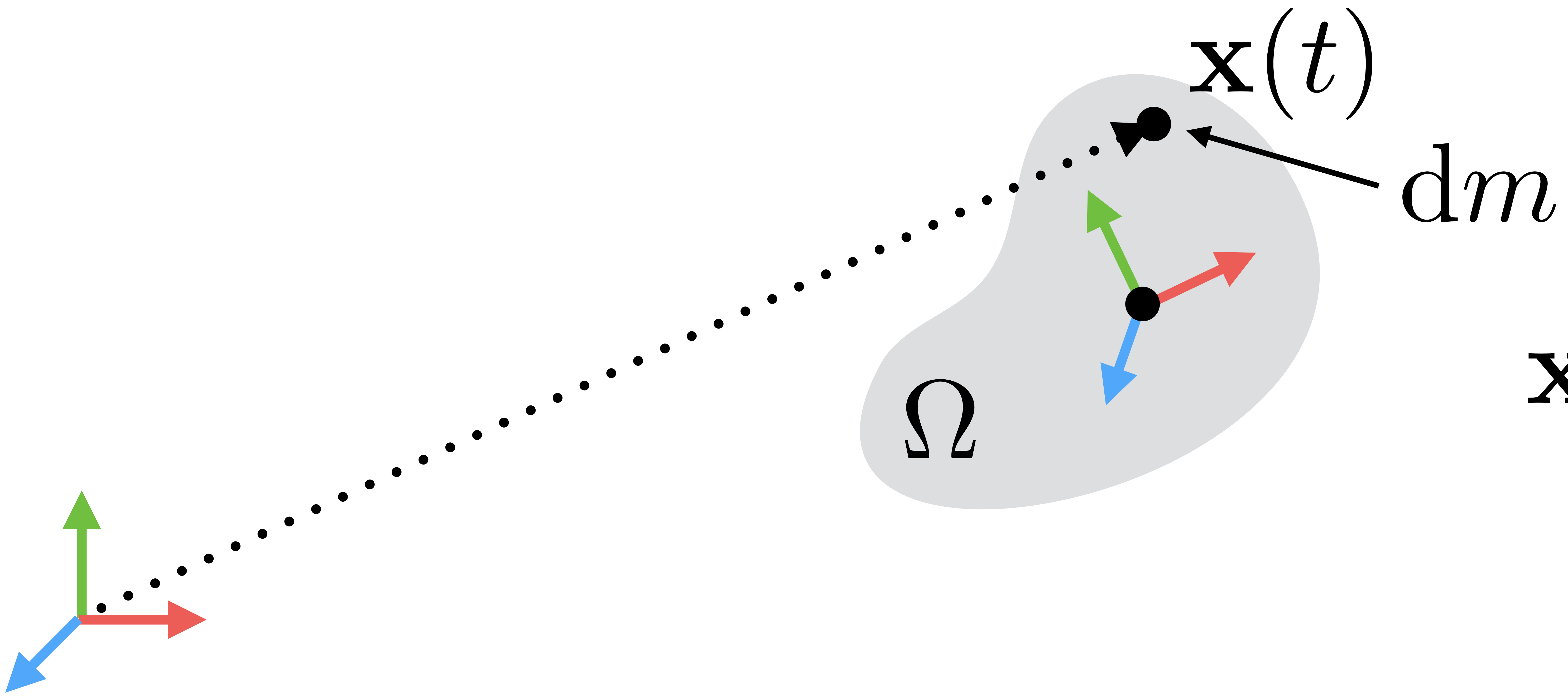
$$[\mathbf{a} + \mathbf{b}]_\times = [\mathbf{a}]_\times + [\mathbf{b}]_\times$$

$$[\mathbf{a}]_\times^T = -[\mathbf{a}]_\times$$

$$[\mathbf{a}]_\times [\mathbf{b}]_\times = \mathbf{b}\mathbf{a}^T - \mathbf{a}^T \mathbf{b} \mathbf{E}_{3\times 3}$$

$\mathbf{E}_{3\times 3}$ *identity matrix*

Mass Properties: Moment of Inertia

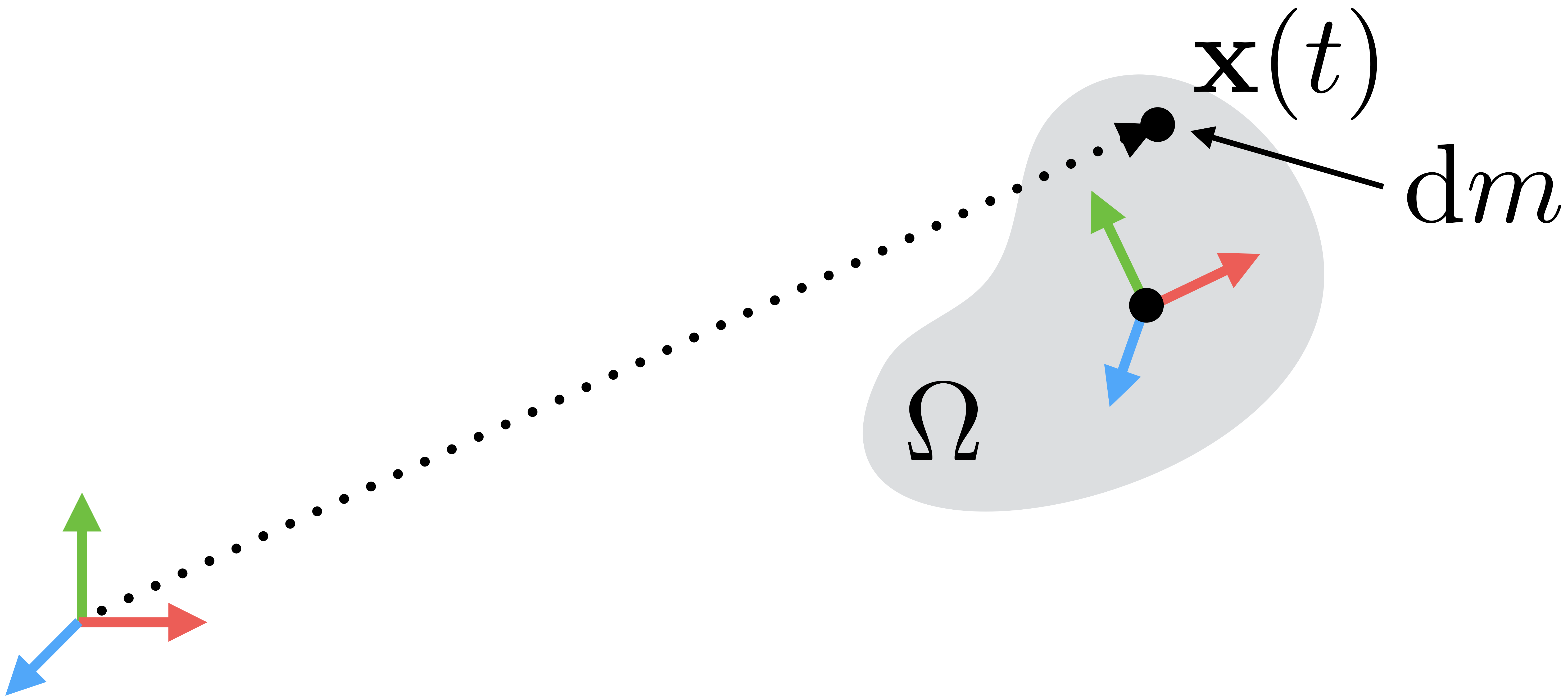


$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\mathbf{I}(t) = \int_{\Omega} \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} \mathrm{d}m$$

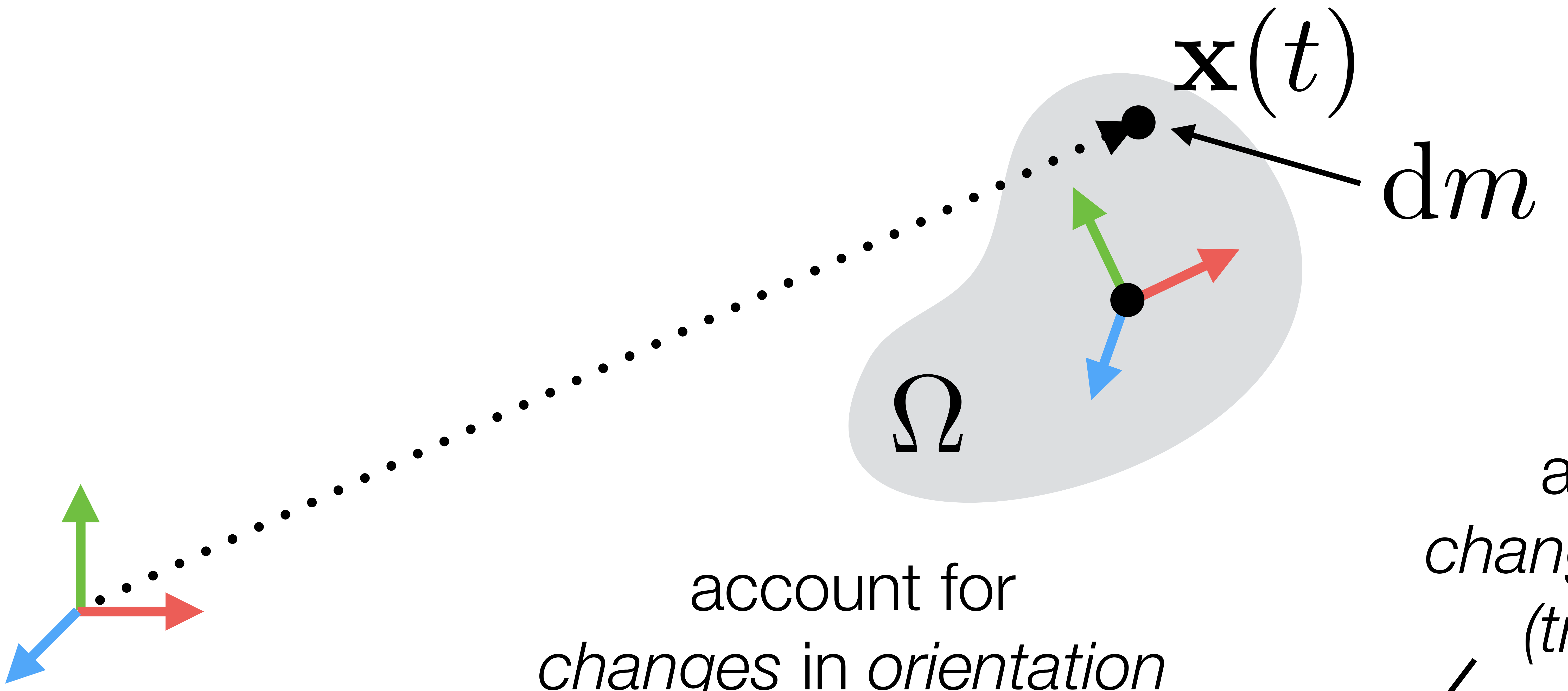
omitting time dependence

Mass Properties: Moment of Inertia



$$\mathbf{I}(t) = \int_{\Omega} [\mathbf{x}(t)]_{\times}^{T} [\mathbf{x}(t)]_{\times} \, \mathrm{d}m$$

Mass Properties: Moment of Inertia



account for
changes in *orientation*
(rotations)

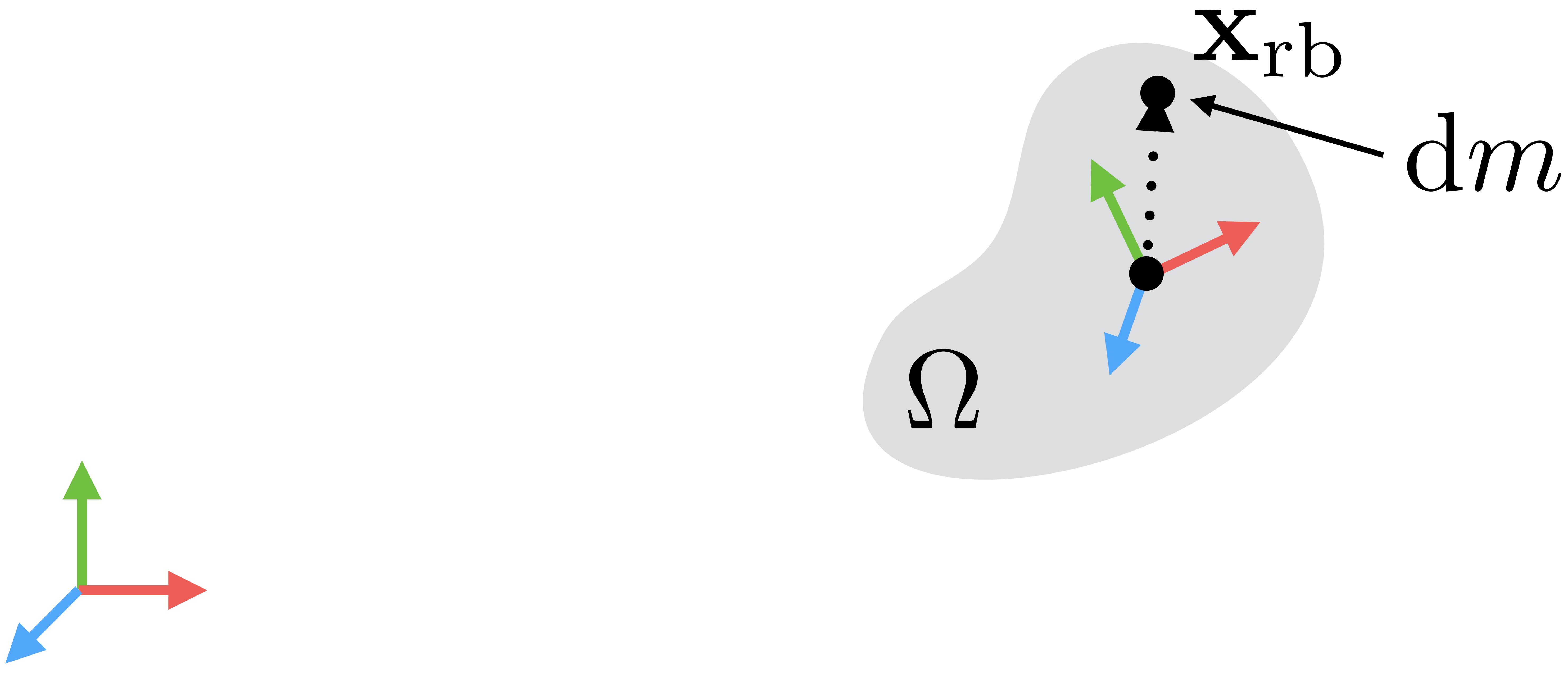
account for
changes in *position*
(translations)

$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{\mathrm{rb}}\mathbf{R}(t)^T - M\left(\mathbf{c}(t)\mathbf{c}(t)^T - \mathbf{c}(t)^T\mathbf{c}(t)\mathbf{E}_{3\times 3}\right)$$

$$\mathbf{I}_{\mathrm{rb}} = \int_{\Omega} [\mathbf{x}_{\mathrm{rb}}]_{\times}^T [\mathbf{x}_{\mathrm{rb}}]_{\times}\, \mathrm{d}m$$

constant!

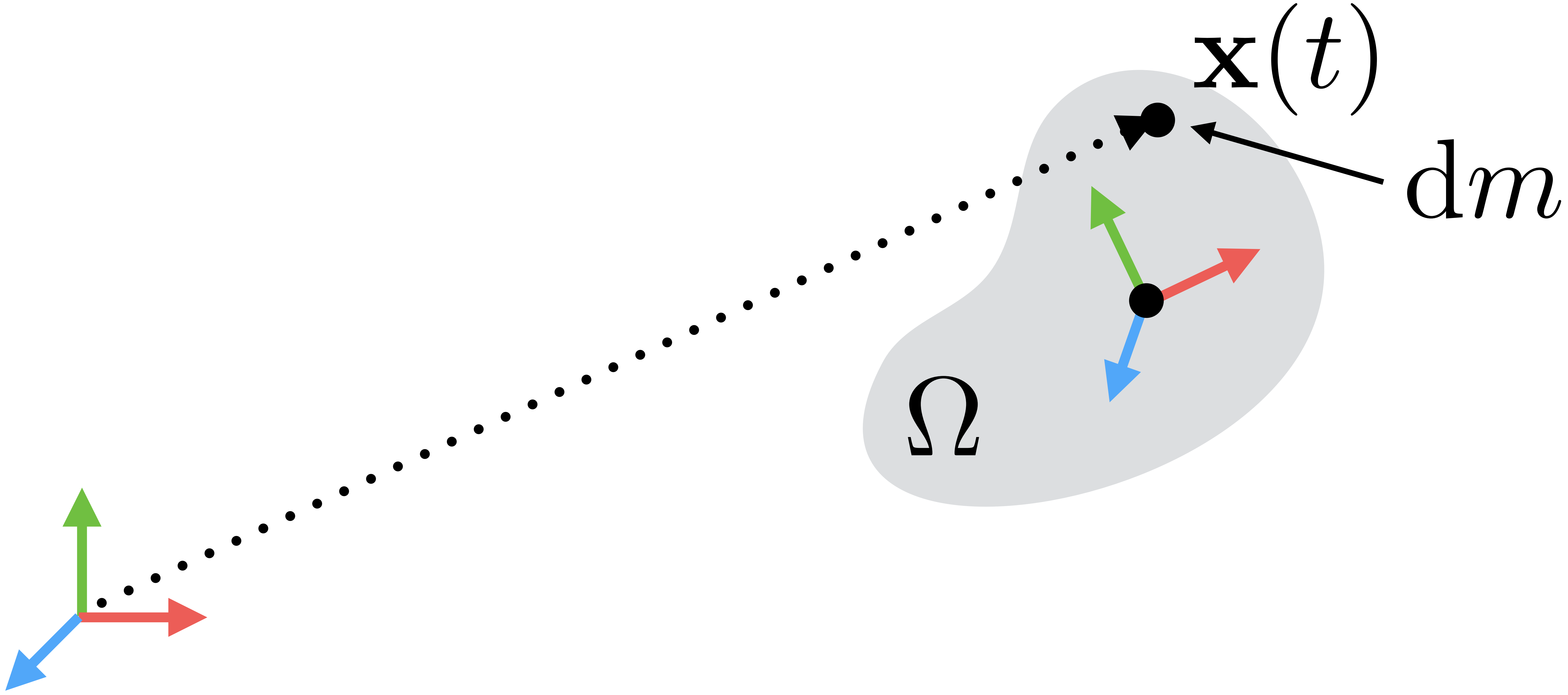
Mass Properties: Moment of Inertia



$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{\mathrm{rb}}\mathbf{R}(t)^T - M\left(\mathbf{c}(t)\mathbf{c}(t)^T - \mathbf{c}(t)^T\mathbf{c}(t)\mathbf{E}_{3\times 3}\right)$$

$$\mathbf{I}_{\mathrm{rb}} = \int_\Omega [\mathbf{x}_{\mathrm{rb}}]_\times^T [\mathbf{x}_{\mathrm{rb}}]_\times \,\mathrm{d}m$$

Mass Properties: Moment of Inertia



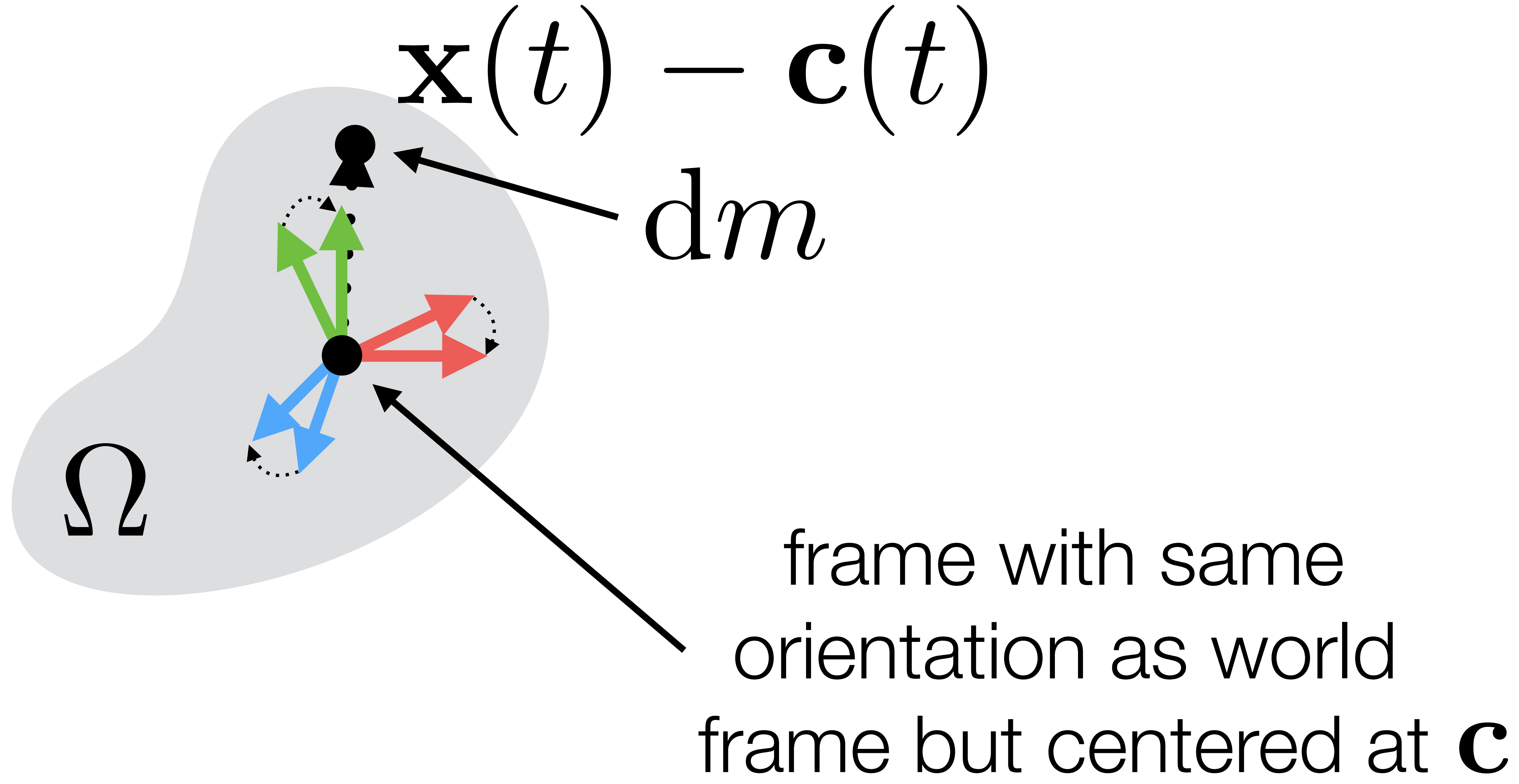
$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{\mathrm{rb}}\mathbf{R}(t)^T - M\left(\mathbf{c}(t)\mathbf{c}(t)^T - \mathbf{c}(t)^T\mathbf{c}(t)\mathbf{E}_{3\times 3}\right)$$

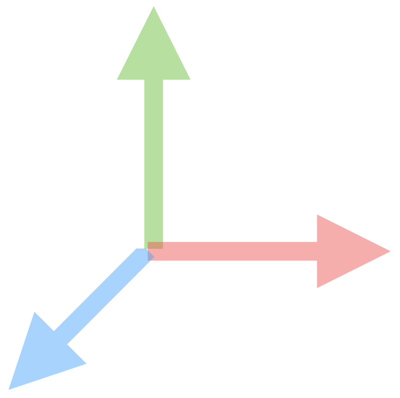
$$\mathbf{I}_{\mathrm{rb}} = \int_{\Omega} [\mathbf{x}_{\mathrm{rb}}]_{\times}^T [\mathbf{x}_{\mathrm{rb}}]_{\times} \, \mathrm{d}m$$

Mass Properties: Moment of Inertia

- *depends* on *orientation* (*rotations*) and *positions* (*translations*)

Mass Properties: Moment of Inertia



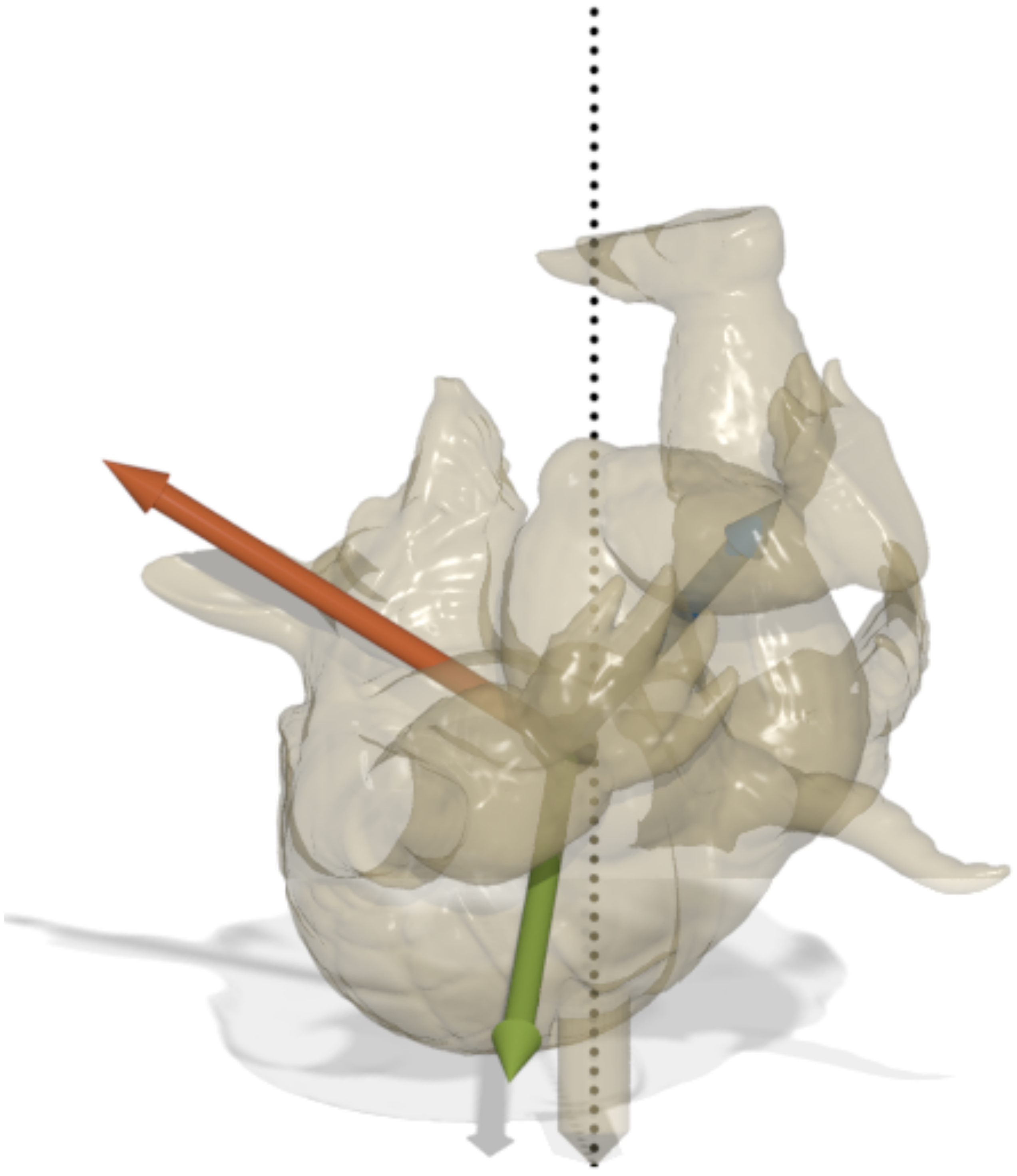


$$\mathbf{I}_{\mathbf{c}}(t) = \mathbf{R}(t)\mathbf{I}_{\mathrm{rb}}\mathbf{R}(t)^T$$

subscript $\mathbf{c}$

Mass Properties: Moment of Inertia

$$\mathbf{s}_\Omega(\rho) = [s_1, s_x, s_y, s_z, s_{xy}, s_{yz}, s_{xz}, s_{x^2}, s_{y^2}, s_{z^2}]^T$$

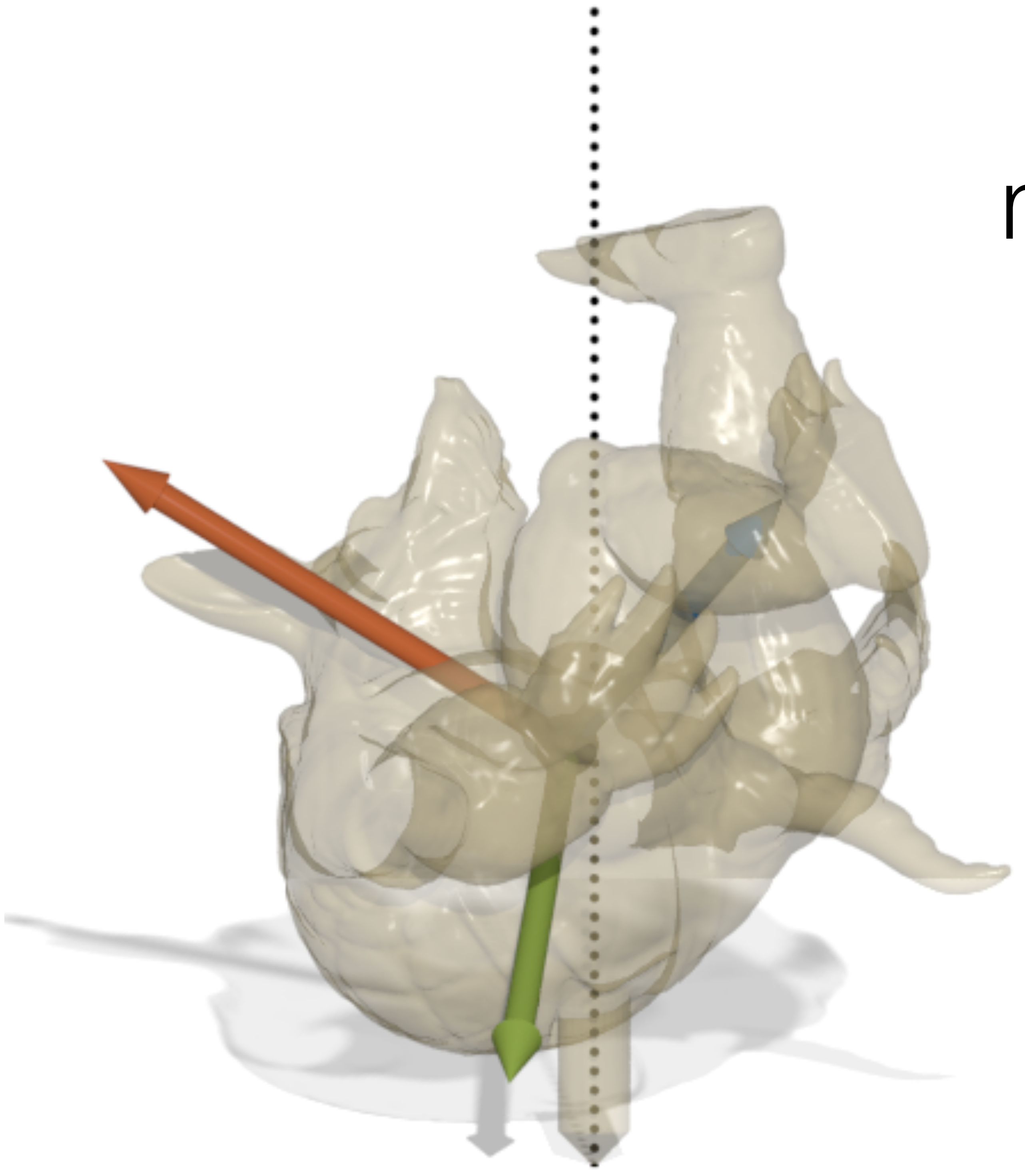


$$s_t = \rho \int_\Omega t \, \mathrm{dV}$$, e.g., $$s_{xy} = \rho \int_\Omega xy \, \mathrm{dV}$$

integral over *monomials*

Mass Properties: Moment of Inertia

$$\mathbf{s}_\Omega(\rho) = [s_1, s_x, s_y, s_z, s_{xy}, s_{yz}, s_{xz}, s_{x^2}, s_{y^2}, s_{z^2}]^T$$

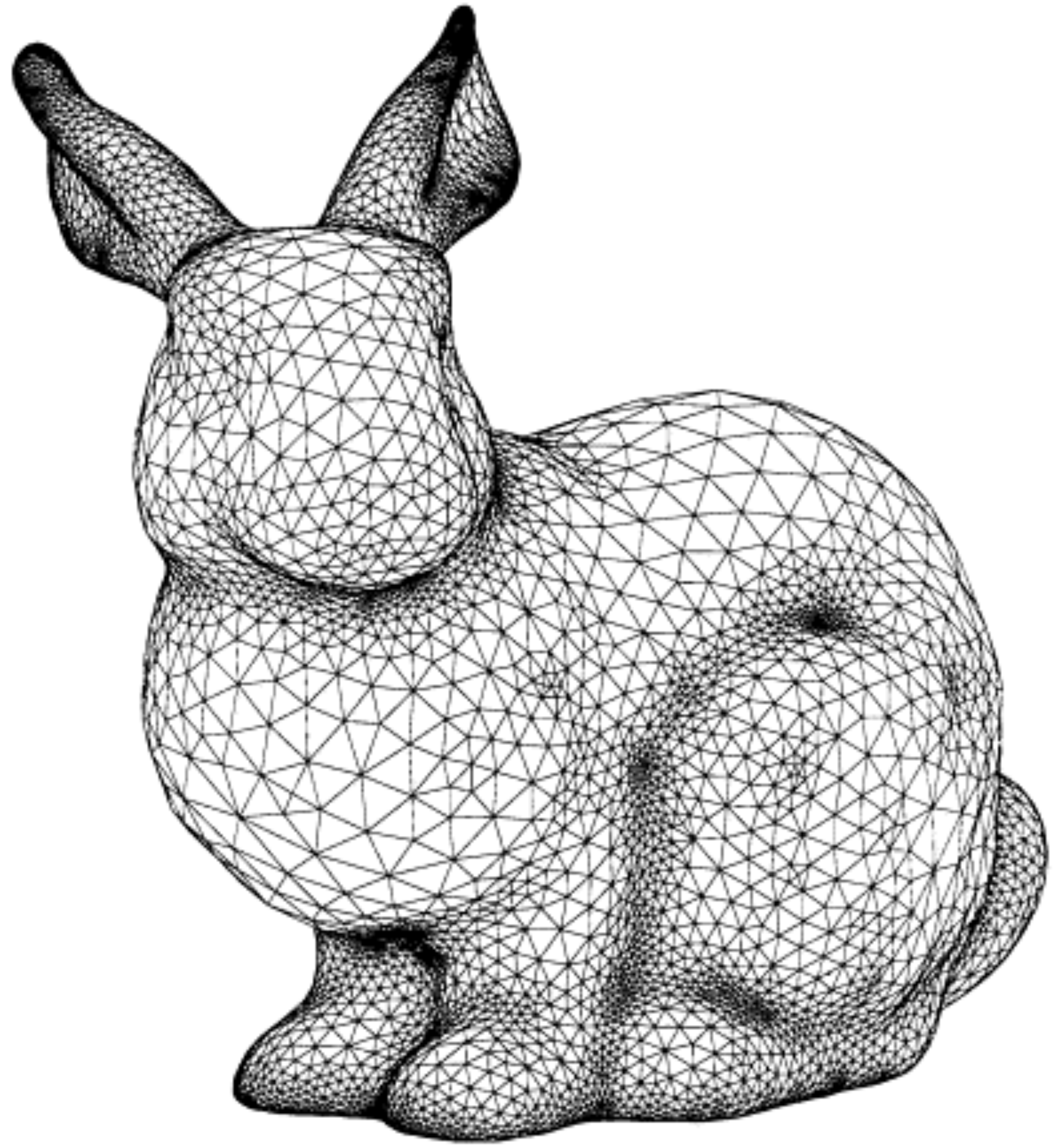


mass $M = s_1$

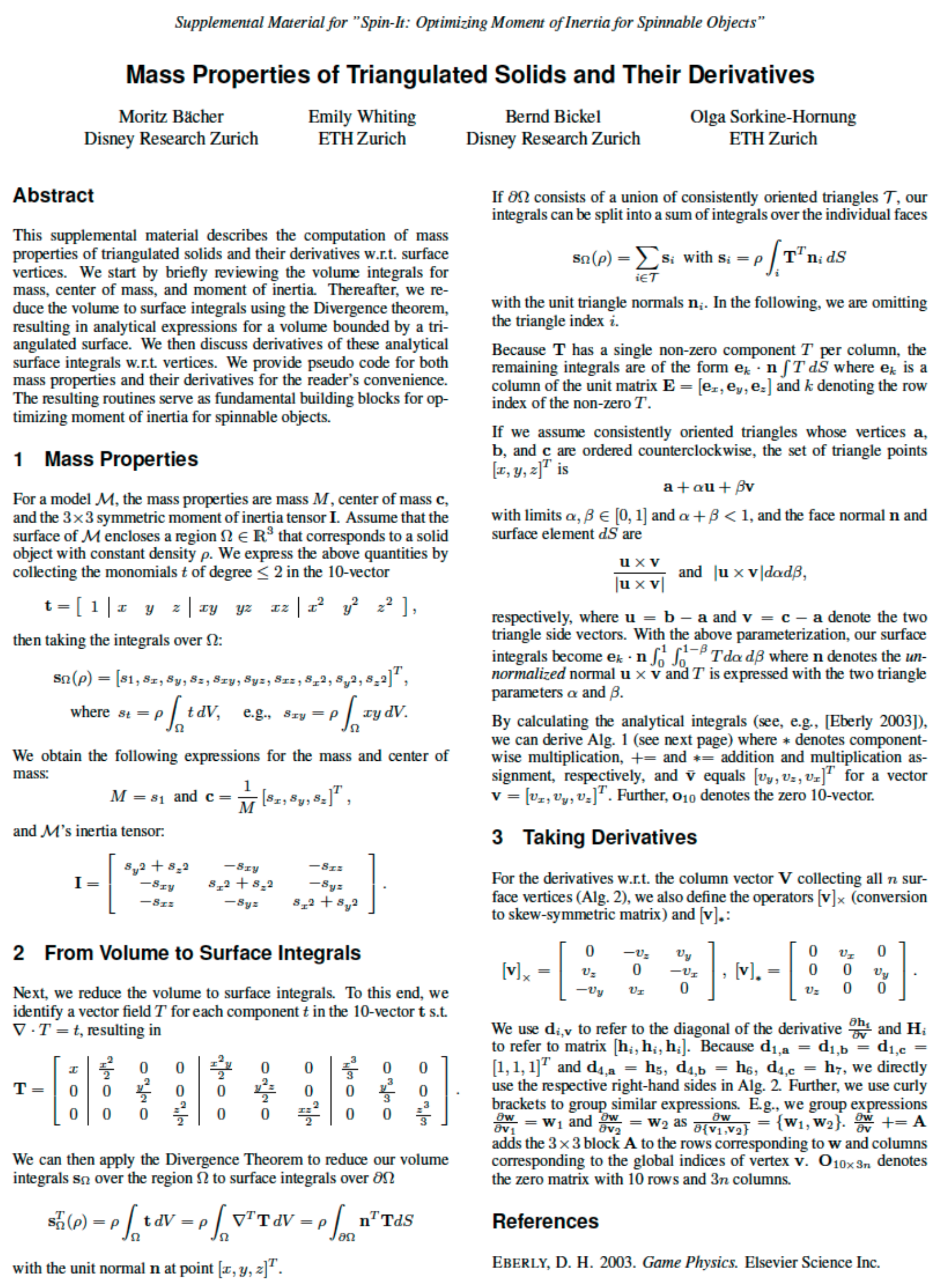
center of mass $\mathbf{c} = \frac{1}{M}\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}$

moment of inertia $\mathbf{I} = \begin{bmatrix} s_{y^2}+s_{z^2} & -s_{xy} & -s_{xz} \\ -s_{xy} & s_{x^2}+s_{z^2} & -s_{yz} \\ -s_{xz} & -s_{yz} & s_{x^2}+s_{y^2} \end{bmatrix}$

Homework: Moment of Inertia of Triangulated Solids



idea: use *Divergence theorem* to recast *volume* as *surface* integrals



Supplemental Material for "Spin-It: Optimizing Moment of Inertia for Spinnable Objects"

Mass Properties of Triangulated Solids and Their Derivatives

Moritz Bächer, Disney Research Zurich
Emily Whiting, ETH Zurich
Bernd Bickel, Disney Research Zurich
Olga Sorkine-Hornung, ETH Zurich

Abstract

This supplemental material describes the computation of mass properties of triangulated solids and their derivatives w.r.t. surface vertices. We start by briefly reviewing the volume integrals for mass, center of mass, and moment of inertia. Thereafter, we reduce the volume to surface integrals using the Divergence theorem, resulting in analytical expressions for a volume bounded by a triangulated surface. We then discuss derivatives of these analytical surface integrals w.r.t. vertices. We provide pseudo code for both mass properties and their derivatives for the reader's convenience. The resulting routines serve as fundamental building blocks for optimizing moment of inertia for spinnable objects.

1 Mass Properties

For a model $\mathcal{M}$, the mass properties are mass M, center of mass $\mathbf{c}$, and the 3×3 symmetric moment of inertia tensor $\mathbf{I}$. Assume that the surface of $\mathcal{M}$ encloses a region $\Omega \in \mathbb{R}^3$ that corresponds to a solid object with constant density ρ. We express the above quantities by collecting the monomials t of degree ≤ 2 in the 10-vector

$$\mathbf{t} = [\ 1 \mid x \quad y \quad z \mid xy \quad yz \quad xz \mid x^2 \quad y^2 \quad z^2\ ],$$

then taking the integrals over Ω:

$$\mathbf{s}_\Omega(\rho) = [s_1, s_x, s_y, s_z, s_{xy}, s_{yz}, s_{xz}, s_{x^2}, s_{y^2}, s_{z^2}]^T,$$
$$\text{where } s_t = \rho \int_\Omega t\, dV, \quad \text{e.g., } s_{xy} = \rho \int_\Omega xy\, dV.$$

We obtain the following expressions for the mass and center of mass:

$$M = s_1 \text{ and } \mathbf{c} = \frac{1}{M}[s_x, s_y, s_z]^T,$$

and $\mathcal{M}$'s inertia tensor:

$$\mathbf{I} = \begin{bmatrix} s_{y^2} + s_{z^2} & -s_{xy} & -s_{xz} \\ -s_{xy} & s_{x^2} + s_{z^2} & -s_{yz} \\ -s_{xz} & -s_{yz} & s_{x^2} + s_{y^2} \end{bmatrix}.$$

2 From Volume to Surface Integrals

Next, we reduce the volume to surface integrals. To this end, we identify a vector field T for each component t in the 10-vector $\mathbf{t}$ s.t. $\nabla \cdot T = t$, resulting in

$$\mathbf{T} = \left[\begin{array}{c|ccc|ccc|ccc} x & \frac{x^2}{2} & 0 & 0 & \frac{x^2y}{2} & 0 & 0 & \frac{x^3}{3} & 0 & 0 \\ 0 & 0 & \frac{y^2}{2} & 0 & 0 & \frac{y^2z}{2} & 0 & 0 & \frac{y^3}{3} & 0 \\ 0 & 0 & 0 & \frac{z^2}{2} & 0 & 0 & \frac{xz^2}{2} & 0 & 0 & \frac{z^3}{3} \end{array}\right].$$

We can then apply the Divergence Theorem to reduce our volume integrals $\mathbf{s}_\Omega$ over the region Ω to surface integrals over $\partial\Omega$

$$\mathbf{s}_\Omega^T(\rho) = \rho \int_\Omega \mathbf{t}\, dV = \rho \int_\Omega \nabla^T \mathbf{T}\, dV = \rho \int_{\partial\Omega} \mathbf{n}^T \mathbf{T} dS$$

with the unit normal $\mathbf{n}$ at point $[x, y, z]^T$.

If $\partial\Omega$ consists of a union of consistently oriented triangles $\mathcal{T}$, our integrals can be split into a sum of integrals over the individual faces

$$\mathbf{s}_\Omega(\rho) = \sum_{i \in \mathcal{T}} \mathbf{s}_i \text{ with } \mathbf{s}_i = \rho \int_i \mathbf{T}^T \mathbf{n}_i\, dS$$

with the unit triangle normals $\mathbf{n}_i$. In the following, we are omitting the triangle index i.

Because $\mathbf{T}$ has a single non-zero component T per column, the remaining integrals are of the form $\mathbf{e}_k \cdot \mathbf{n} \int T\, dS$ where $\mathbf{e}_k$ is a column of the unit matrix $\mathbf{E} = [\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]$ and k denoting the row index of the non-zero T.

If we assume consistently oriented triangles whose vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are ordered counterclockwise, the set of triangle points $[x, y, z]^T$ is

$$\mathbf{a} + \alpha\mathbf{u} + \beta\mathbf{v}$$

with limits $\alpha, \beta \in [0, 1]$ and $\alpha + \beta < 1$, and the face normal $\mathbf{n}$ and surface element dS are

$$\frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} \text{ and } |\mathbf{u} \times \mathbf{v}| d\alpha d\beta,$$

respectively, where $\mathbf{u} = \mathbf{b} - \mathbf{a}$ and $\mathbf{v} = \mathbf{c} - \mathbf{a}$ denote the two triangle side vectors. With the above parameterization, our surface integrals become $\mathbf{e}_k \cdot \mathbf{n} \int_0^1 \int_0^{1-\beta} T d\alpha\, d\beta$ where $\mathbf{n}$ denotes the *unnormalized* normal $\mathbf{u} \times \mathbf{v}$ and T is expressed with the two triangle parameters α and β.

By calculating the analytical integrals (see, e.g., [Eberly 2003]), we can derive Alg. 1 (see next page) where $*$ denotes component-wise multiplication, $+=$ and $*=$ addition and multiplication assignment, respectively, and $\bar{\mathbf{v}}$ equals $[v_y, v_z, v_x]^T$ for a vector $\mathbf{v} = [v_x, v_y, v_z]^T$. Further, $\mathbf{o}_{10}$ denotes the zero 10-vector.

3 Taking Derivatives

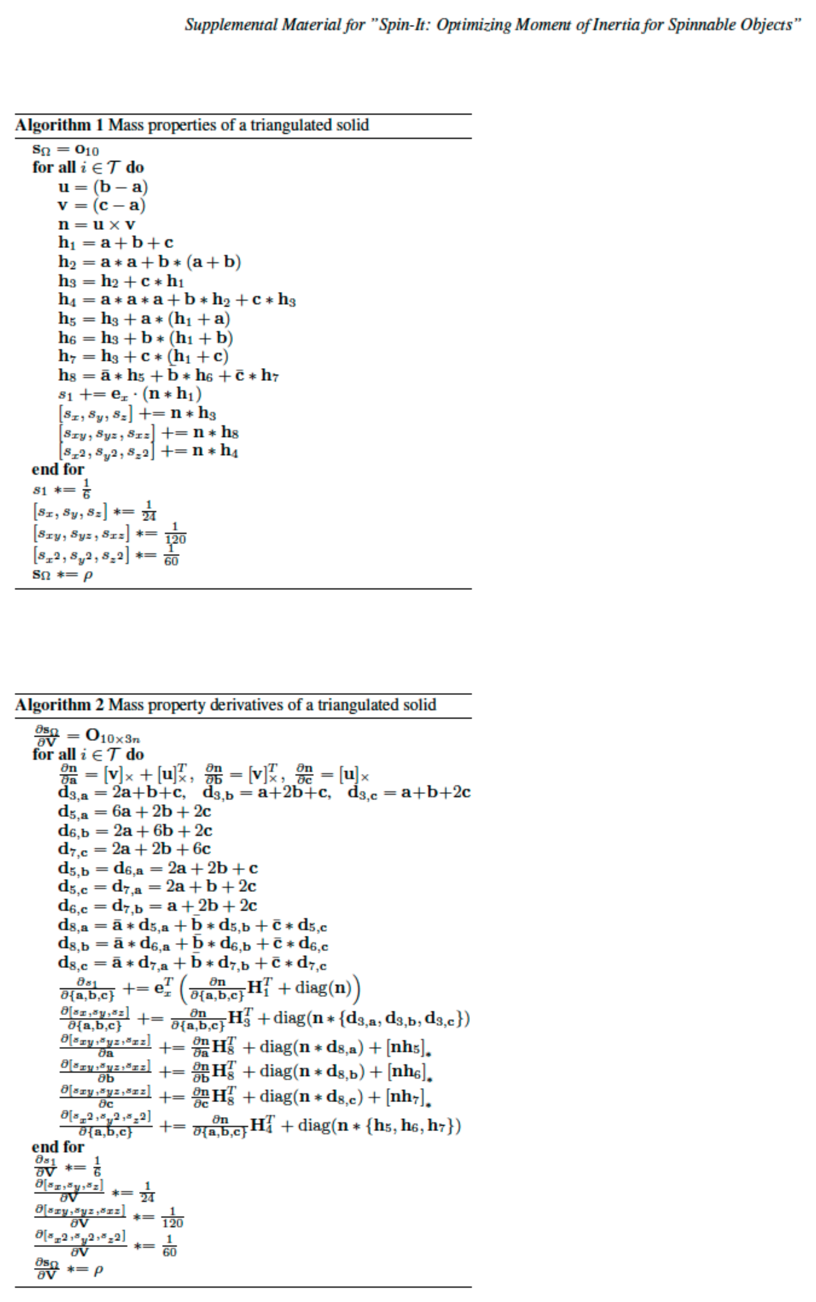
For the derivatives w.r.t. the column vector $\mathbf{V}$ collecting all n surface vertices (Alg. 2), we also define the operators $[\mathbf{v}]_\times$ (conversion to skew-symmetric matrix) and $[\mathbf{v}]_*$:

$$[\mathbf{v}]_\times = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}, [\mathbf{v}]_* = \begin{bmatrix} 0 & v_x & 0 \\ 0 & 0 & v_y \\ v_z & 0 & 0 \end{bmatrix}.$$

We use $\mathbf{d}_{i,\mathbf{v}}$ to refer to the diagonal of the derivative $\frac{\partial \mathbf{h}_i}{\partial \mathbf{v}}$ and $\mathbf{H}_i$ to refer to matrix $[\mathbf{h}_i, \mathbf{h}_i, \mathbf{h}_i]$. Because $\mathbf{d}_{1,\mathbf{a}} = \mathbf{d}_{1,\mathbf{b}} = \mathbf{d}_{1,\mathbf{c}} = [1, 1, 1]^T$ and $\mathbf{d}_{4,\mathbf{a}} = \mathbf{h}_5$, $\mathbf{d}_{4,\mathbf{b}} = \mathbf{h}_6$, $\mathbf{d}_{4,\mathbf{c}} = \mathbf{h}_7$, we directly use the respective right-hand sides in Alg. 2. Further, we use curly brackets to group similar expressions. E.g., we group expressions $\frac{\partial \mathbf{w}}{\partial \mathbf{v}_1} = \mathbf{w}_1$ and $\frac{\partial \mathbf{w}}{\partial \mathbf{v}_2} = \mathbf{w}_2$ as $\frac{\partial \mathbf{w}}{\partial \{\mathbf{v}_1, \mathbf{v}_2\}} = \{\mathbf{w}_1, \mathbf{w}_2\}$. $\frac{\partial \mathbf{w}}{\partial \mathbf{v}} += \mathbf{A}$ adds the 3×3 block $\mathbf{A}$ to the rows corresponding to $\mathbf{w}$ and columns corresponding to the global indices of vertex $\mathbf{v}$. $\mathbf{O}_{10\times3n}$ denotes the zero matrix with 10 rows and $3n$ columns.

References

EBERLY, D. H. 2003. *Game Physics.* Elsevier Science Inc.



Supplemental Material for "Spin-It: Optimizing Moment of Inertia for Spinnable Objects"

Algorithm 1 Mass properties of a triangulated solid

$\mathbf{s}_\Omega = \mathbf{o}_{10}$
for all $i \in \mathcal{T}$ **do**
  $\mathbf{u} = (\mathbf{b} - \mathbf{a})$
  $\mathbf{v} = (\mathbf{c} - \mathbf{a})$
  $\mathbf{n} = \mathbf{u} \times \mathbf{v}$
  $\mathbf{h}_1 = \mathbf{a} + \mathbf{b} + \mathbf{c}$
  $\mathbf{h}_2 = \mathbf{a} * \mathbf{a} + \mathbf{b} * (\mathbf{a} + \mathbf{b})$
  $\mathbf{h}_3 = \mathbf{h}_2 + \mathbf{c} * \mathbf{h}_1$
  $\mathbf{h}_4 = \mathbf{a} * \mathbf{a} * \mathbf{a} + \mathbf{b} * \mathbf{h}_2 + \mathbf{c} * \mathbf{h}_3$
  $\mathbf{h}_5 = \mathbf{h}_3 + \mathbf{a} * (\mathbf{h}_1 + \mathbf{a})$
  $\mathbf{h}_6 = \mathbf{h}_3 + \mathbf{b} * (\mathbf{h}_1 + \mathbf{b})$
  $\mathbf{h}_7 = \mathbf{h}_3 + \mathbf{c} * (\mathbf{h}_1 + \mathbf{c})$
  $\mathbf{h}_8 = \bar{\mathbf{a}} * \mathbf{h}_5 + \bar{\mathbf{b}} * \mathbf{h}_6 + \bar{\mathbf{c}} * \mathbf{h}_7$
  $s_1 += \mathbf{e}_x \cdot (\mathbf{n} * \mathbf{h}_1)$
  $[s_x, s_y, s_z] += \mathbf{n} * \mathbf{h}_3$
  $[s_{xy}, s_{yz}, s_{xz}] += \mathbf{n} * \mathbf{h}_8$
  $[s_{x^2}, s_{y^2}, s_{z^2}] += \mathbf{n} * \mathbf{h}_4$
end for
$s_1 *= \frac{1}{6}$
$[s_x, s_y, s_z] *= \frac{1}{24}$
$[s_{xy}, s_{yz}, s_{xz}] *= \frac{1}{120}$
$[s_{x^2}, s_{y^2}, s_{z^2}] *= \frac{1}{60}$
$\mathbf{s}_\Omega *= \rho$

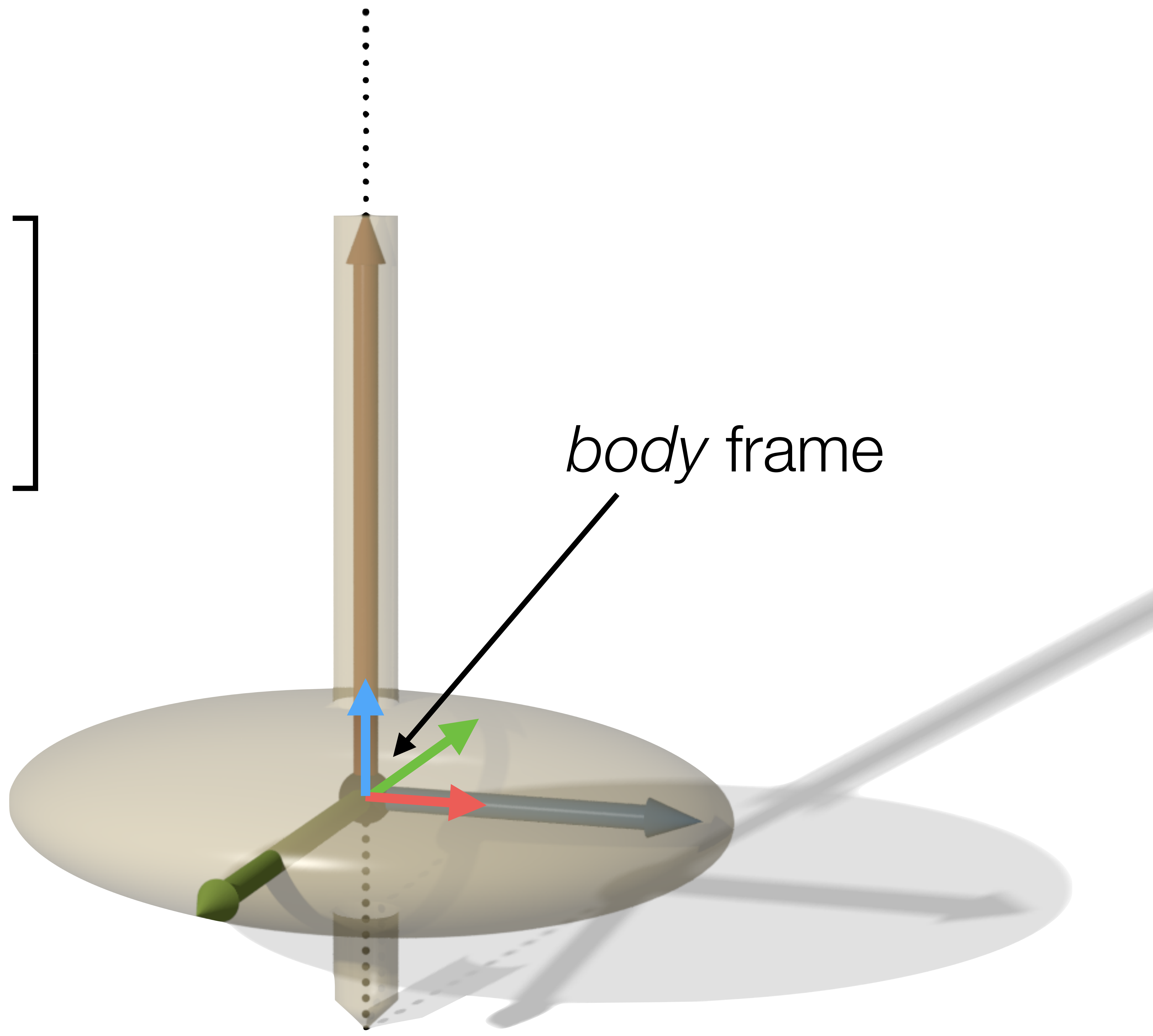
Algorithm 2 Mass property derivatives of a triangulated solid

$\frac{\partial \mathbf{s}_\Omega}{\partial \mathbf{V}} = \mathbf{O}_{10\times3n}$
for all $i \in \mathcal{T}$ **do**
  $\frac{\partial \mathbf{n}}{\partial \mathbf{a}} = [\mathbf{v}]_\times + [\mathbf{u}]_\times^T$, $\frac{\partial \mathbf{n}}{\partial \mathbf{b}} = [\mathbf{v}]_\times^T$, $\frac{\partial \mathbf{n}}{\partial \mathbf{c}} = [\mathbf{u}]_\times$
  $\mathbf{d}_{3,\mathbf{a}} = 2\mathbf{a}+\mathbf{b}+\mathbf{c}$, $\mathbf{d}_{3,\mathbf{b}} = \mathbf{a}+2\mathbf{b}+\mathbf{c}$, $\mathbf{d}_{3,\mathbf{c}} = \mathbf{a}+\mathbf{b}+2\mathbf{c}$
  $\mathbf{d}_{5,\mathbf{a}} = 6\mathbf{a} + 2\mathbf{b} + 2\mathbf{c}$
  $\mathbf{d}_{6,\mathbf{b}} = 2\mathbf{a} + 6\mathbf{b} + 2\mathbf{c}$
  $\mathbf{d}_{7,\mathbf{c}} = 2\mathbf{a} + 2\mathbf{b} + 6\mathbf{c}$
  $\mathbf{d}_{5,\mathbf{b}} = \mathbf{d}_{6,\mathbf{a}} = 2\mathbf{a} + 2\mathbf{b} + \mathbf{c}$
  $\mathbf{d}_{5,\mathbf{c}} = \mathbf{d}_{7,\mathbf{a}} = 2\mathbf{a} + \mathbf{b} + 2\mathbf{c}$
  $\mathbf{d}_{6,\mathbf{c}} = \mathbf{d}_{7,\mathbf{b}} = \mathbf{a} + 2\mathbf{b} + 2\mathbf{c}$
  $\mathbf{d}_{8,\mathbf{a}} = \bar{\mathbf{a}} * \mathbf{d}_{5,\mathbf{a}} + \bar{\mathbf{b}} * \mathbf{d}_{5,\mathbf{b}} + \bar{\mathbf{c}} * \mathbf{d}_{5,\mathbf{c}}$
  $\mathbf{d}_{8,\mathbf{b}} = \bar{\mathbf{a}} * \mathbf{d}_{6,\mathbf{a}} + \bar{\mathbf{b}} * \mathbf{d}_{6,\mathbf{b}} + \bar{\mathbf{c}} * \mathbf{d}_{6,\mathbf{c}}$
  $\mathbf{d}_{8,\mathbf{c}} = \bar{\mathbf{a}} * \mathbf{d}_{7,\mathbf{a}} + \bar{\mathbf{b}} * \mathbf{d}_{7,\mathbf{b}} + \bar{\mathbf{c}} * \mathbf{d}_{7,\mathbf{c}}$
  $\frac{\partial s_1}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} += \mathbf{e}_x^T \left(\frac{\partial \mathbf{n}}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} \mathbf{H}_1^T + \mathrm{diag}(\mathbf{n})\right)$
  $\frac{\partial [s_x, s_y, s_z]}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} += \frac{\partial \mathbf{n}}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} \mathbf{H}_3^T + \mathrm{diag}(\mathbf{n} * \{\mathbf{d}_{3,\mathbf{a}}, \mathbf{d}_{3,\mathbf{b}}, \mathbf{d}_{3,\mathbf{c}}\})$
  $\frac{\partial [s_{xy}, s_{yz}, s_{xz}]}{\partial \mathbf{a}} += \frac{\partial \mathbf{n}}{\partial \mathbf{a}} \mathbf{H}_8^T + \mathrm{diag}(\mathbf{n} * \mathbf{d}_{8,\mathbf{a}}) + [\mathbf{n}\mathbf{h}_5]_*$
  $\frac{\partial [s_{xy}, s_{yz}, s_{xz}]}{\partial \mathbf{b}} += \frac{\partial \mathbf{n}}{\partial \mathbf{b}} \mathbf{H}_8^T + \mathrm{diag}(\mathbf{n} * \mathbf{d}_{8,\mathbf{b}}) + [\mathbf{n}\mathbf{h}_6]_*$
  $\frac{\partial [s_{xy}, s_{yz}, s_{xz}]}{\partial \mathbf{c}} += \frac{\partial \mathbf{n}}{\partial \mathbf{c}} \mathbf{H}_8^T + \mathrm{diag}(\mathbf{n} * \mathbf{d}_{8,\mathbf{c}}) + [\mathbf{n}\mathbf{h}_7]_*$
  $\frac{\partial [s_{x^2}, s_{y^2}, s_{z^2}]}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} += \frac{\partial \mathbf{n}}{\partial \{\mathbf{a},\mathbf{b},\mathbf{c}\}} \mathbf{H}_4^T + \mathrm{diag}(\mathbf{n} * \{\mathbf{h}_5, \mathbf{h}_6, \mathbf{h}_7\})$
end for
$\frac{\partial s_1}{\partial \mathbf{V}} *= \frac{1}{6}$
$\frac{\partial [s_x, s_y, s_z]}{\partial \mathbf{V}} *= \frac{1}{24}$
$\frac{\partial [s_{xy}, s_{yz}, s_{xz}]}{\partial \mathbf{V}} *= \frac{1}{120}$
$\frac{\partial [s_{x^2}, s_{y^2}, s_{z^2}]}{\partial \mathbf{V}} *= \frac{1}{60}$
$\frac{\partial \mathbf{s}_\Omega}{\partial \mathbf{V}} *= \rho$

http://www.baecher.info/publications/spin_it_sup_mat_sig14.pdf

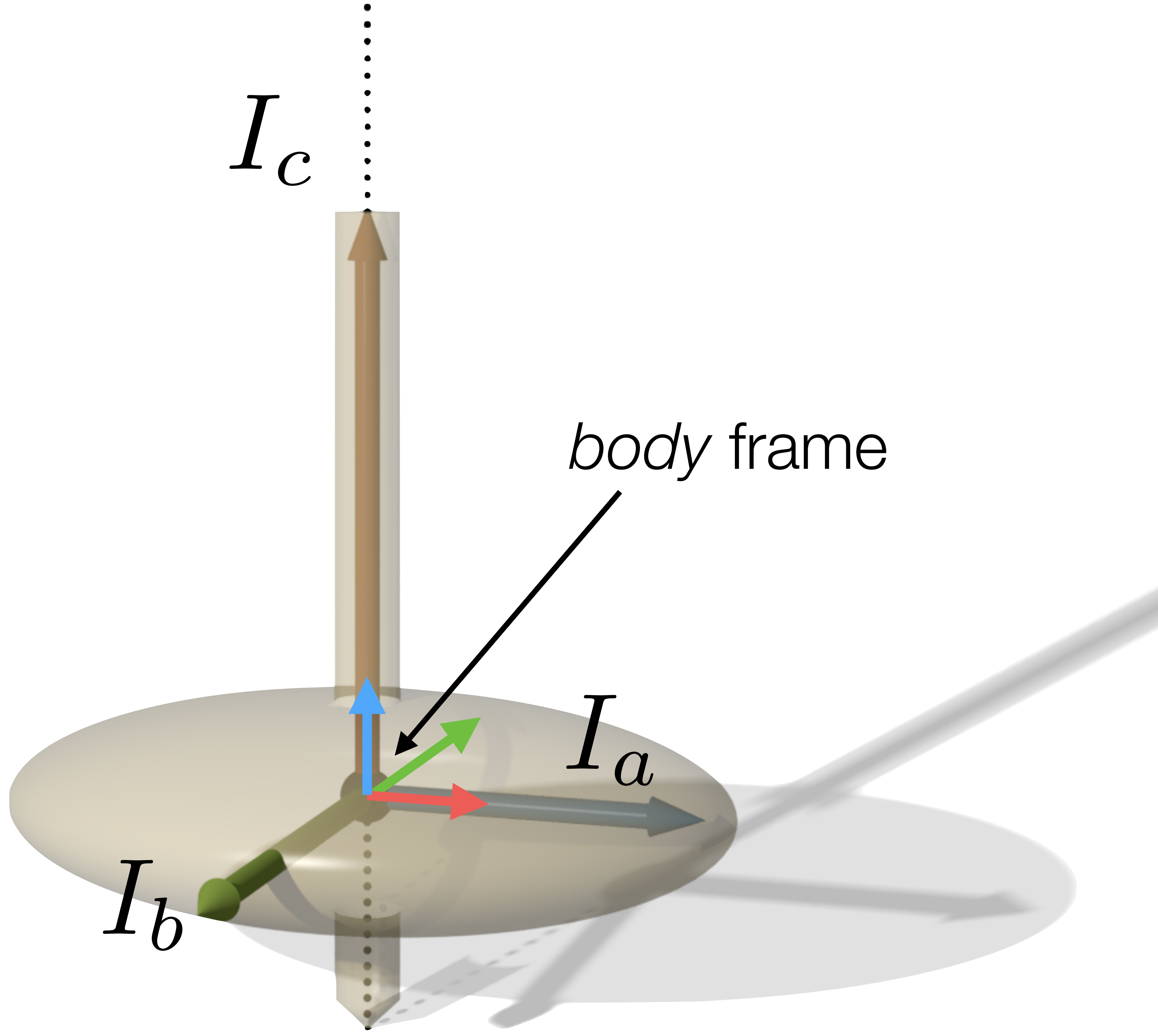
Mass Properties: Moment of Inertia

$$\mathbf{I}_{\mathrm{rb}} = \begin{bmatrix} s_{y^2} + s_{z^2} & & \\ & s_{x^2} + s_{z^2} & \\ & & s_{x^2} + s_{y^2} \end{bmatrix}$$



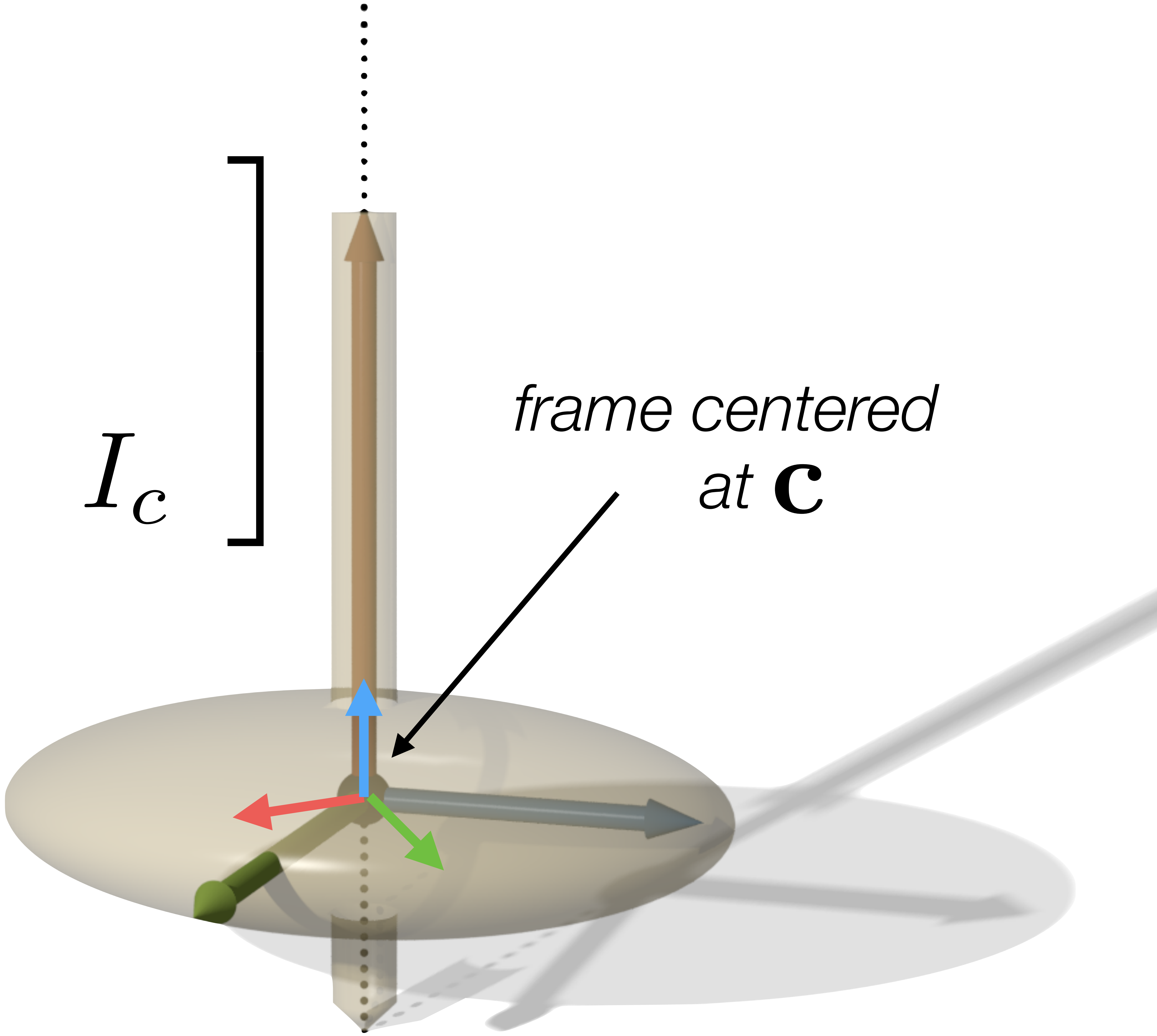
Mass Properties: Moment of Inertia

$$\mathbf{I}_{\mathrm{rb}} = \begin{bmatrix} I_a & & \\ & I_b & \\ & & I_c \end{bmatrix}$$



Mass Properties: Moment of Inertia

$$\mathbf{I_c} = \begin{bmatrix} s_{y^2} + s_{z^2} & -s_{xy} & \\ -s_{xy} & s_{x^2} + s_{z^2} & \\ & & I_c \end{bmatrix}$$

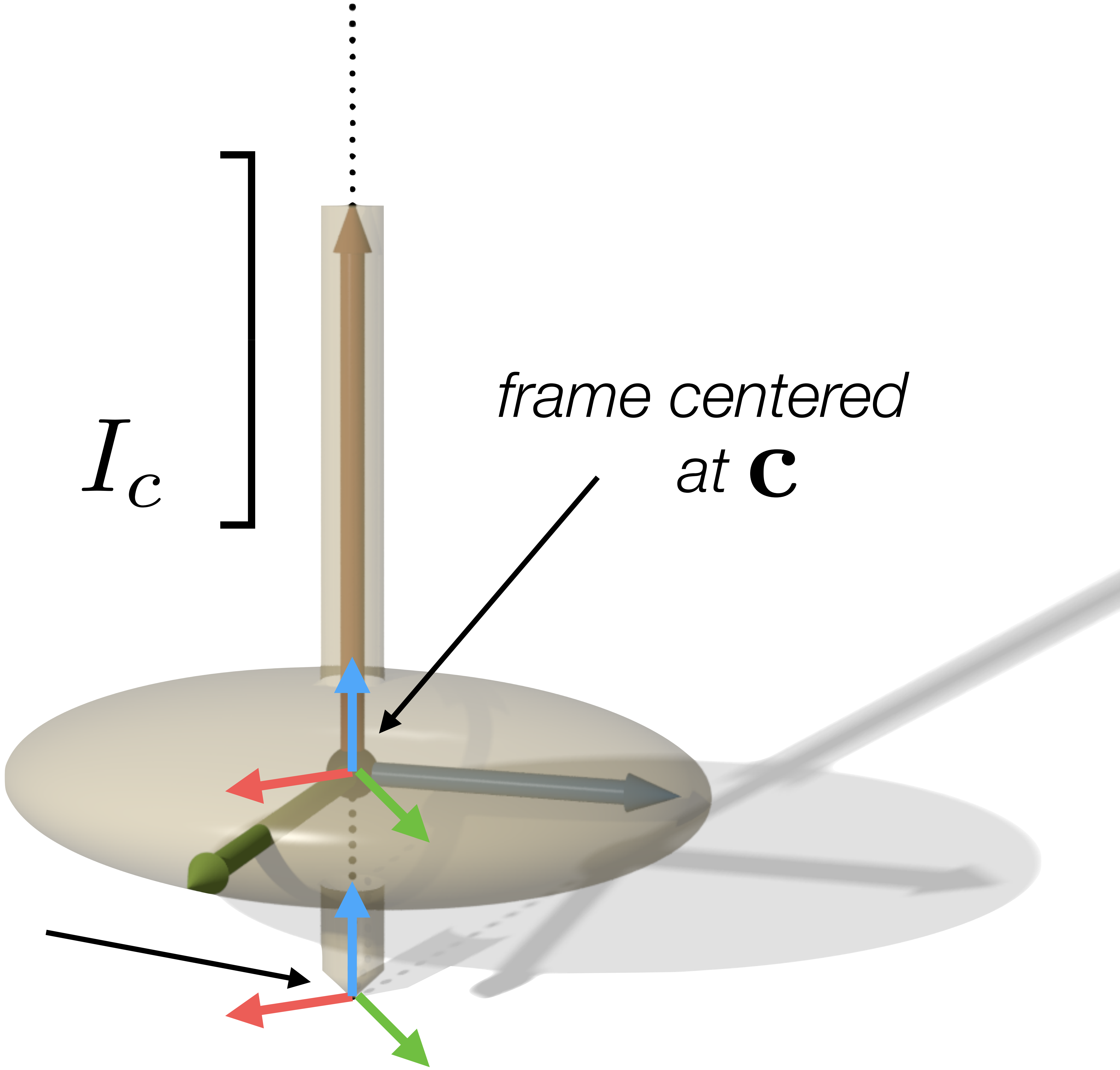


Mass Properties: Moment of Inertia

world frame

$$\mathbf{I_c} = \begin{bmatrix} s_{y^2} + s_{z^2} & -s_{xy} & \\ -s_{xy} & s_{x^2} + s_{z^2} & \\ & & I_c \end{bmatrix}$$

$$+M\left(\mathbf{c}\mathbf{c}^T - \mathbf{c}^T\mathbf{c}\mathbf{E}_{3\times 3}\right)$$



frame centered at $\mathbf{c}$

world frame

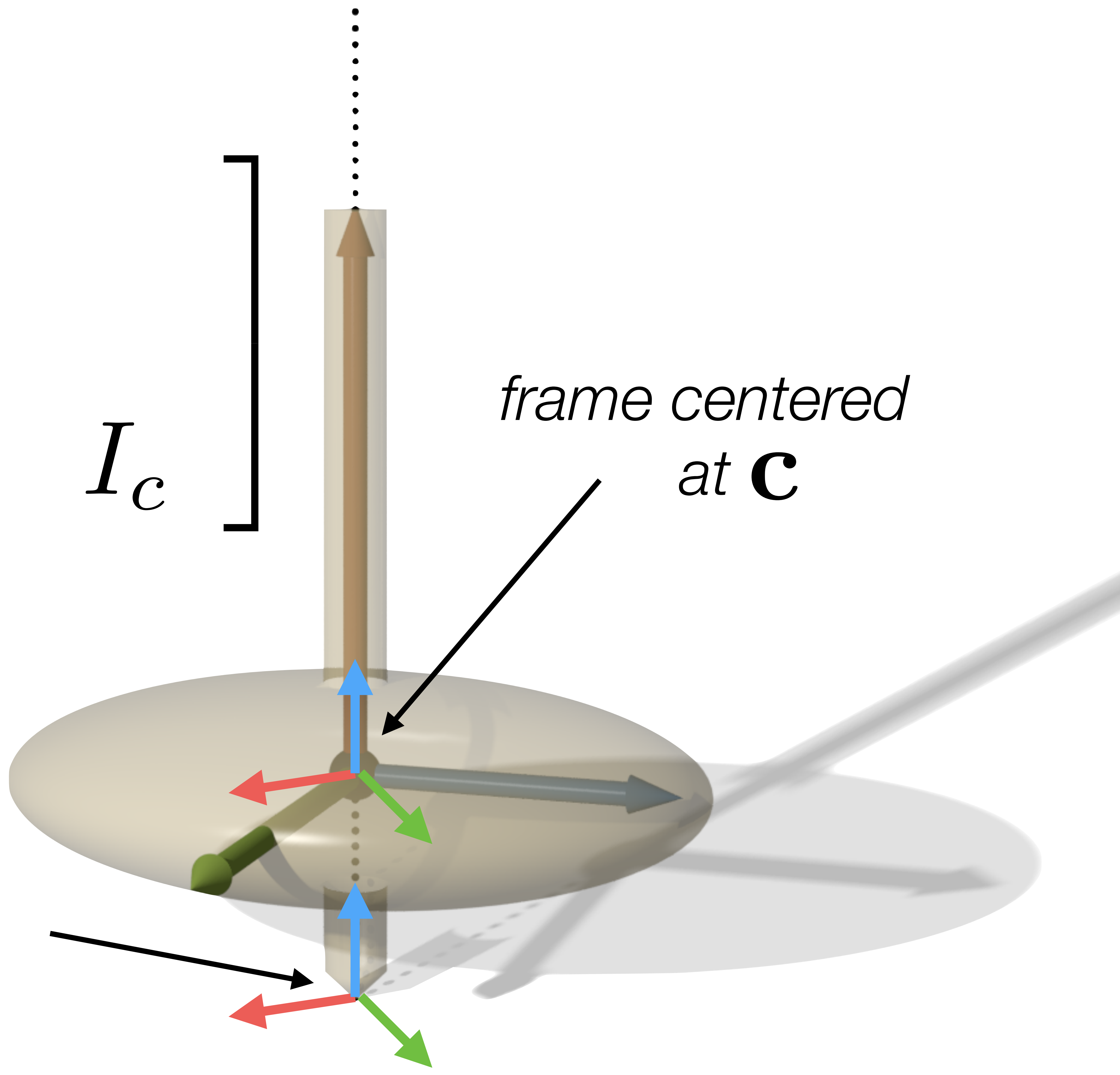
Mass Properties: Moment of Inertia

world frame

$$\mathbf{I_c} = \begin{bmatrix} s_{y^2} + s_{z^2} & -s_{xy} & \\ -s_{xy} & s_{x^2} + s_{z^2} & \\ & & I_c \end{bmatrix}$$

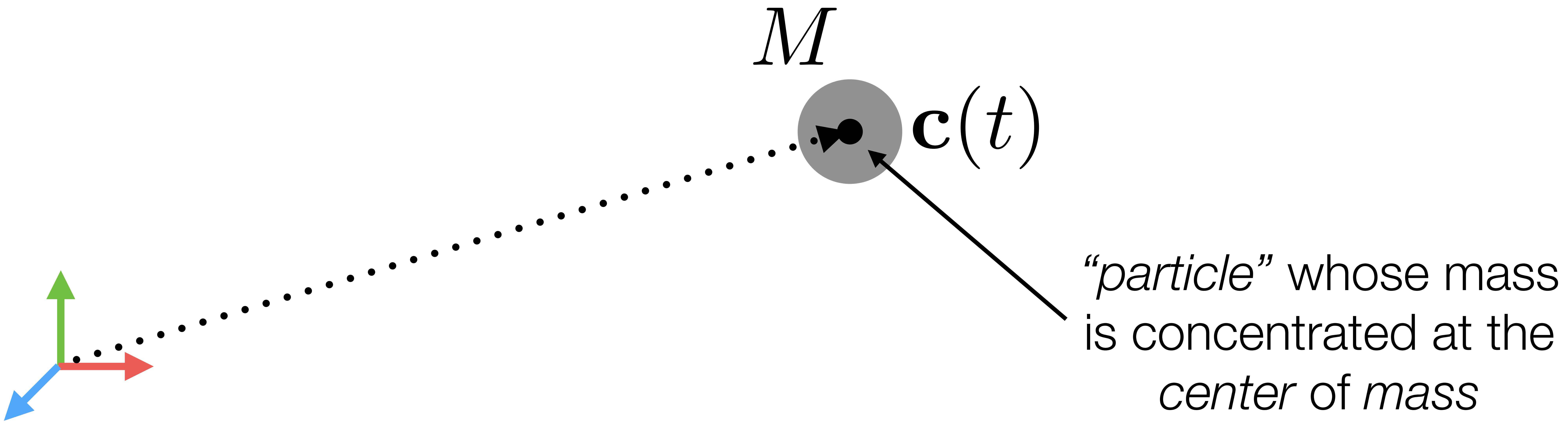
$$+M\left(\mathbf{c}\mathbf{c}^T - \mathbf{c}^T\mathbf{c}\mathbf{E}_{3\times 3}\right)$$

$$\mathbf{c} = \frac{1}{M}[0, 0, s_z]^T$$



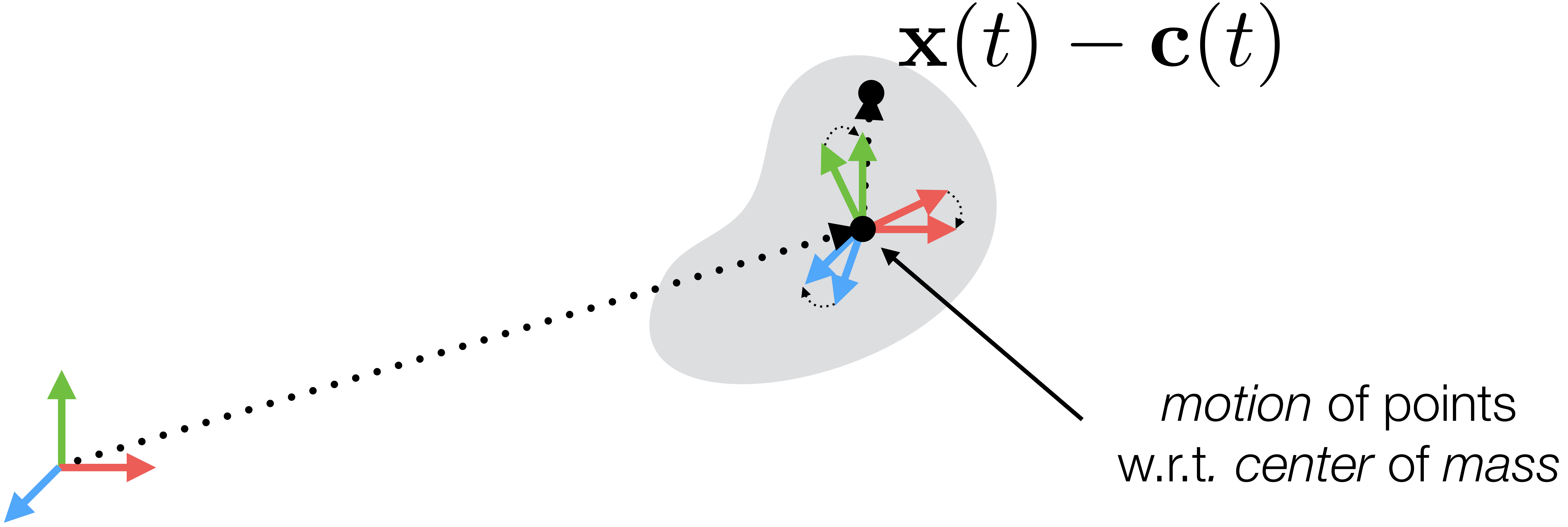
Linear vs. Angular Motion

- *linear* motion: motion of *center* of *mass*

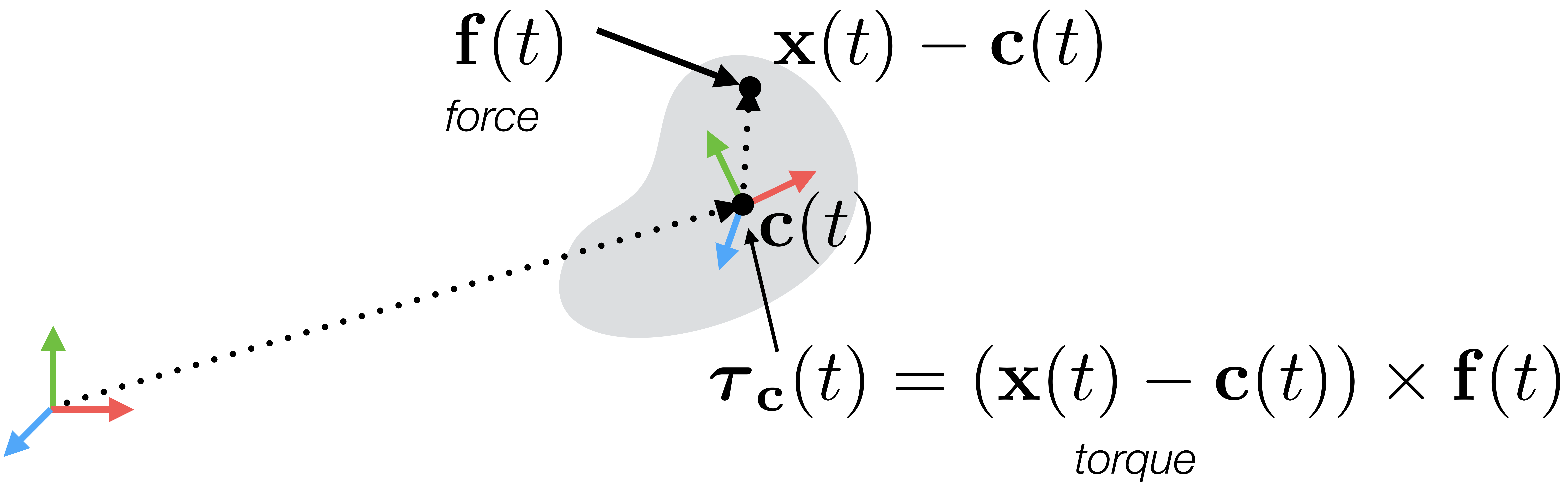


Linear vs. **Angular** Motion

- *linear* motion*:* motion of *center* of *mass*
- *angular* motion: motion of points *about* center of mass



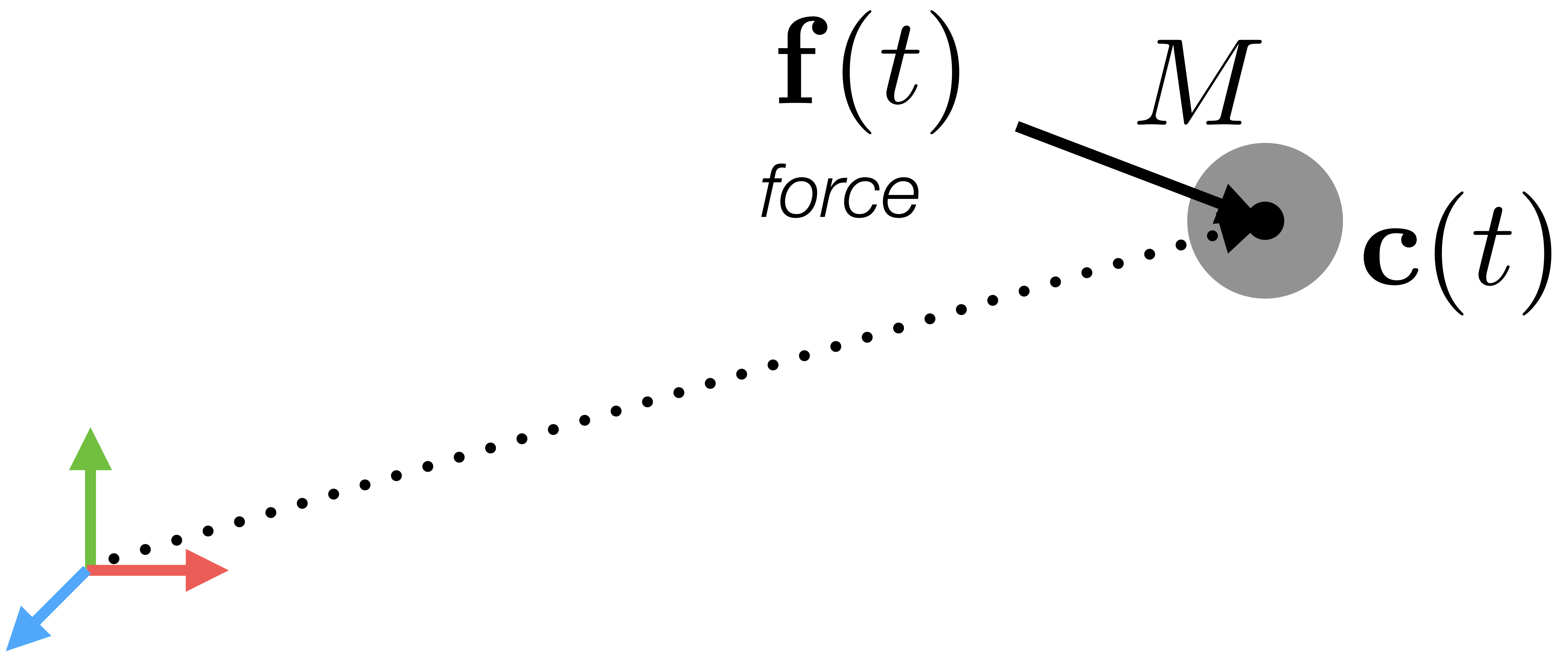
Linear vs. **Angular** Motion



force $\mathbf{f}(t)$ ➡ *change of linear motion*

torque $\boldsymbol{\tau}_{\mathbf{c}}(t)$ ➡ *change of angular motion*

Linear vs. Angular Motion



Newton's 2nd Law

$$\mathbf{f}(t) = M\mathbf{a}(t)$$

2nd order ODE

$$\ddot{\mathbf{c}}(t) = \frac{\mathbf{f}(t)}{M}$$

1st order ODE

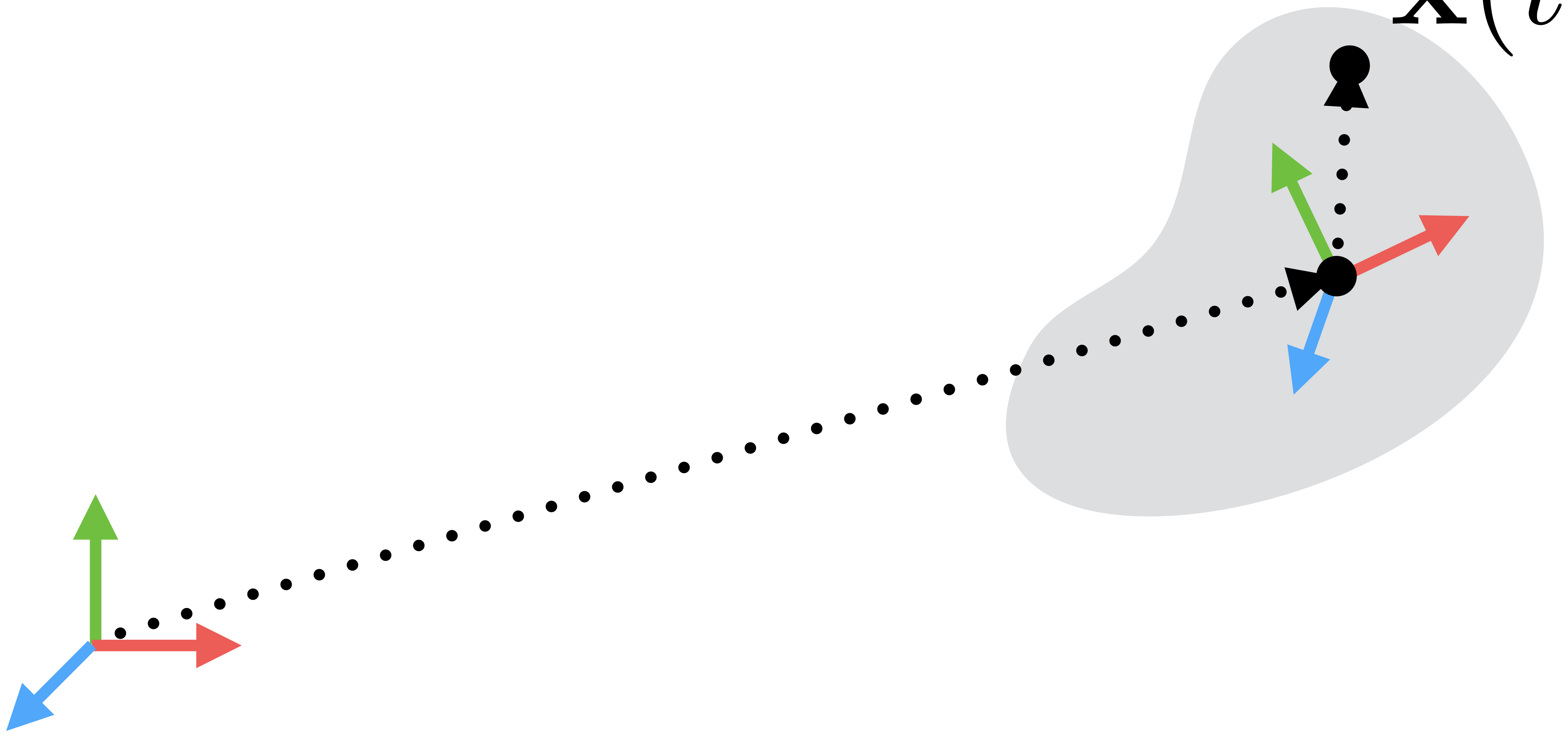
$$\mathbf{v}(t) = \dot{\mathbf{c}}(t)$$

$$\dot{\mathbf{v}}(t) = \frac{\mathbf{f}(t)}{M}$$

introduce *auxiliary variable (velocity)*

Linear vs. **Angular** Motion

$$\mathbf{x}(t) - \mathbf{c}(t)$$



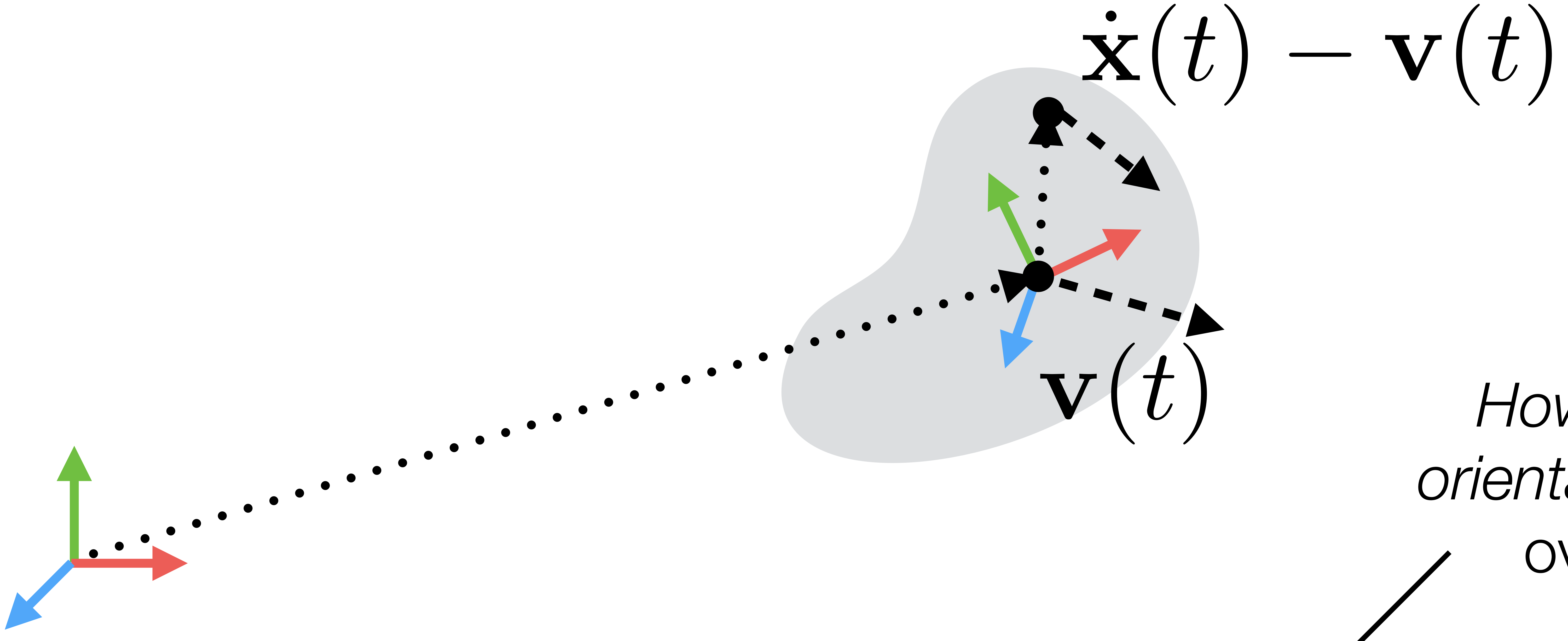
$$\mathbf{x}(t) - \mathbf{c}(t) = \mathbf{R}(t)\mathbf{x}_{\mathrm{rb}}$$

position

position of body

orientation of body

Linear vs. **Angular** Motion



How does the orientation change over time?

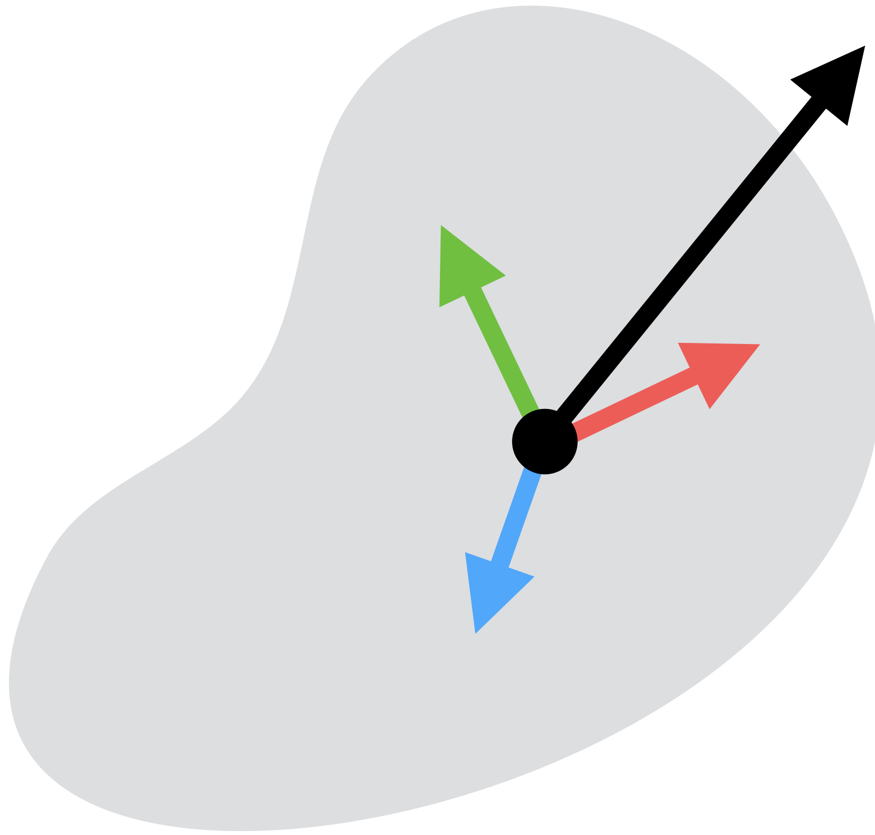
$$\dot{\mathbf{x}}(t) - \mathbf{v}(t) = \dot{\mathbf{R}}(t)\mathbf{x}_{\mathrm{rb}}$$

velocity

velocity of center of *mass*

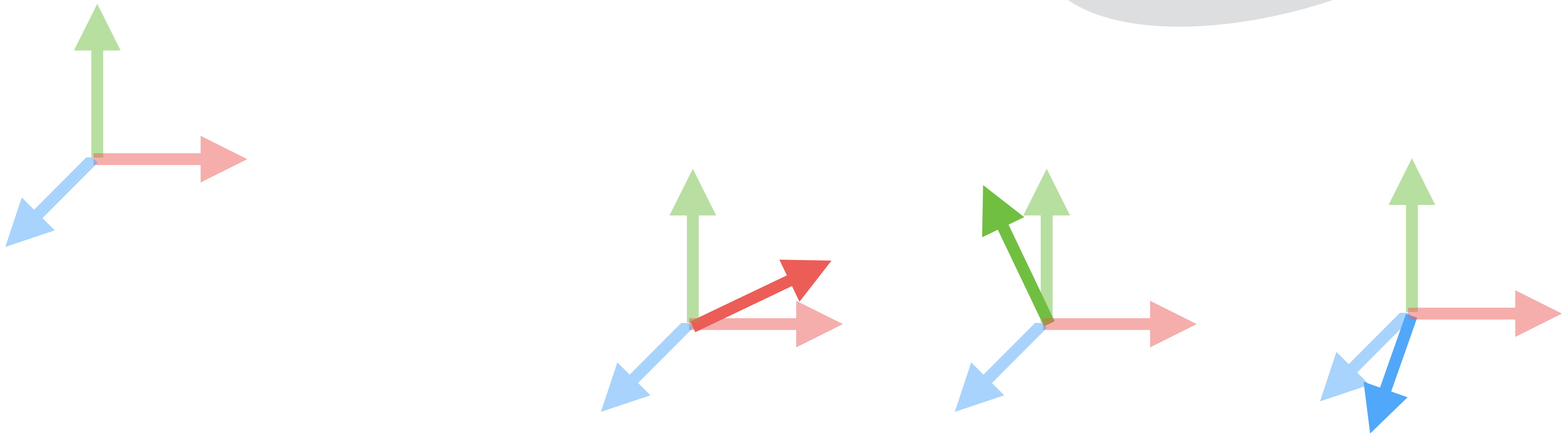
1st time derivative of *orientation*

Linear vs. **Angular** Motion



$\boldsymbol{\omega}(t)$ *angular velocity*

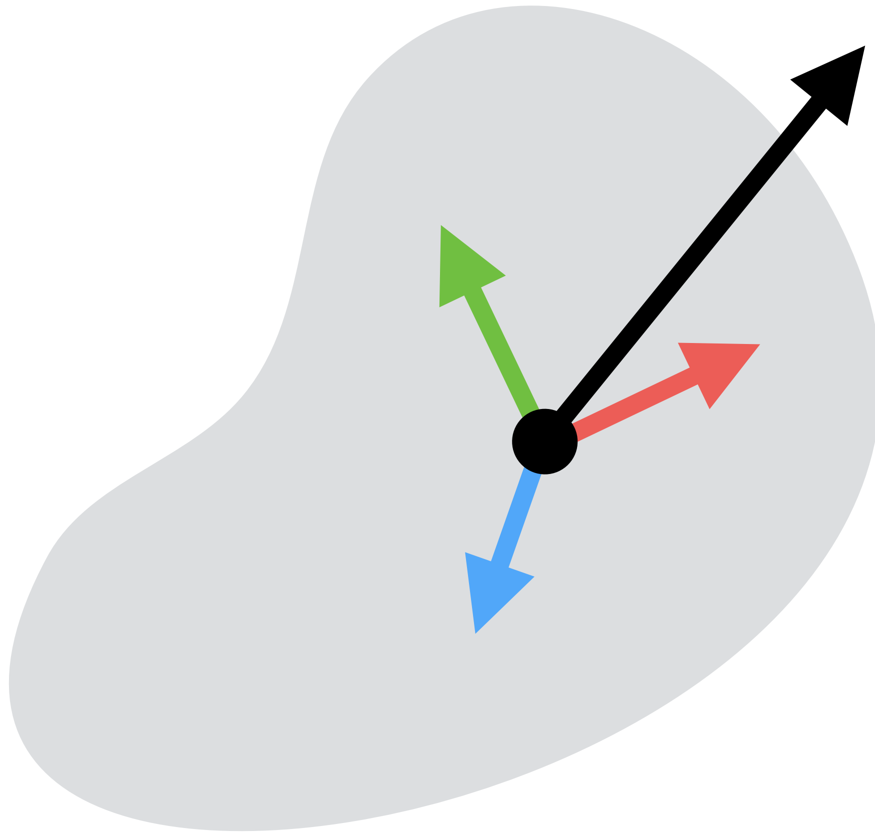
- vector quantity (translation-invariant)
- direction: axis body is spinning about
- magnitude: how fast body is spinning about axis [rads/s]



$$\mathbf{R}(t) = [\mathbf{r}_x(t), \mathbf{r}_y(t), \mathbf{r}_z(t)]$$

columns: *axes*

Linear vs. **Angular** Motion



$\boldsymbol{\omega}(t)$ *angular velocity*

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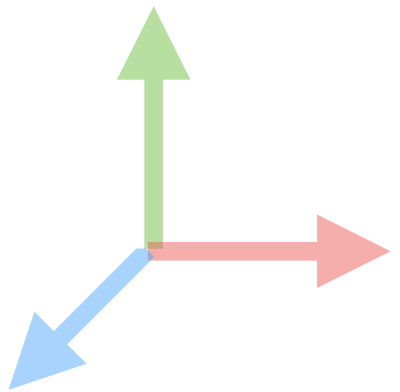
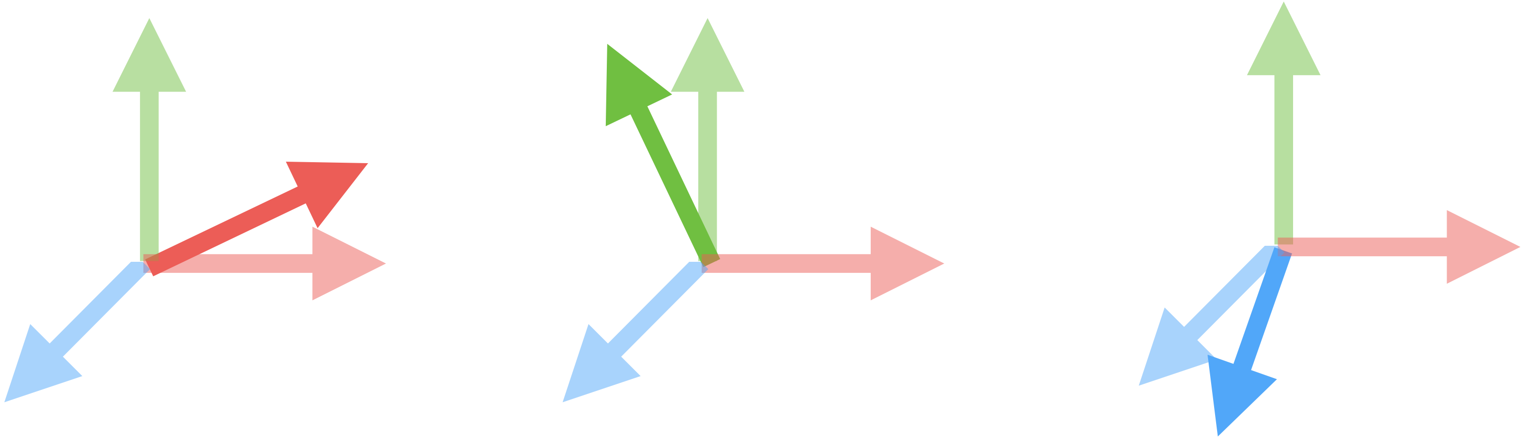
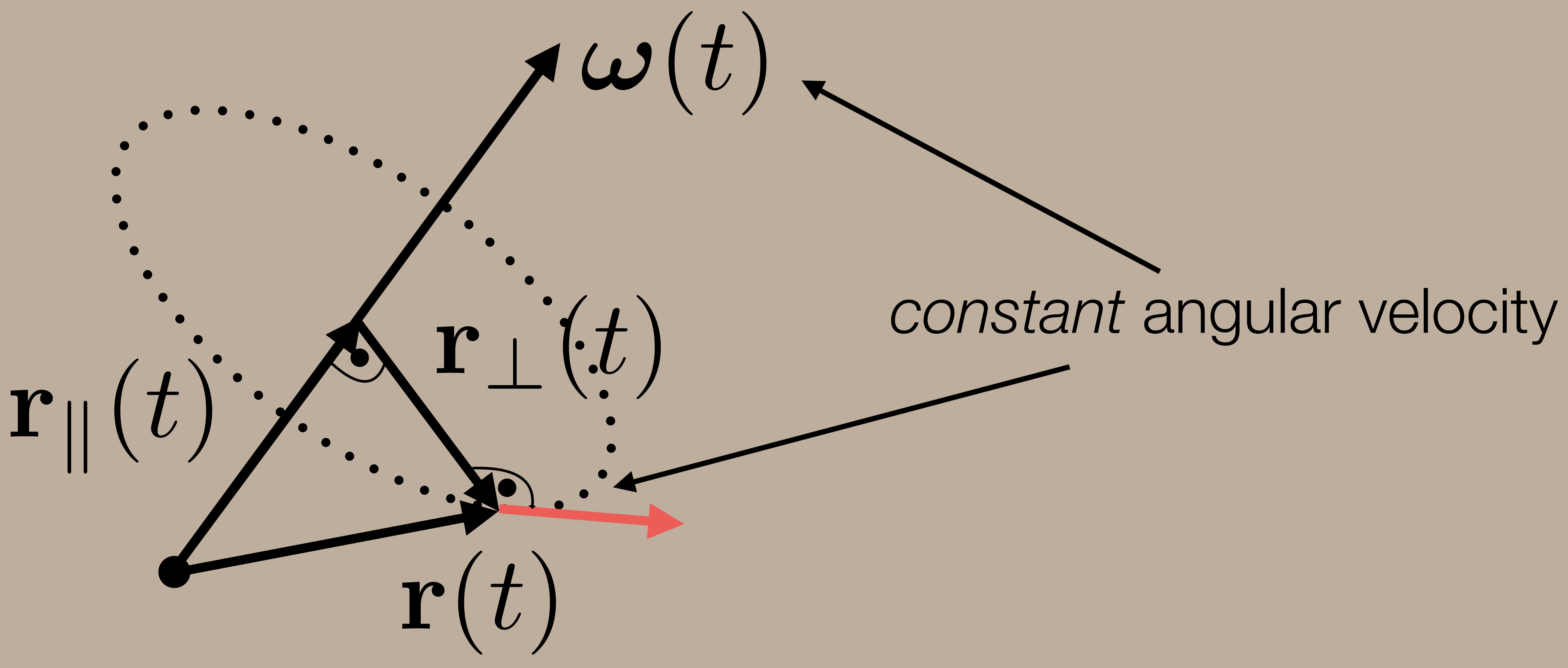




$$\dot{\mathbf{R}}(t) = [\dot{\mathbf{r}}_x(t), \dot{\mathbf{r}}_y(t), \dot{\mathbf{r}}_z(t)] \Leftarrow \dot{\mathbf{r}}(t) = \boldsymbol{\omega}(t) \times \mathbf{r}(t)$$

columns: *"velocities"* of *axes*

Angular Velocity

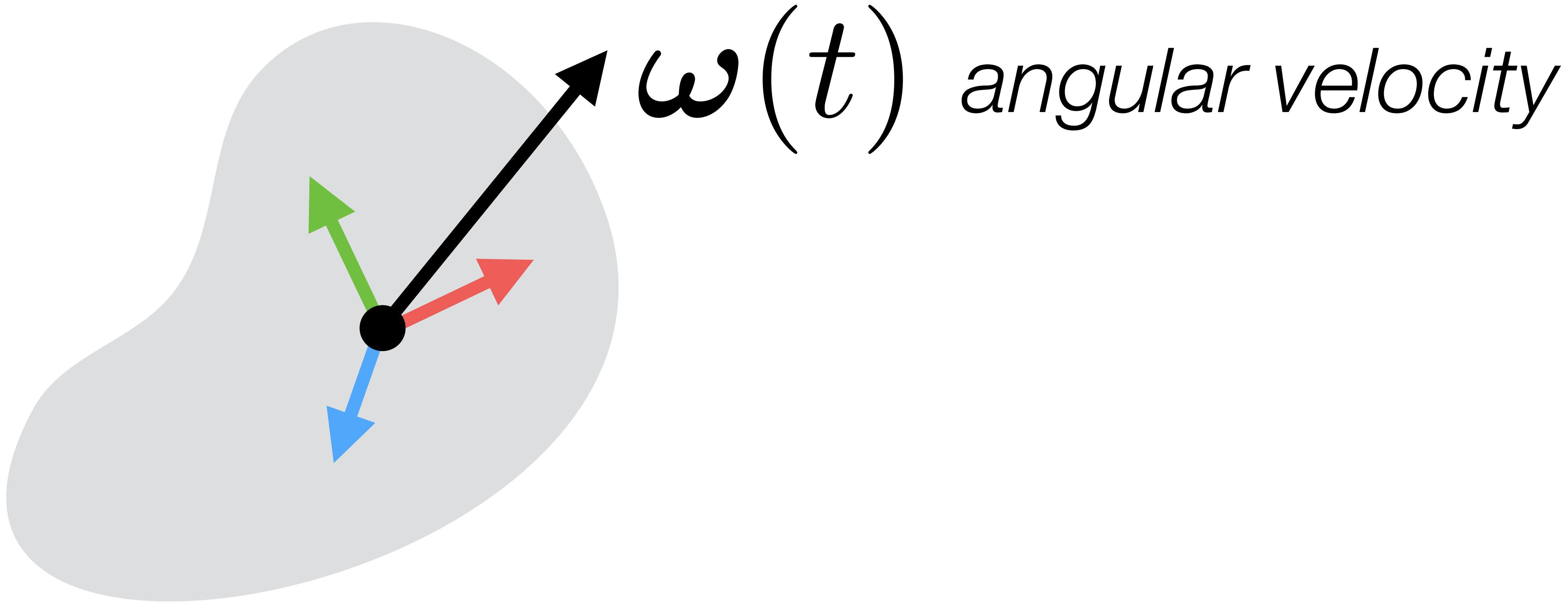


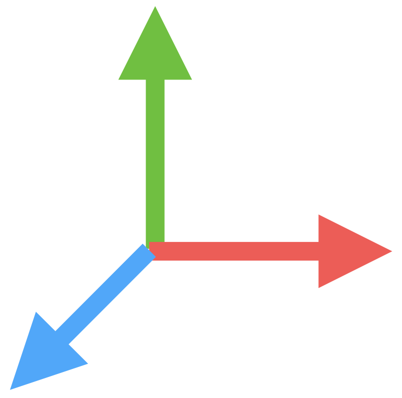
$$\dot{\mathbf{r}}(t) = \boldsymbol{\omega}(t) \times \mathbf{r}(t) = \boldsymbol{\omega}(t) \times \mathbf{r}_{\parallel}(t) + \boldsymbol{\omega}(t) \times \mathbf{r}_{\perp}(t)$$

$$\|\dot{\mathbf{r}}(t)\| = \|\boldsymbol{\omega}(t) \times \mathbf{r}(t)\| = \|\boldsymbol{\omega}(t)\|\|\mathbf{r}_{\perp}(t)\|$$

angular velocity — *radius* of circle

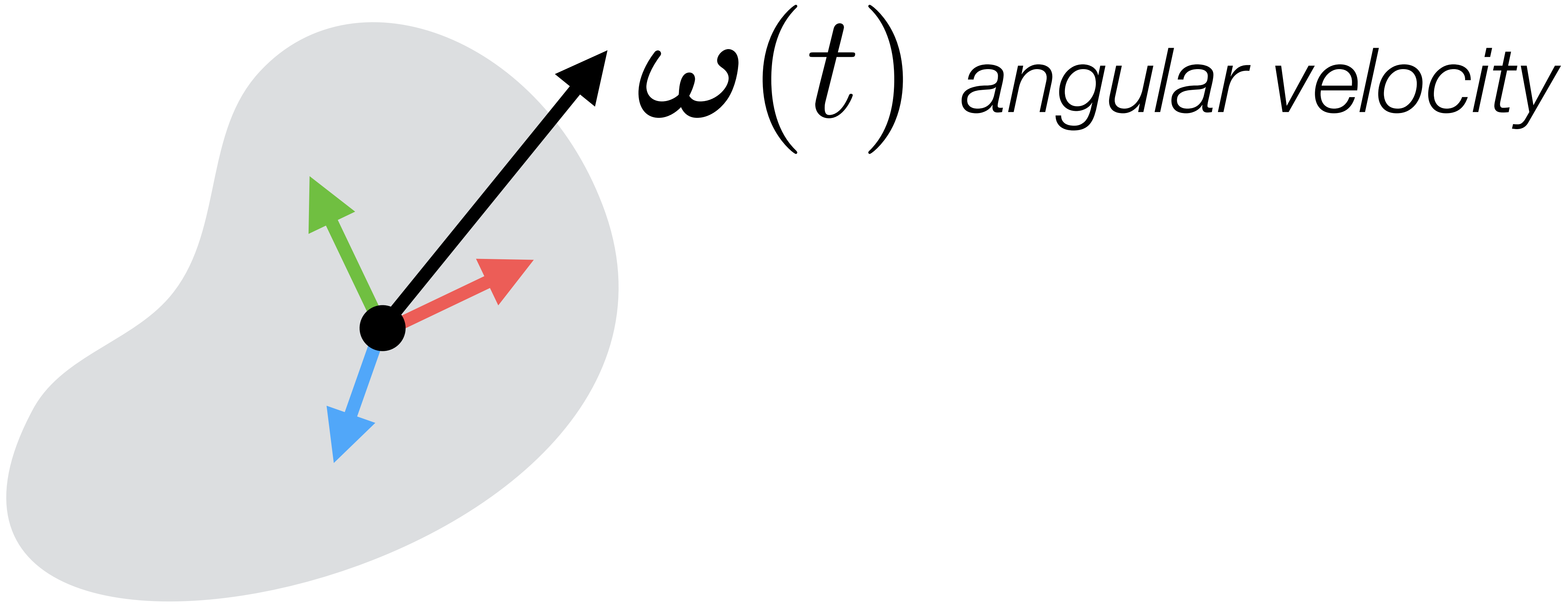
Linear vs. **Angular** Motion

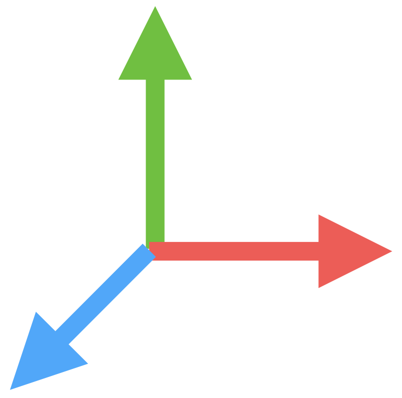




$$\dot{\mathbf{R}}(t) = [\dot{\mathbf{r}}_x(t), \dot{\mathbf{r}}_y(t), \dot{\mathbf{r}}_z(t)] \Leftarrow \dot{\mathbf{r}}(t) = \boldsymbol{\omega}(t) \times \mathbf{r}(t)$$

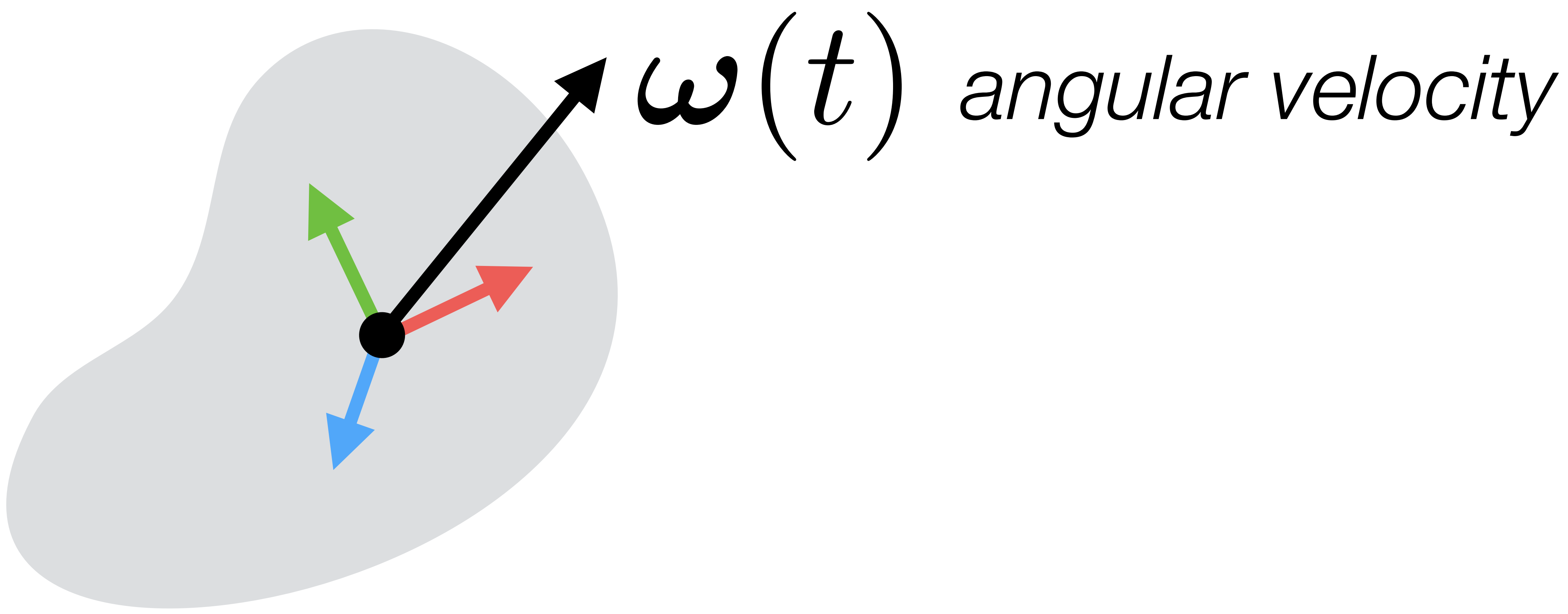
Linear vs. **Angular** Motion

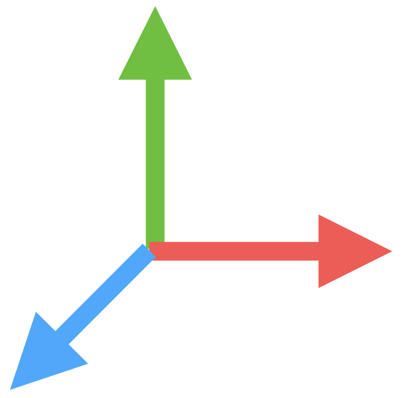




$$\dot{\mathbf{R}}(t) = [\boldsymbol{\omega}(t) \times \mathbf{r}_x(t), \boldsymbol{\omega}(t) \times \mathbf{r}_y(t), \boldsymbol{\omega}(t) \times \mathbf{r}_z(t)]$$

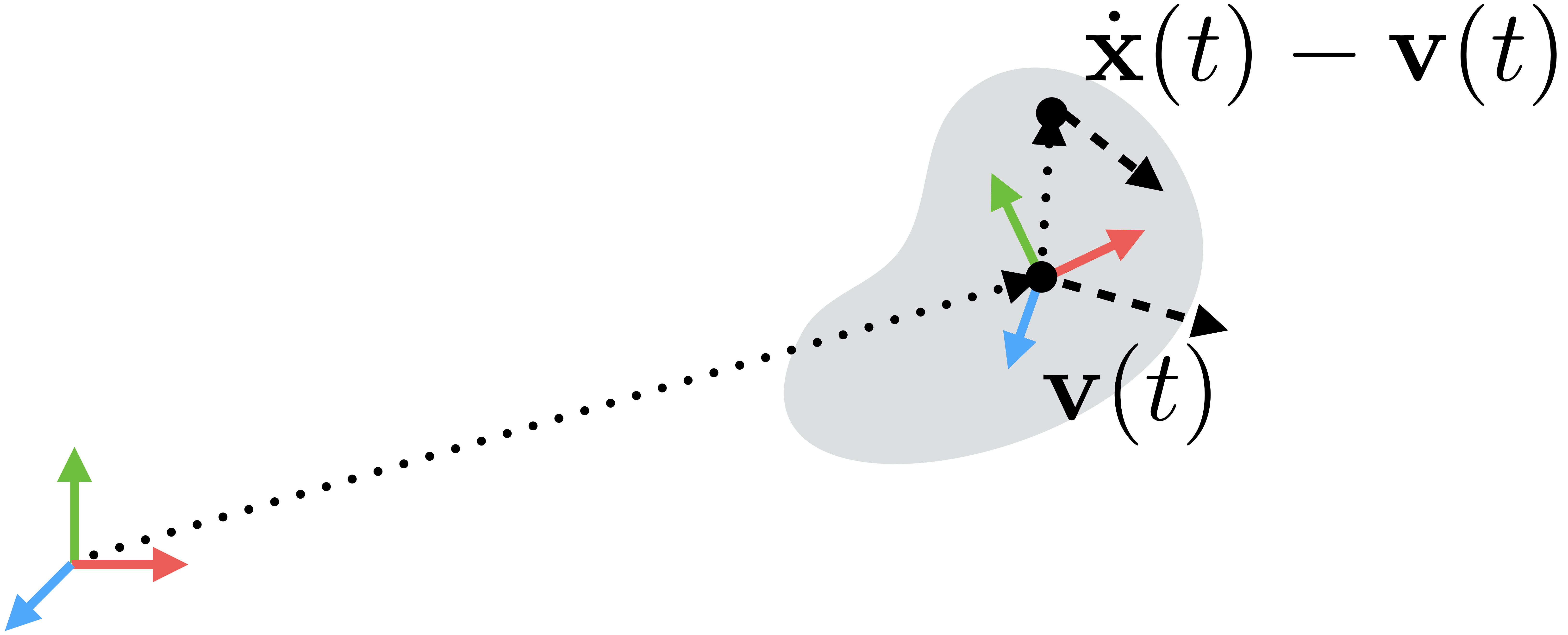
Linear vs. **Angular** Motion





$$\dot{\mathbf{R}}(t) = [\boldsymbol{\omega}(t)]_\times \mathbf{R}(t)$$

Linear vs. **Angular** Motion

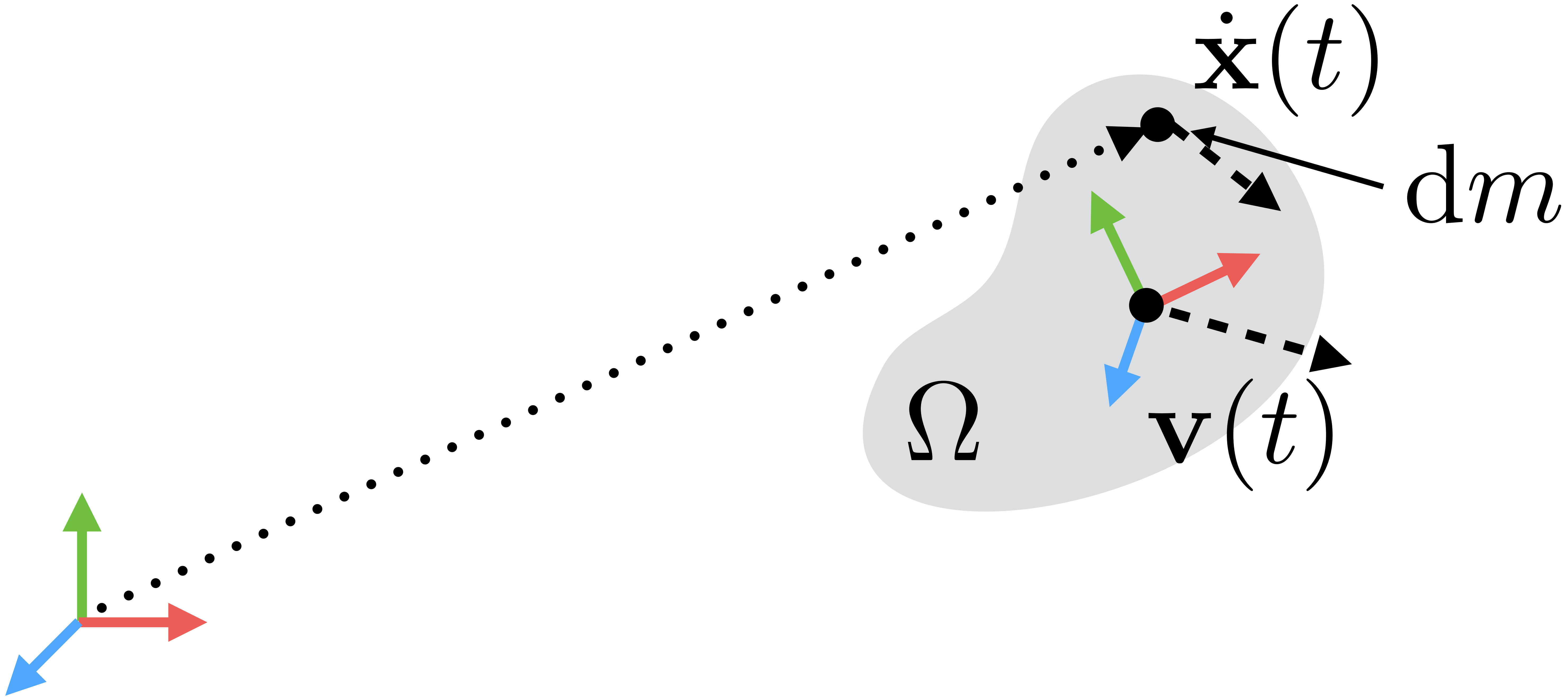


$$\dot{\mathbf{x}}(t) - \mathbf{v}(t) = [\boldsymbol{\omega}]_\times \mathbf{R}(t)\mathbf{x}_{\mathrm{rb}} = [\boldsymbol{\omega}]_\times (\mathbf{x}(t) - \mathbf{c}(t))$$

velocity *relative* to CoM

position *relative* to CoM

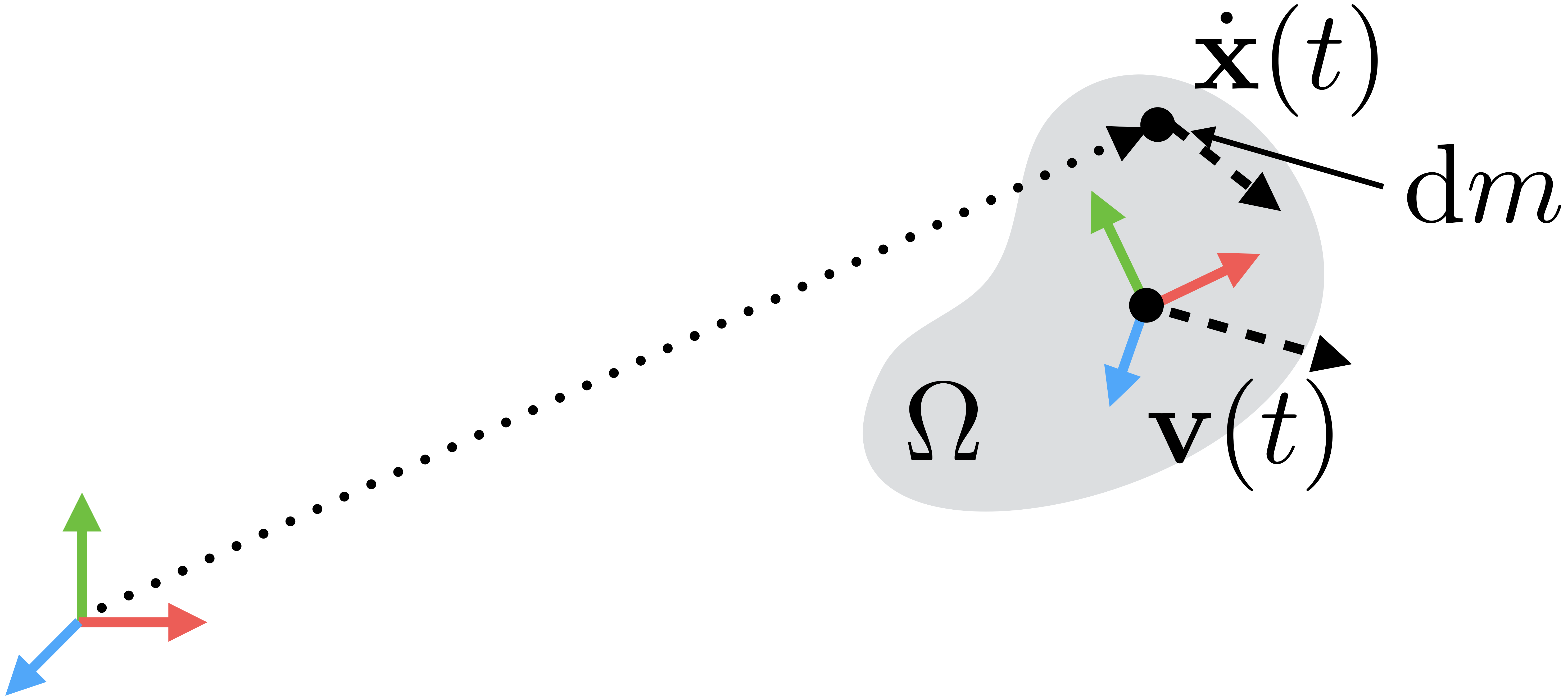
Linear vs. Angular Momentum



$$\mathbf{P}(t) = \int_\Omega \dot{\mathbf{x}}(t)\,\mathrm{d}m$$

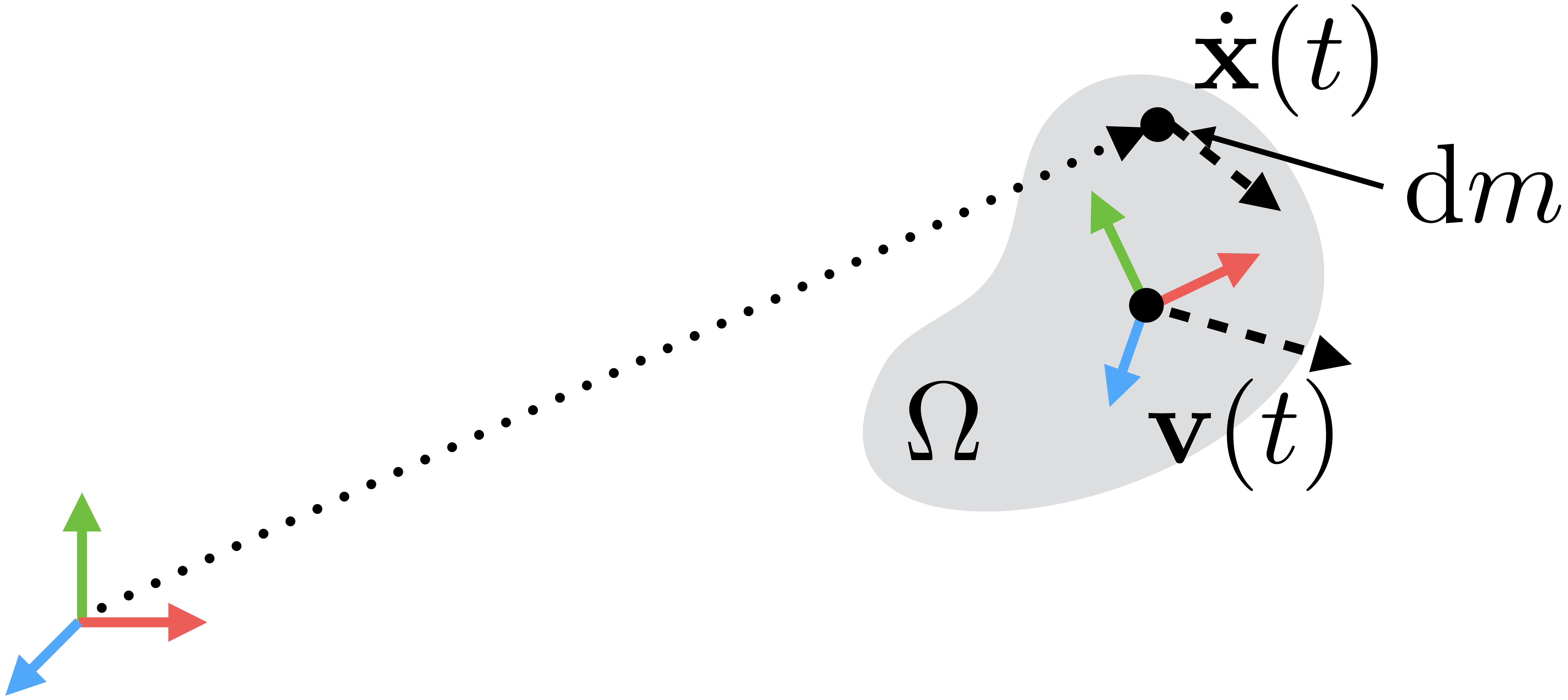
linear momentum

Linear vs. Angular Momentum



$$\mathbf{P}(t) = \int_\Omega [\boldsymbol{\omega}]_\times (\mathbf{x}(t) - \mathbf{c}(t)) + \mathbf{v}(t)\,\mathrm{d}m$$

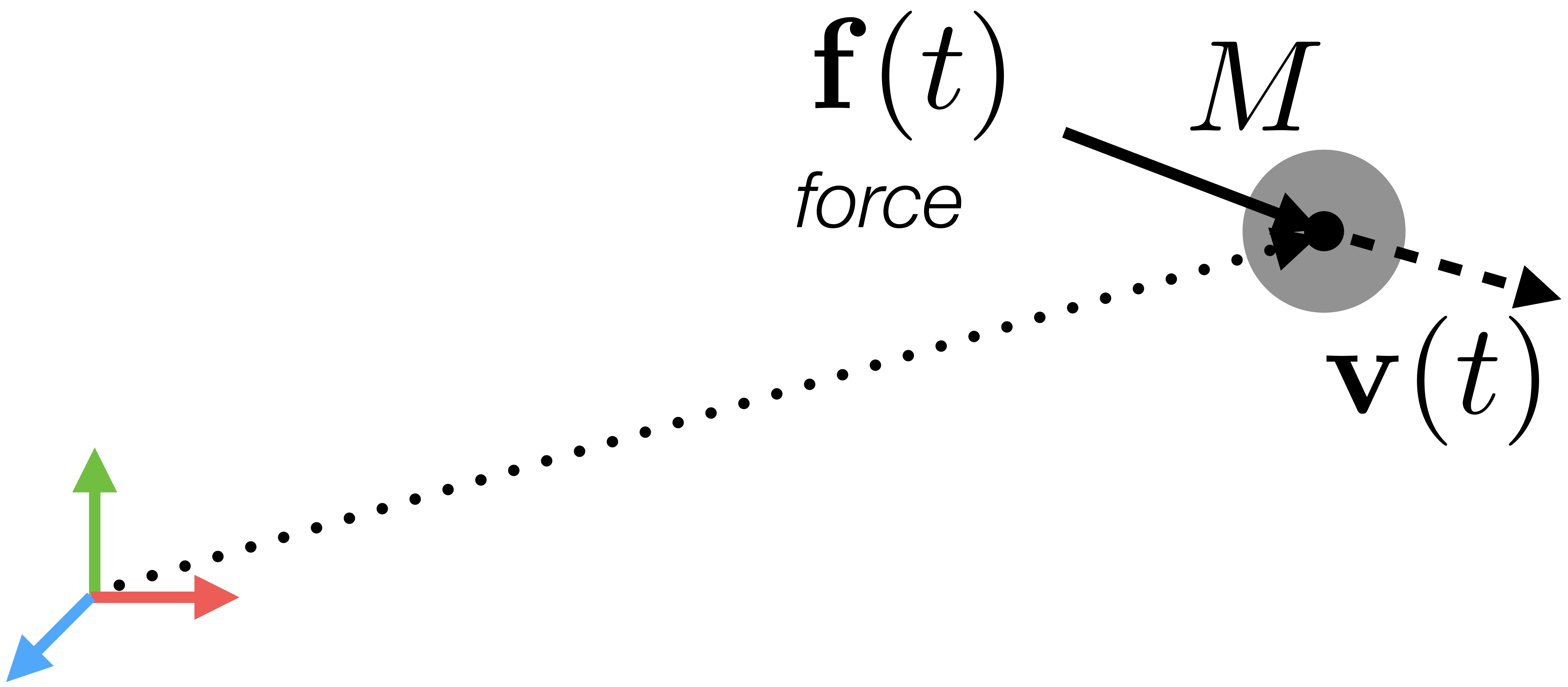
Linear vs. Angular Momentum



$$\mathbf{P}(t) = [\boldsymbol{\omega}]_\times \left(\int_\Omega \mathbf{x}(t) - \mathbf{c}(t) \, \mathrm{d}m \right) + \left(\int_\Omega \mathrm{d}m \right) \mathbf{v}(t)$$

zero (CoM definition) *mass*

Linear vs. Angular Momentum

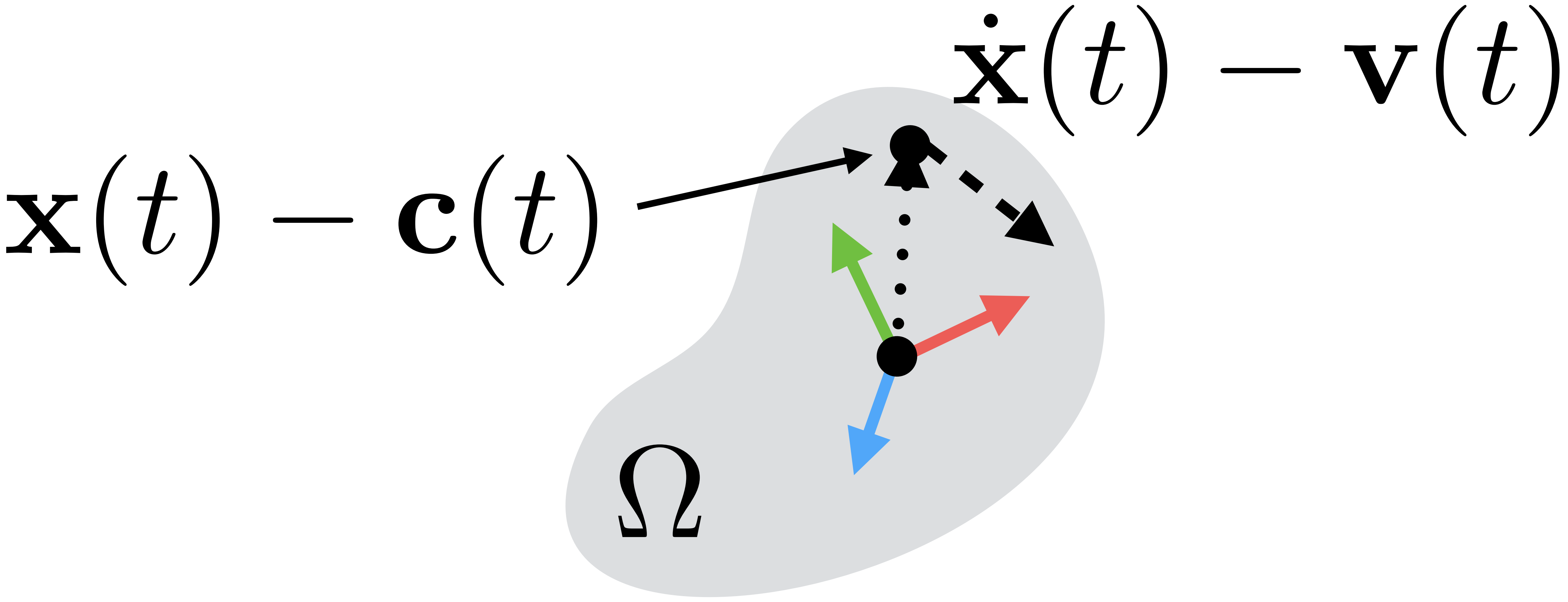


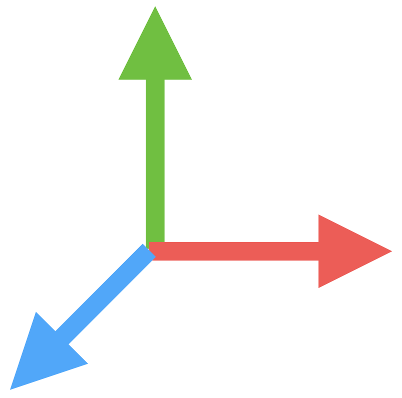
Newton's 2nd Law

$$\mathbf{P}(t) = M\mathbf{v}(t) \;\Rightarrow\; \mathbf{f}(t) = \dot{\mathbf{P}}(t) \;\Rightarrow\; \mathbf{f}(t) = M\mathbf{a}(t)$$

linear momentum is *preserved* if *net force* is *zero*

Linear vs. **Angular** Momentum



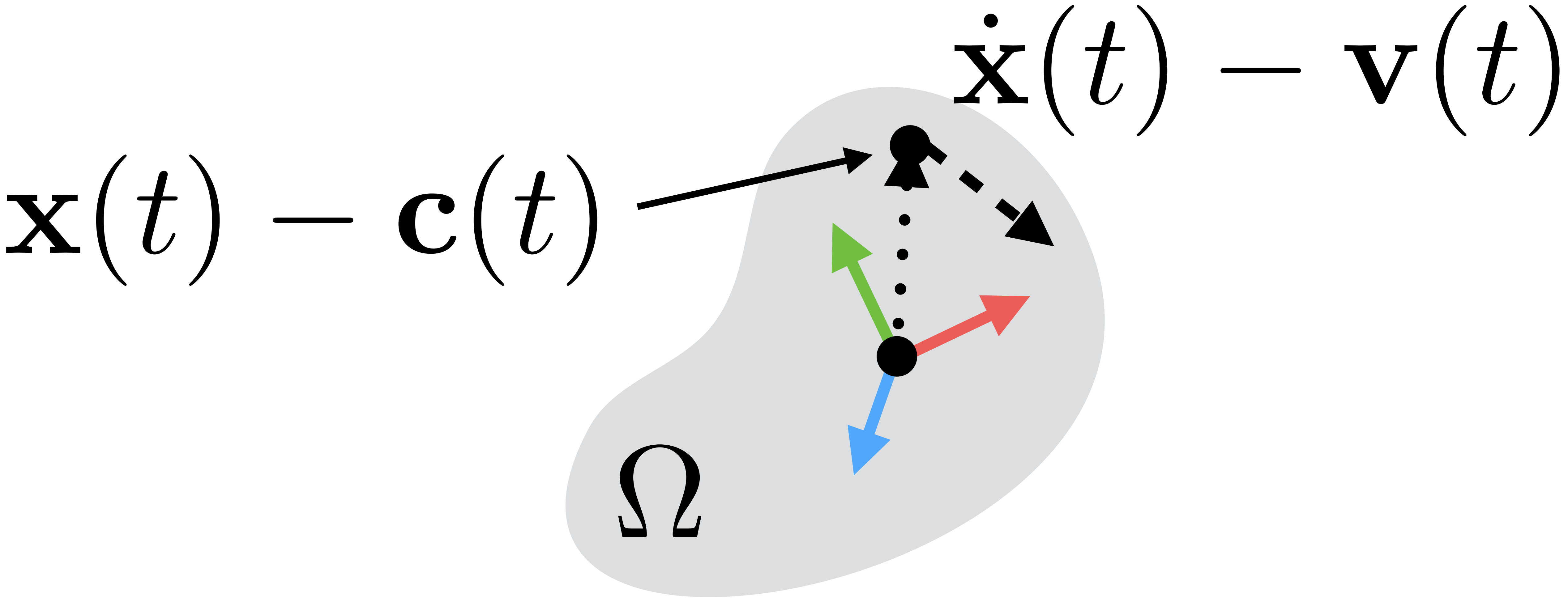


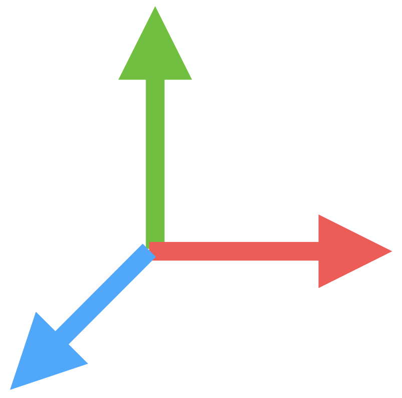
angular momentum

$$\mathbf{L}_{\mathbf{c}}(t) = \int_{\Omega} (\mathbf{x}(t) - \mathbf{c}(t)) \times (\dot{\mathbf{x}}(t) - \mathbf{v}(t)) \, \mathrm{d}m$$

relative positions *relative velocities*

Linear vs. **Angular** Momentum

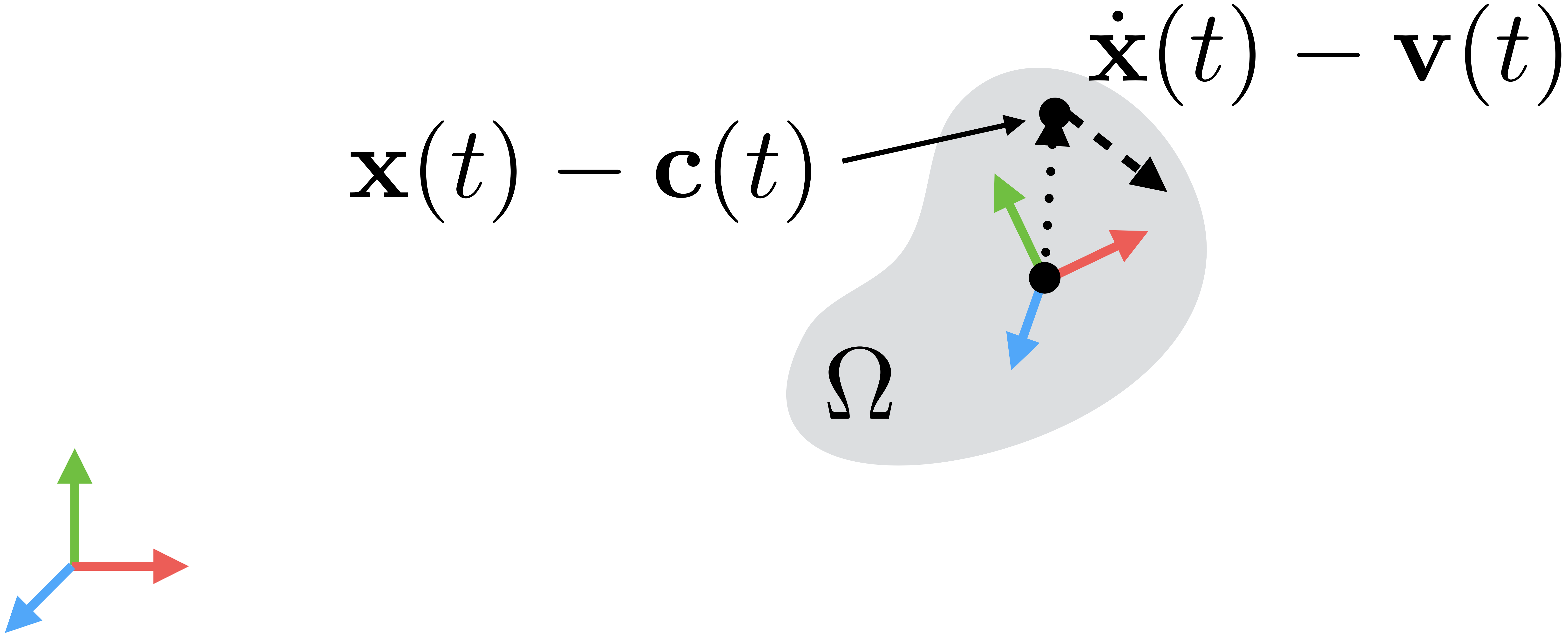




$$\mathbf{L}_{\mathbf{c}}(t) = \int_{\Omega} (\mathbf{x}(t) - \mathbf{c}(t)) \times \boldsymbol{\omega}(t) \times (\mathbf{x}(t) - \mathbf{c}(t)) \, \mathrm{d}m$$

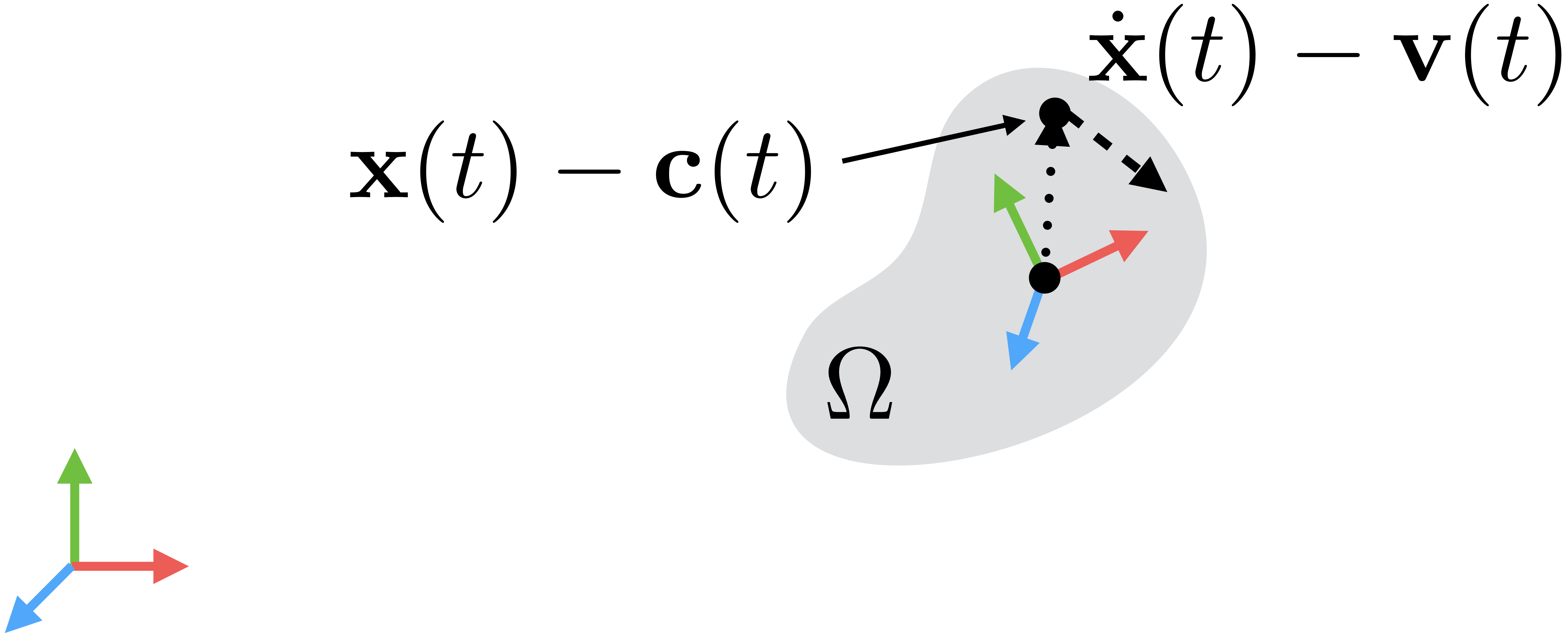
relative positions — *relative velocities*

Linear vs. **Angular** Momentum



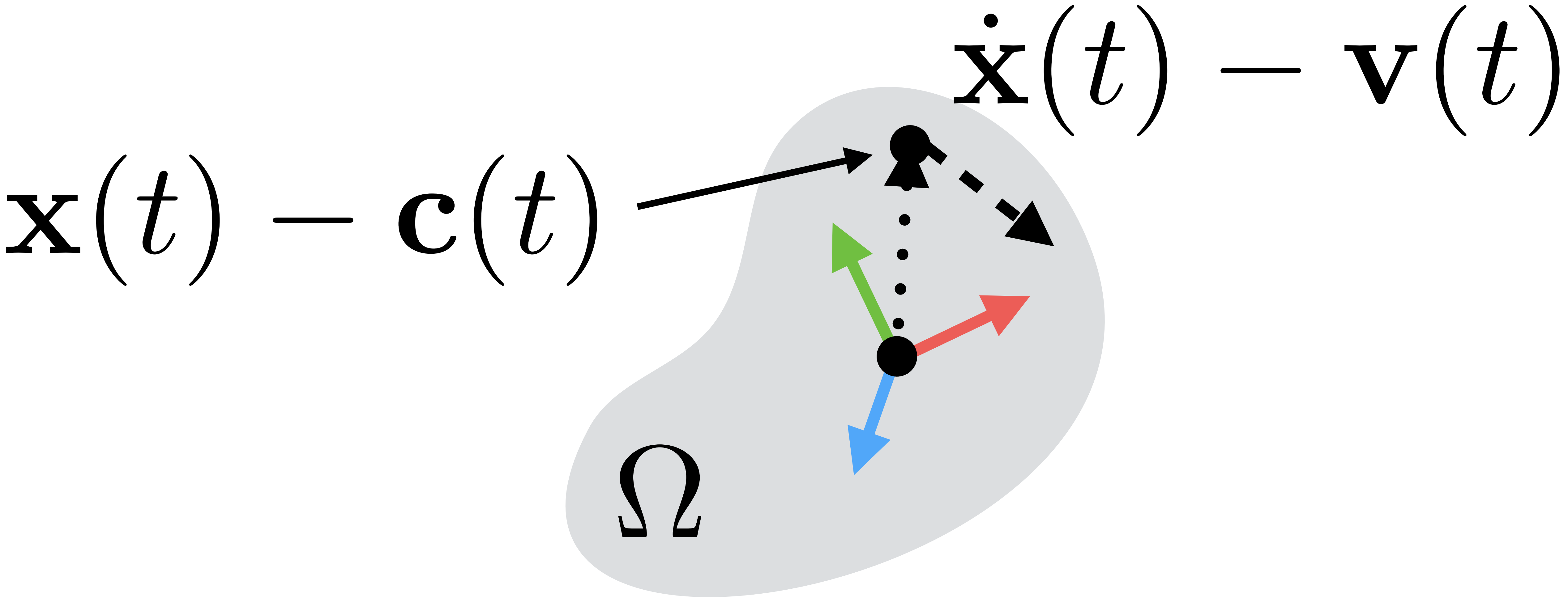
$$\mathbf{L}_{\mathbf{c}}(t) = -\int_{\Omega} (\mathbf{x}(t) - \mathbf{c}(t)) \times (\mathbf{x}(t) - \mathbf{c}(t)) \times \boldsymbol{\omega}(t) \, \mathrm{d}m$$

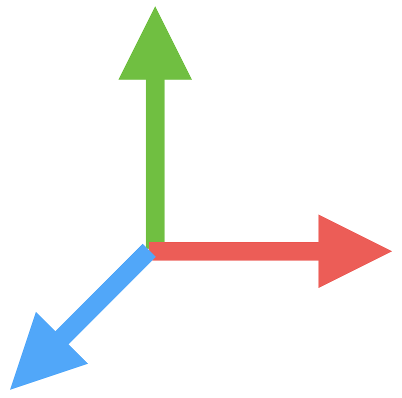
Linear vs. **Angular** Momentum



$$\mathbf{L_c}(t) = \left(\int_\Omega [\mathbf{x}(t) - \mathbf{c}(t)]_\times^T [\mathbf{x}(t) - \mathbf{c}(t)]_\times \,\mathrm{d}m\right) \boldsymbol{\omega}(t)$$

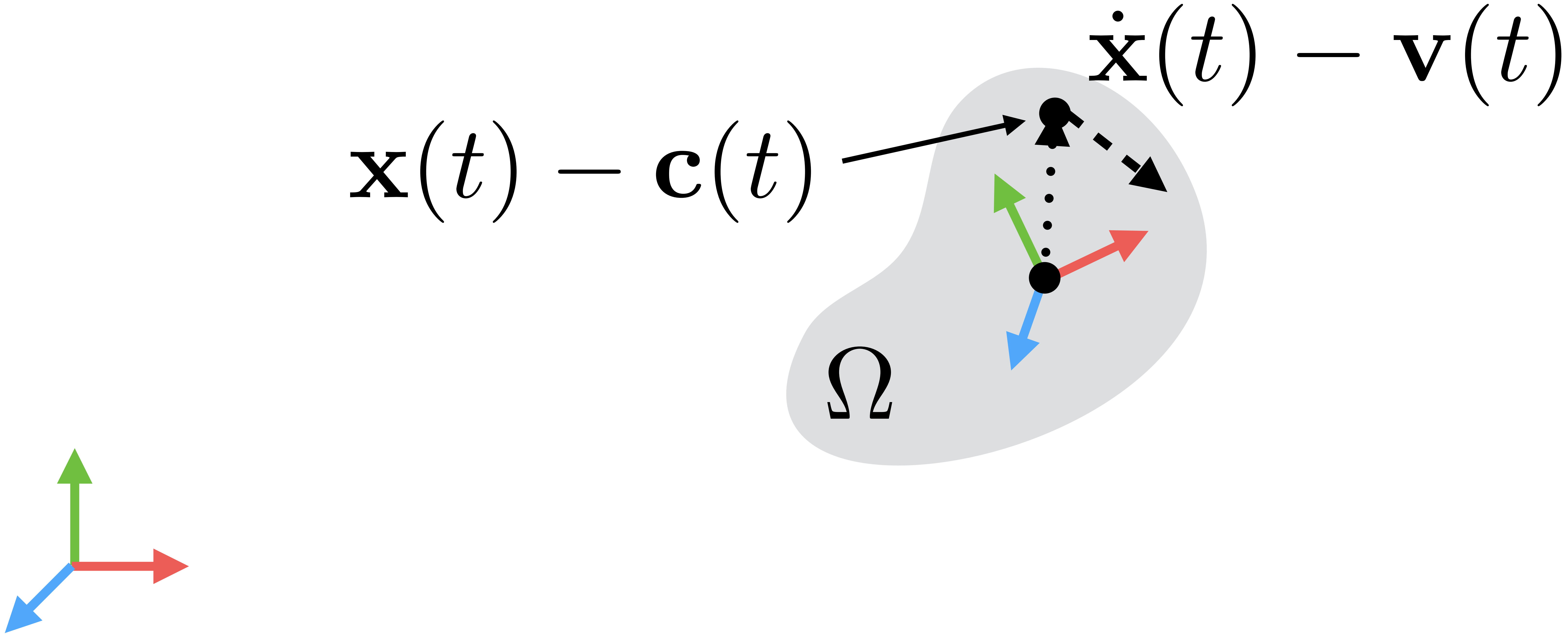
Linear vs. **Angular** Momentum





$$\mathbf{L}_{\mathbf{c}}(t) = \mathbf{R}(t)\mathbf{I}_{\mathrm{rb}}\mathbf{R}(t)^T\boldsymbol{\omega}(t)$$

Linear vs. **Angular** Momentum

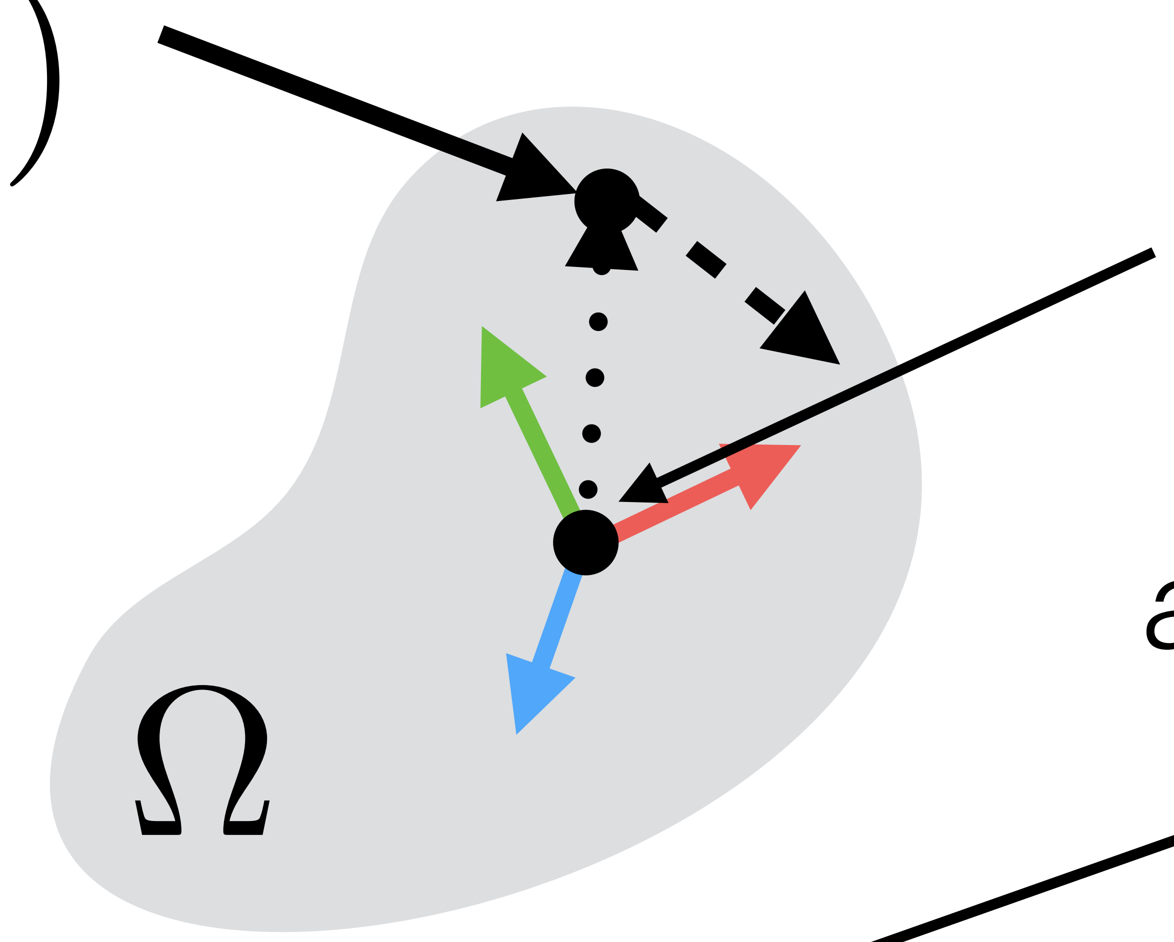


$$\mathbf{L_c}(t) = \mathbf{I_c}(t)\boldsymbol{\omega}(t)$$

Linear vs. **Angular** Momentum

$\mathbf{f}(t)$

force

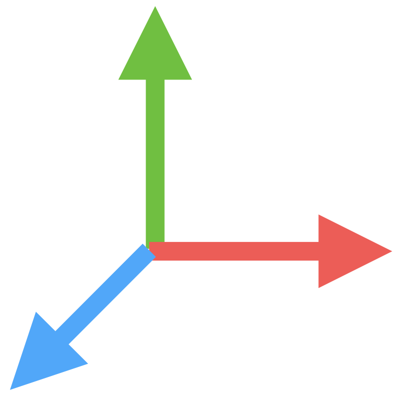


$\boldsymbol{\tau}_{\mathbf{c}}(t)$

torque

angular acceleration

moment of inertia

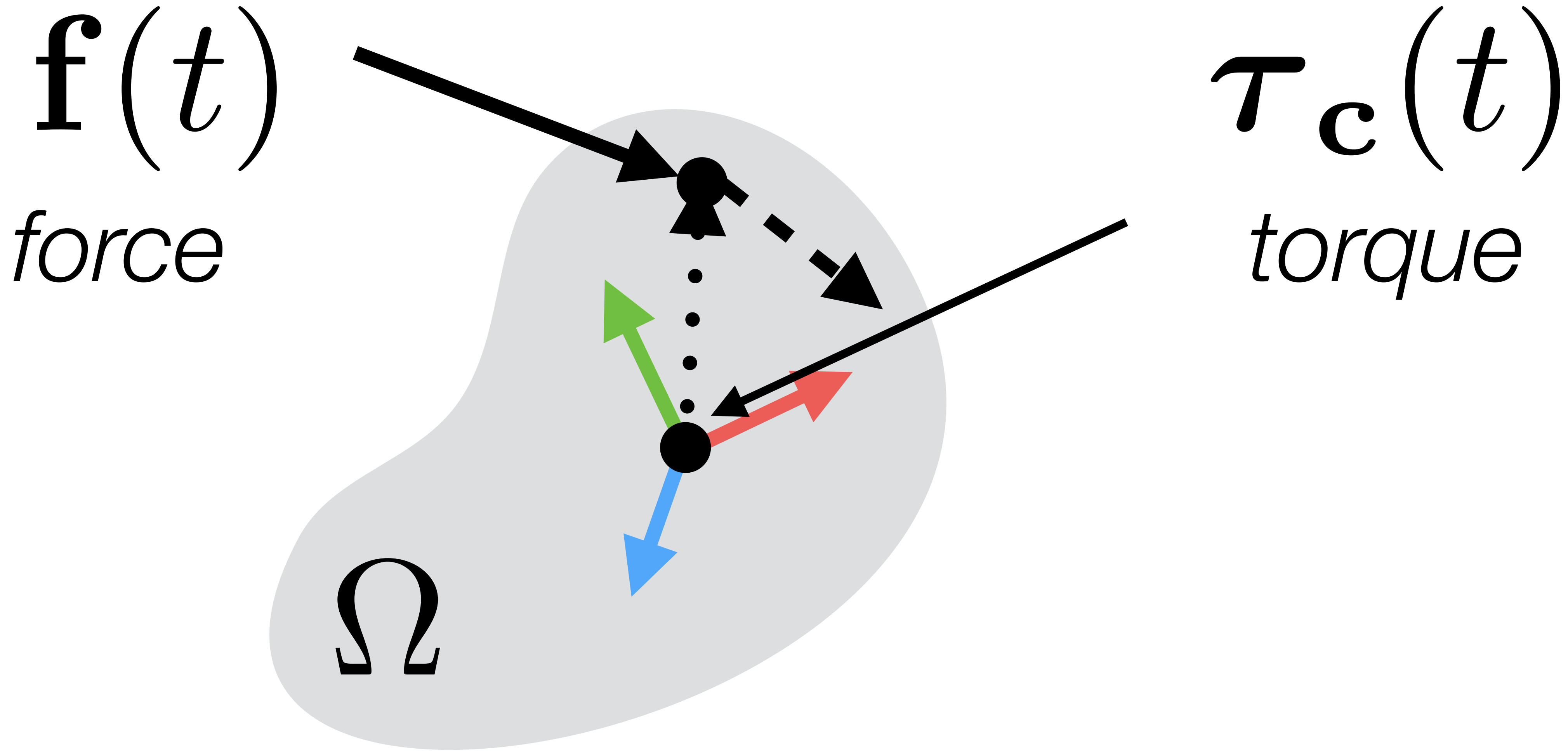


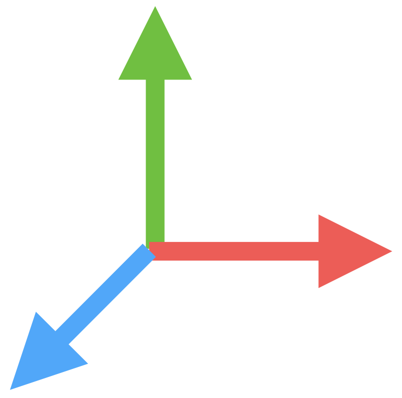
$$\mathbf{I}_{\mathbf{c}}(t)\dot{\boldsymbol{\omega}}(t) = \mathbf{L}_{\mathbf{c}}(t) \times \boldsymbol{\omega}(t) + \boldsymbol{\tau}_{\mathbf{c}}(t)$$

$$\mathbf{L}_{\mathbf{c}}(t) = \mathbf{I}_{\mathbf{c}}(t)\boldsymbol{\omega}(t) \Rightarrow \boldsymbol{\tau}_{\mathbf{c}}(t) = \dot{\mathbf{L}}_{\mathbf{c}}(t)$$

angular momentum is *preserved* if *net* torque is *zero*

Equations of Motion





$$\begin{bmatrix} M\mathbf{E}_{3\times 3} & \\ & \mathbf{I}_{\mathbf{c}}(t) \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{c}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{L}_{\mathbf{c}}(t) \times \boldsymbol{\omega}(t) + \boldsymbol{\tau}_{\mathbf{c}}(t) \end{bmatrix}$$

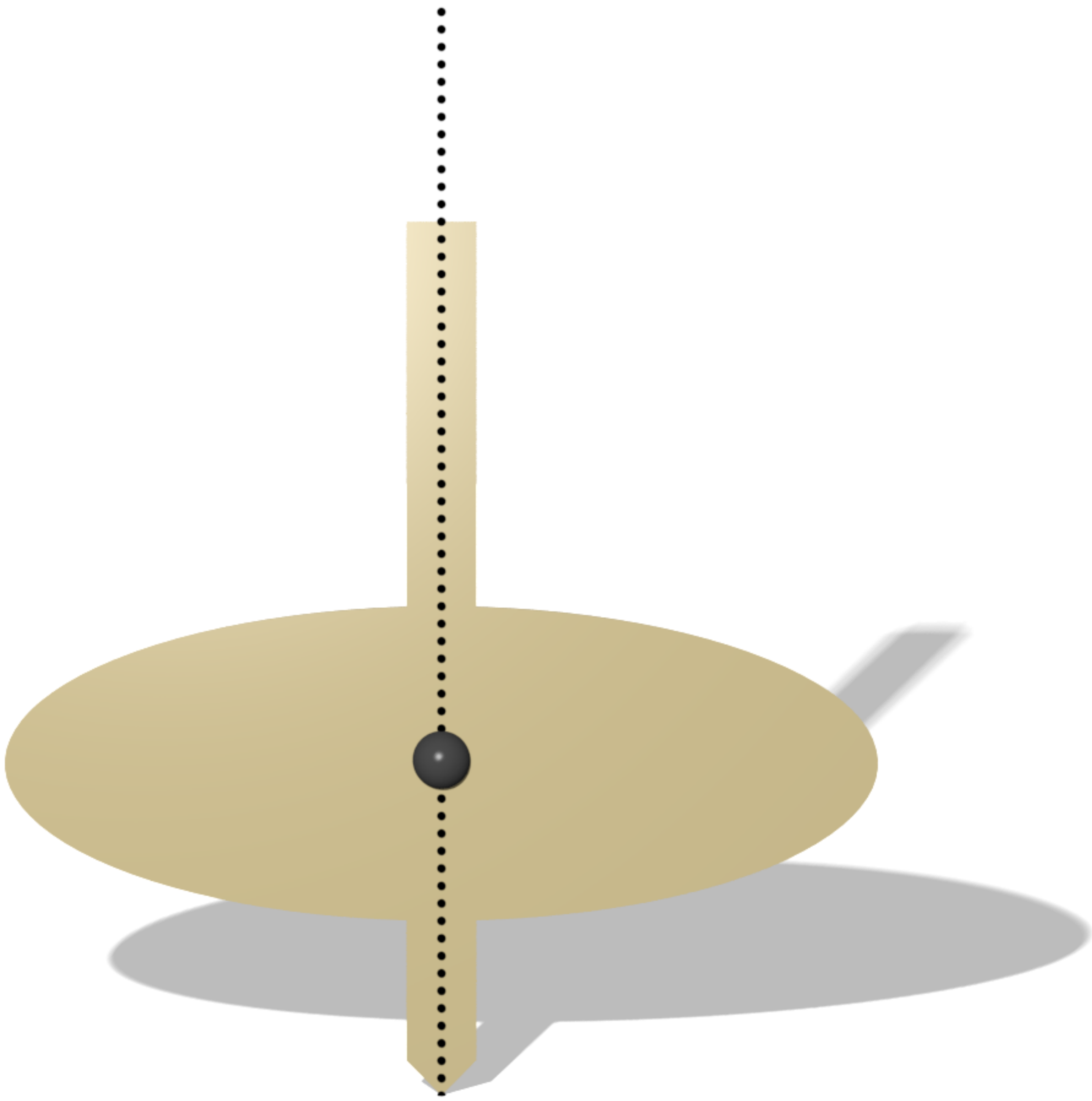
generalized *mass* — generalized *acceleration* — generalized *force*

Agenda

- Motivation
- Coordinate Frames / Motion of a Point
- Mass Properties (Mass, Center of Mass, Moment of Inertia)
- Linear vs. Angular Motion / Momentum
- Equations of Motion
- Optimizing Mass Distribution

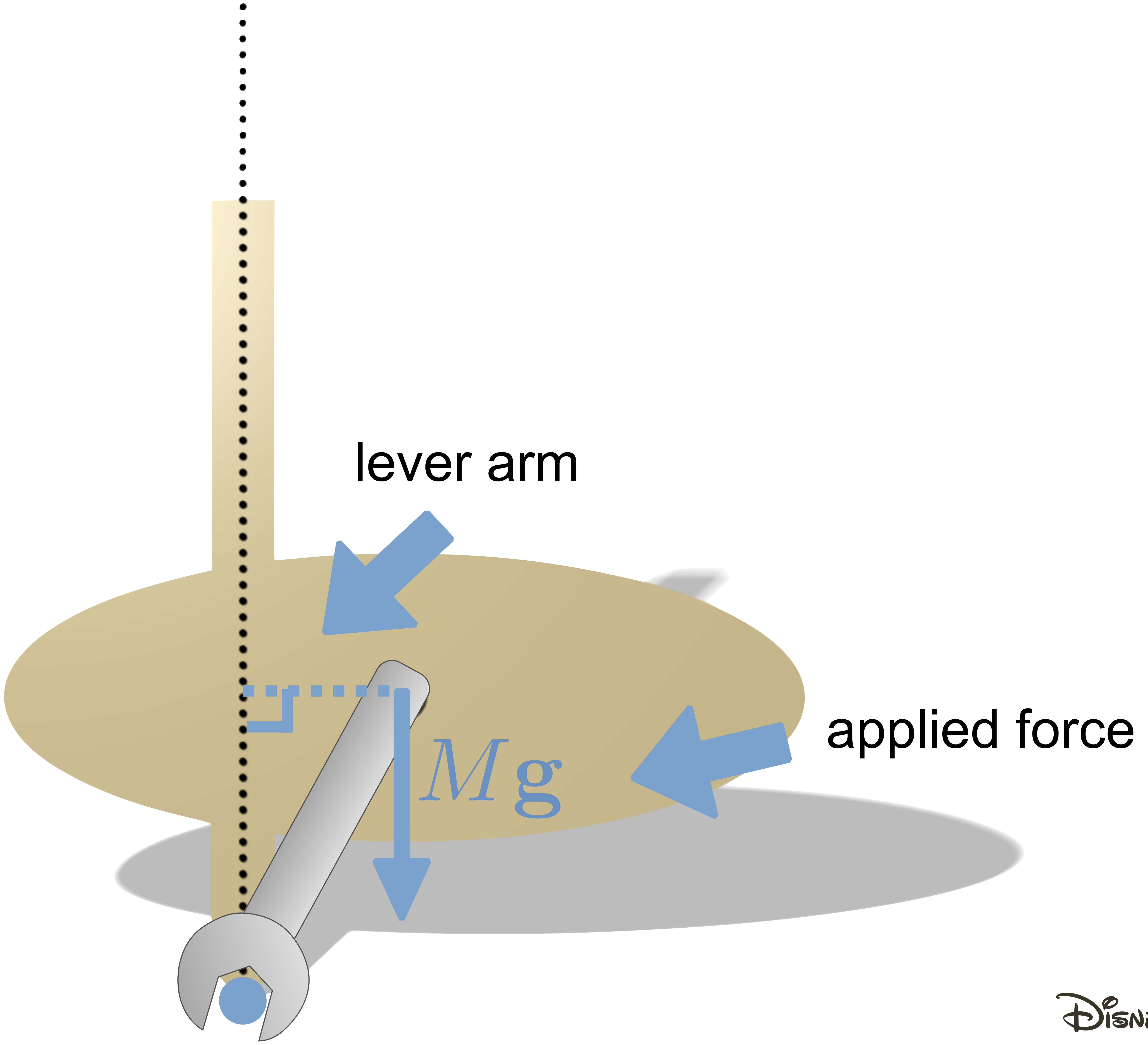
Measuring Spin Quality

- Center of mass
  - on spinning axis



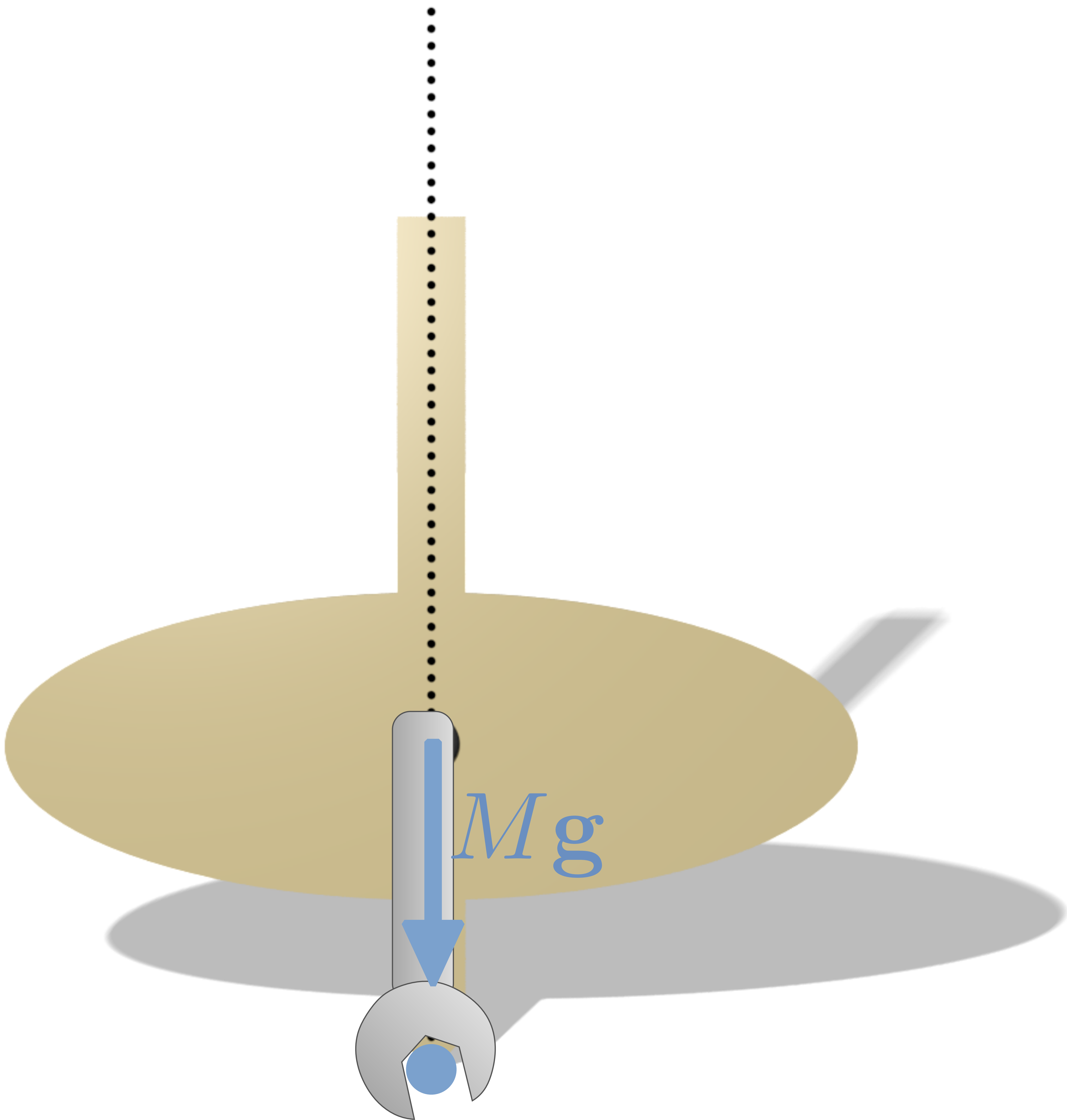
Measuring Spin Quality

- Center of mass
  - on spinning axis



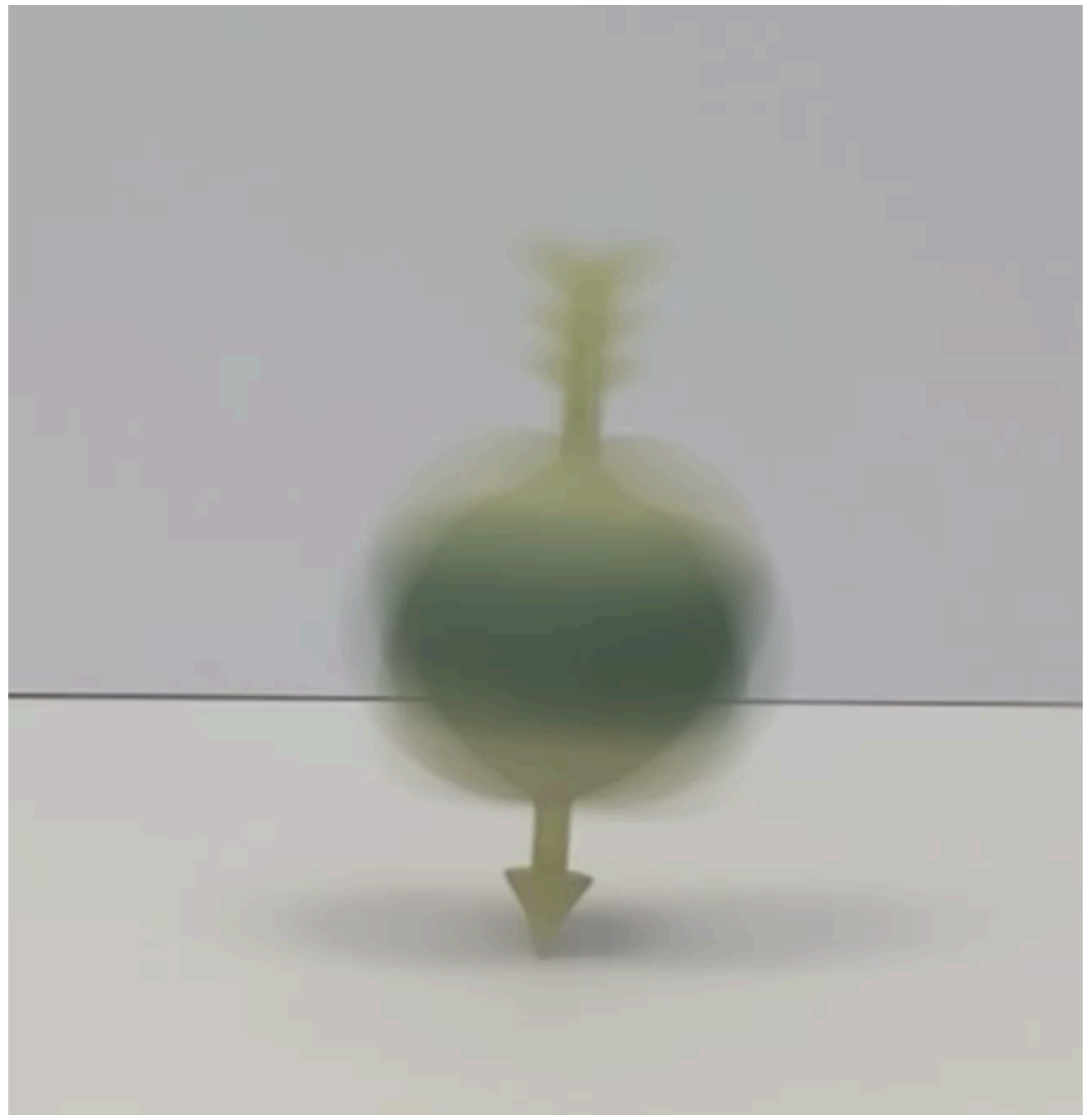
Measuring Spin Quality

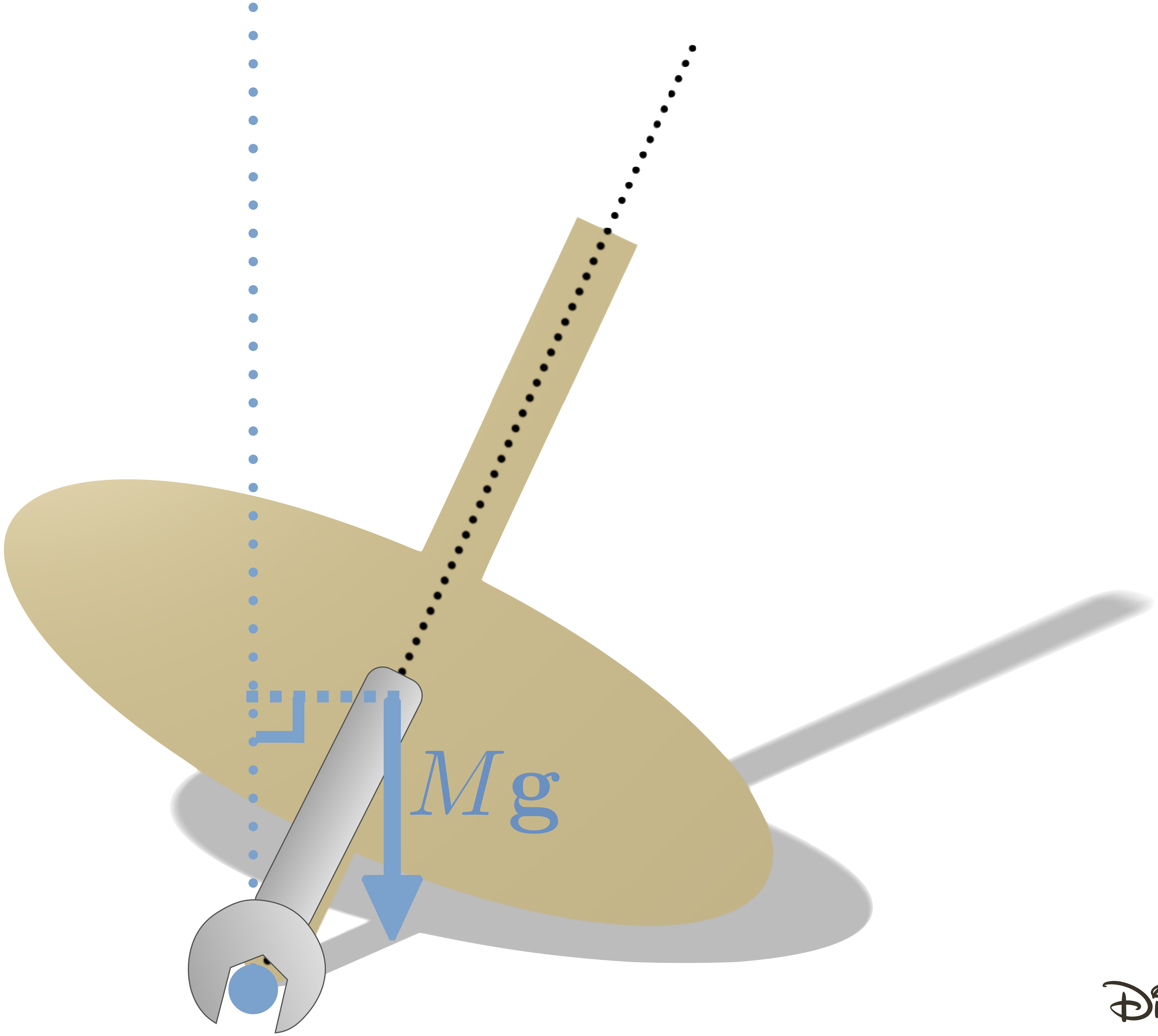
- Center of mass
  - on spinning axis



Measuring Spin Quality

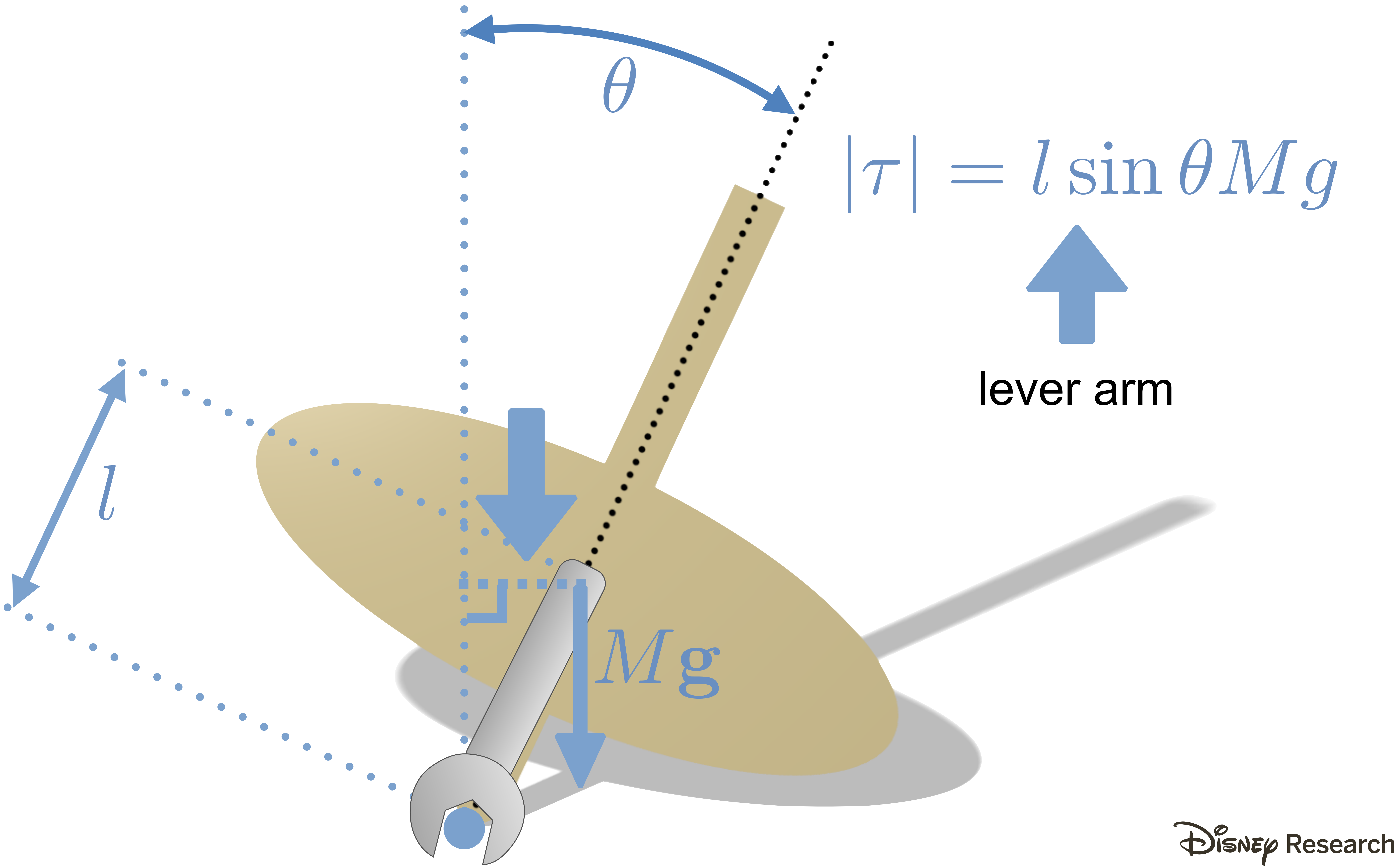
- Center of mass
  - on spinning axis





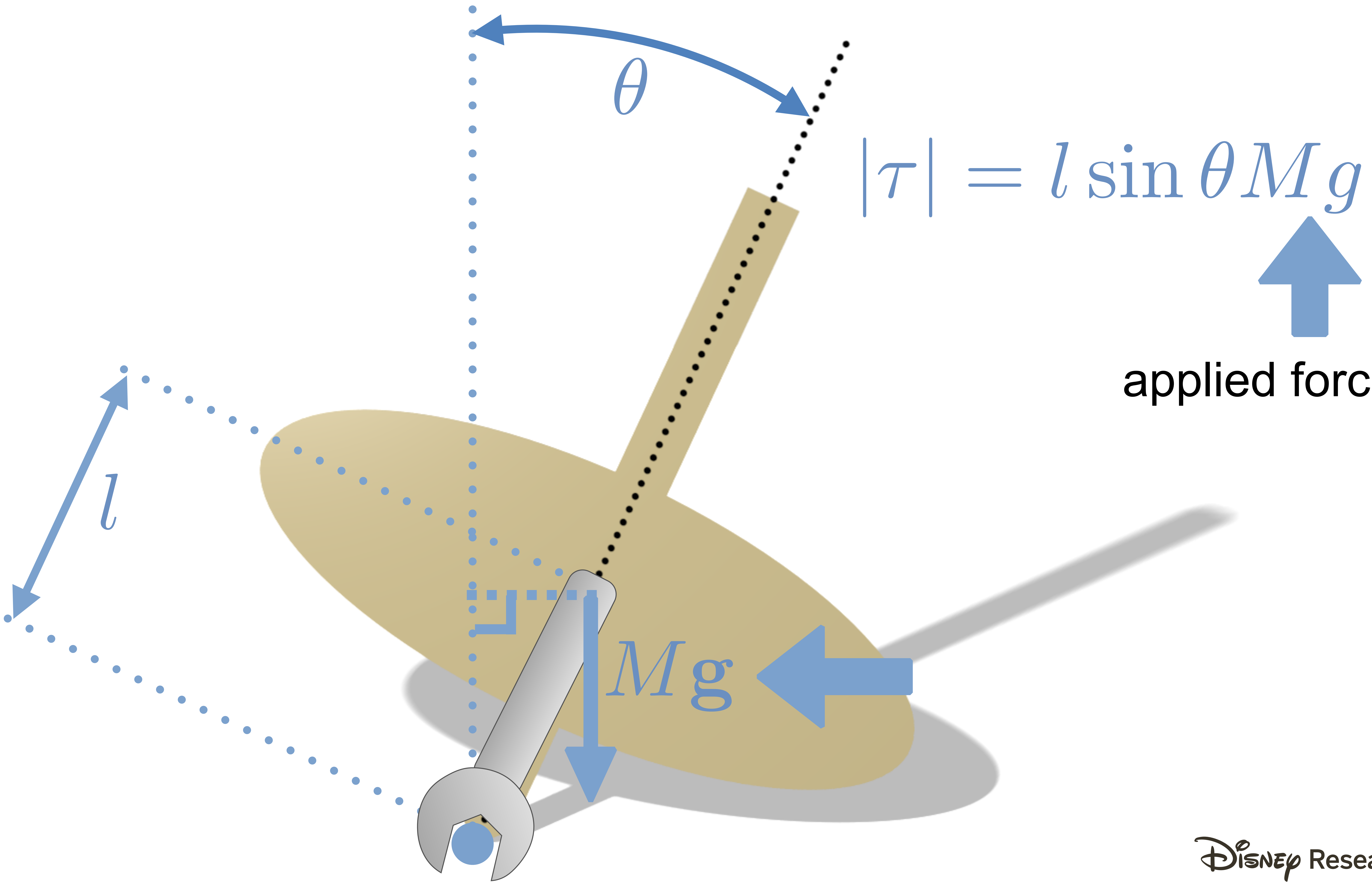
Measuring Spin Quality

- Center of mass
  - on spinning axis



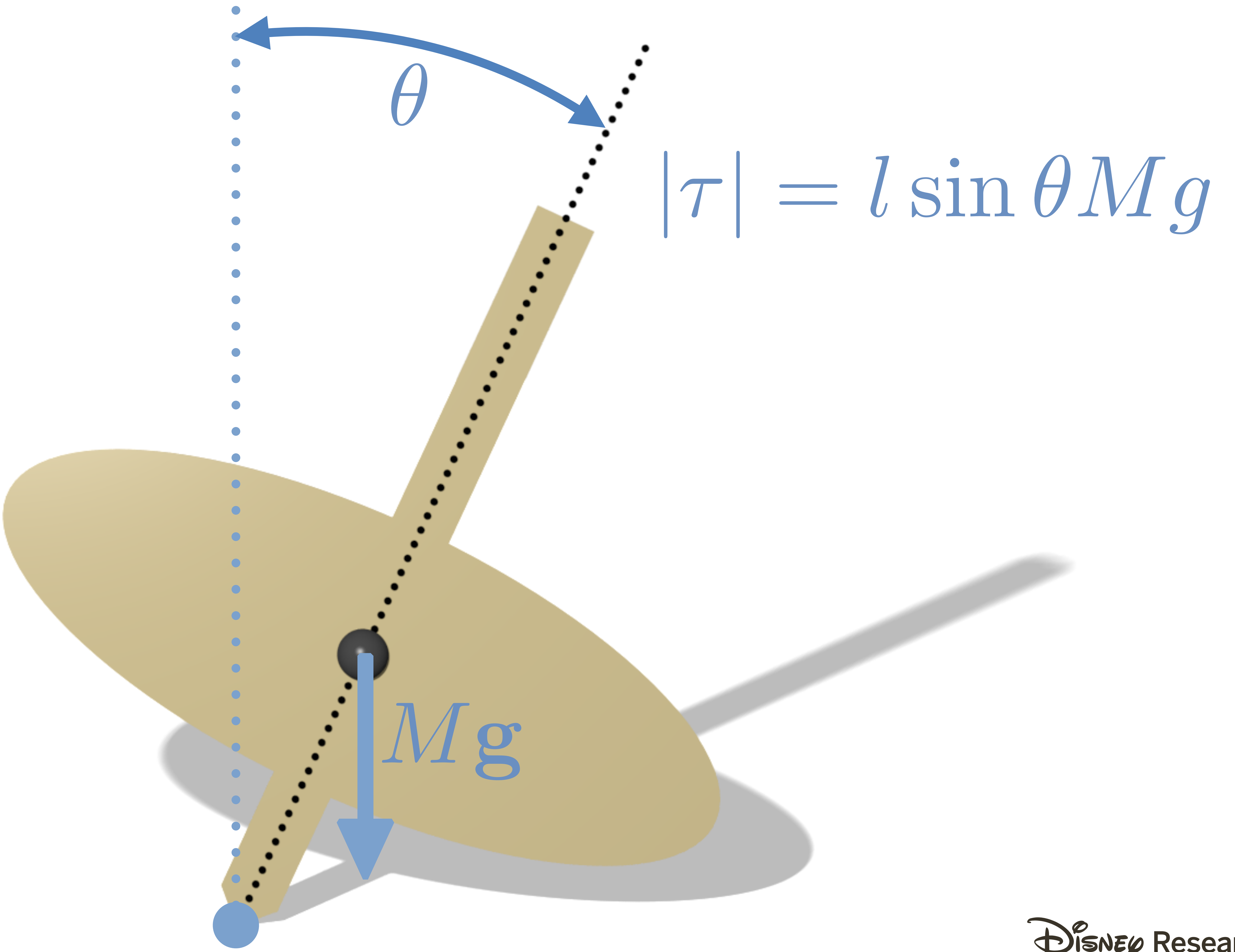
Measuring Spin Quality

- Center of mass
  - on spinning axis



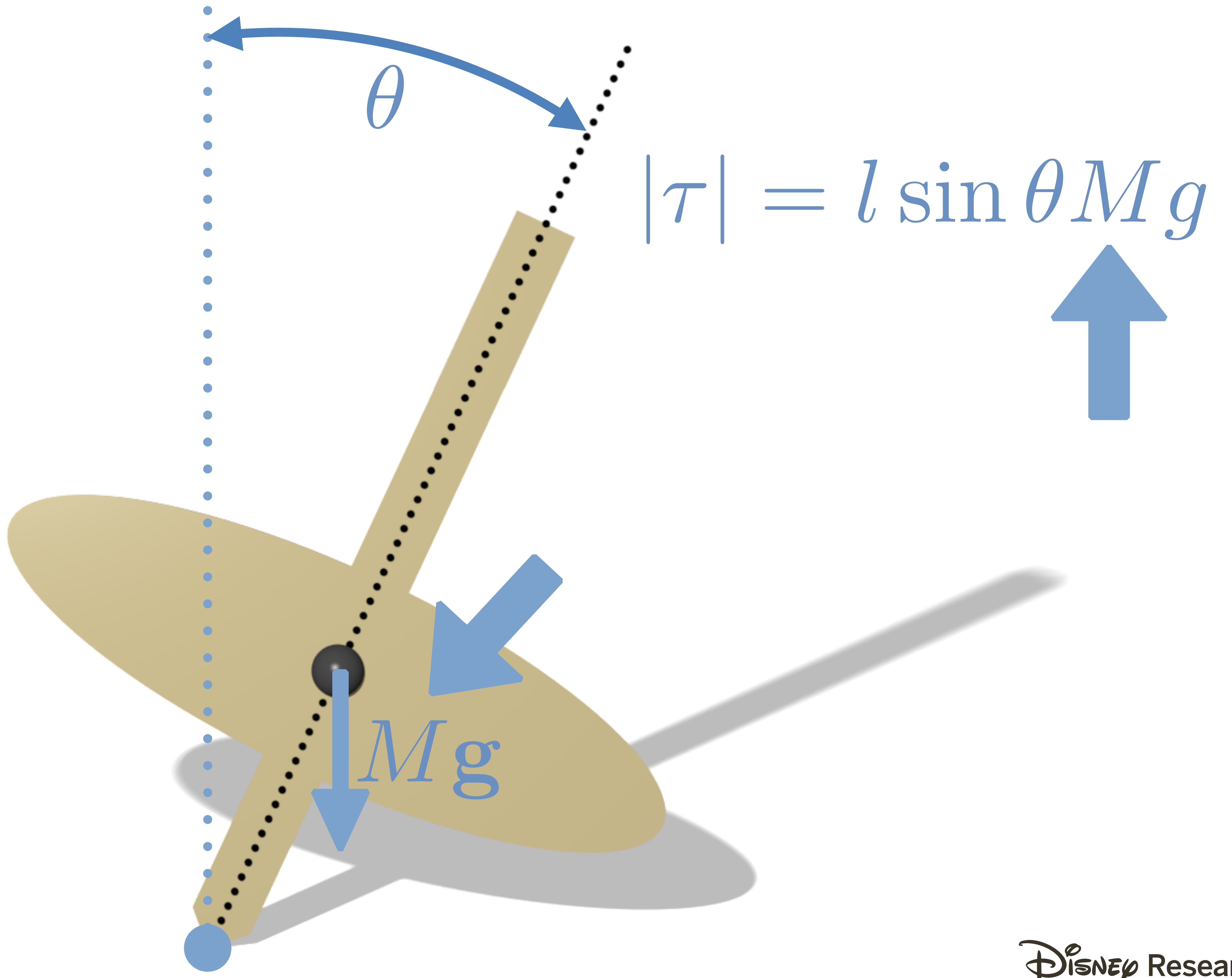
Measuring Spin Quality

- Center of mass
  - on spinning axis



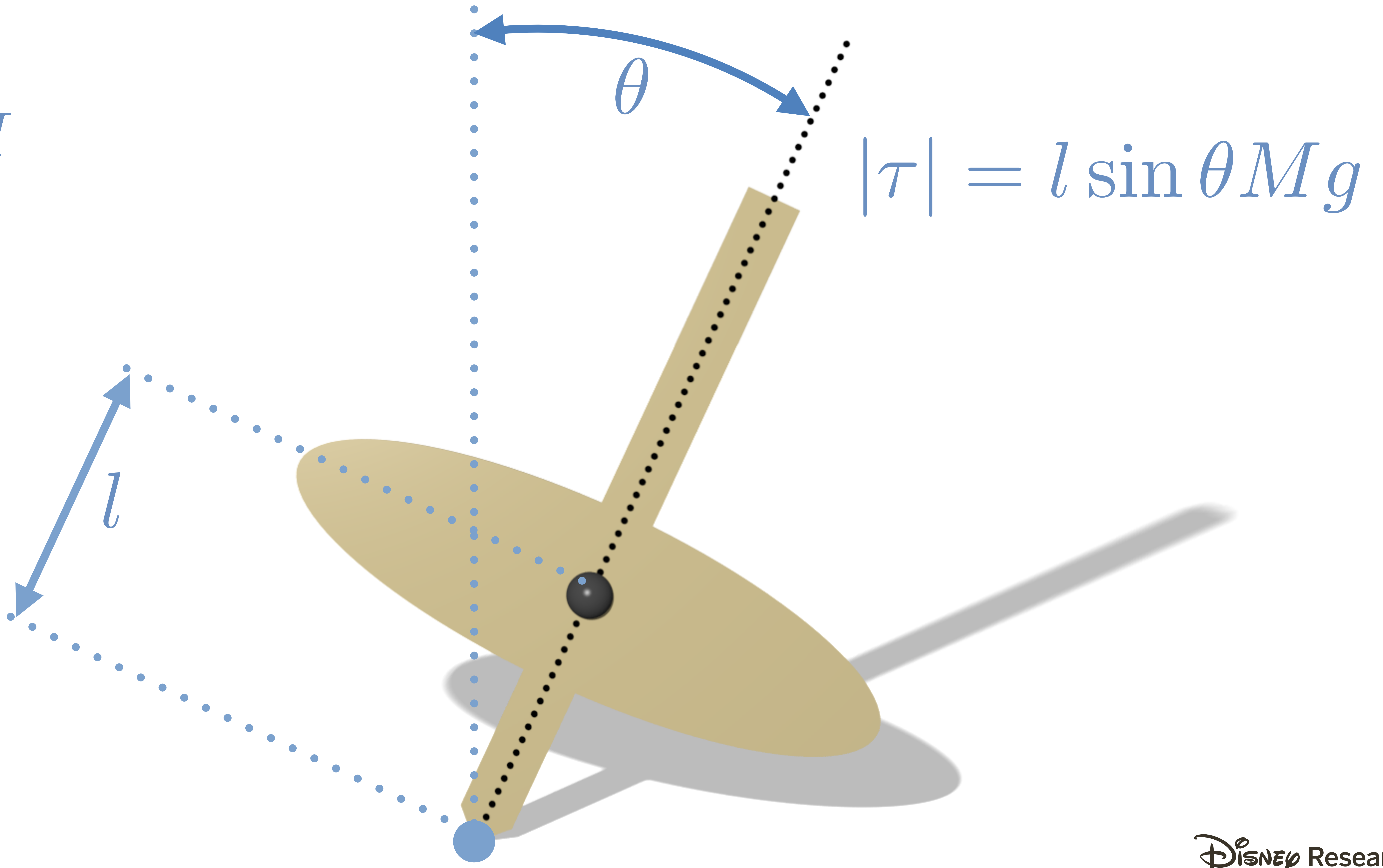
Measuring Spin Quality

- Center of mass
  - on spinning axis
  - reduce mass M



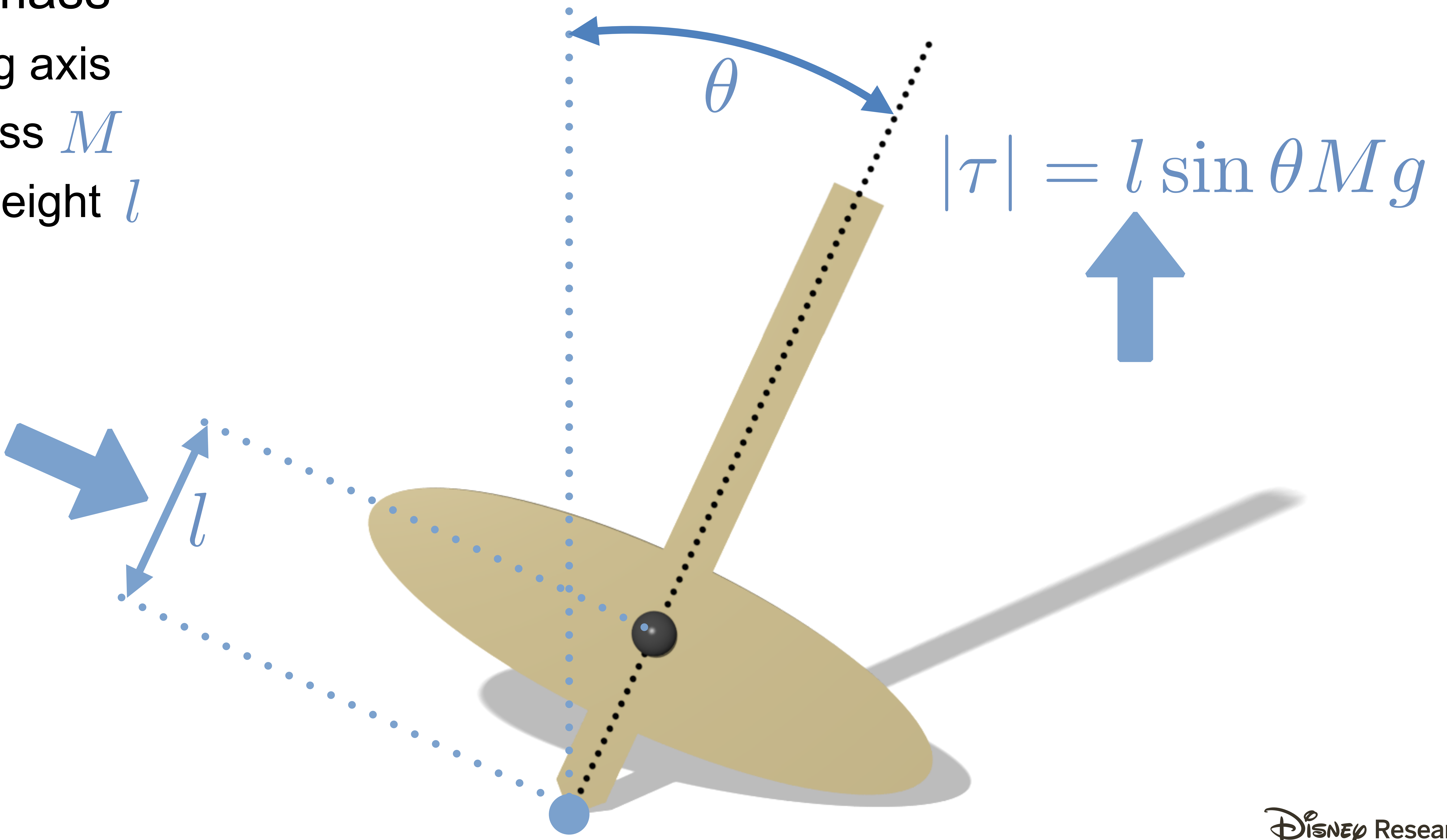
Measuring Spin Quality

- Center of mass
  - on spinning axis
  - reduce mass M



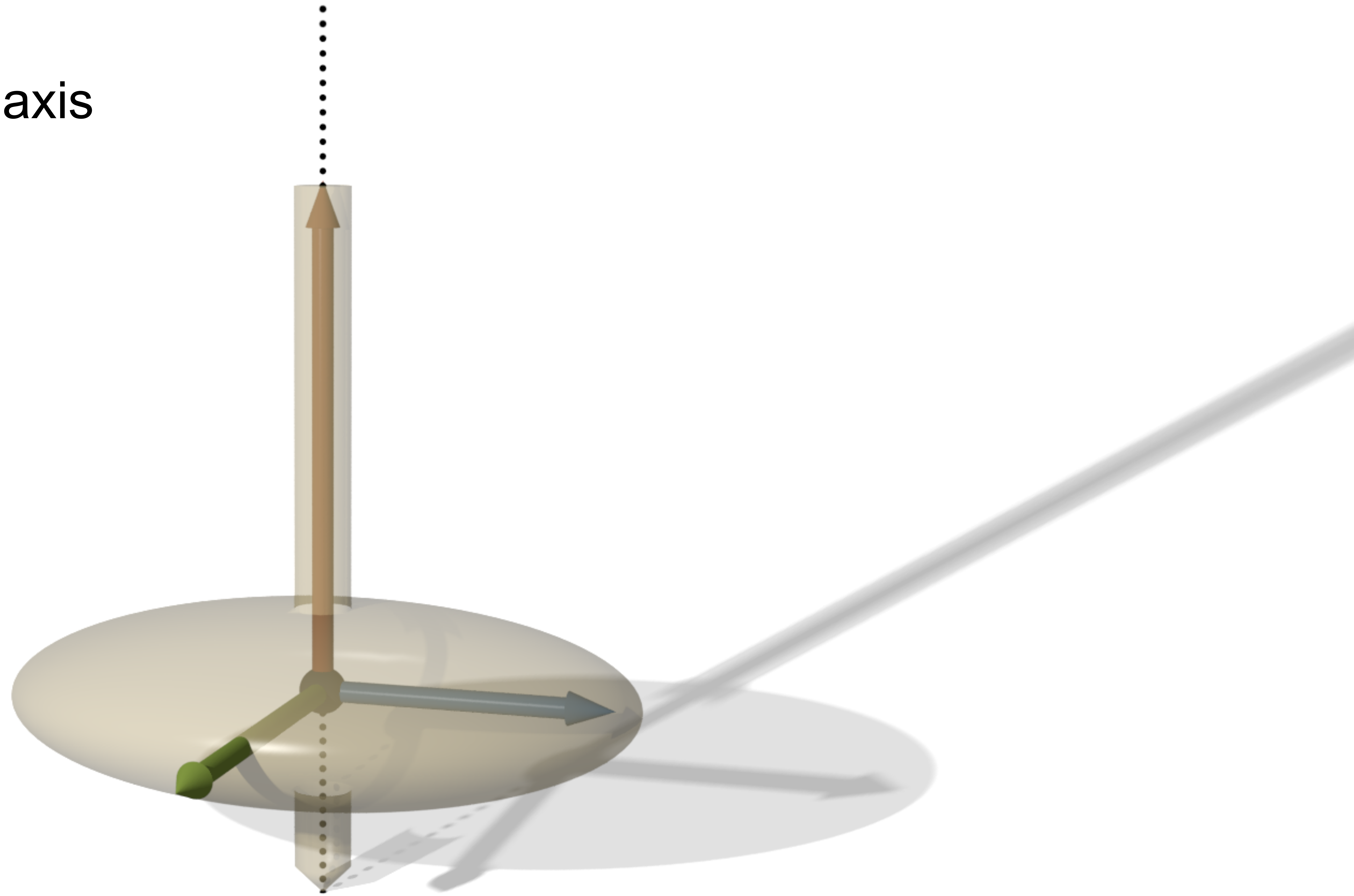
Measuring Spin Quality

- Center of mass
  - on spinning axis
  - reduce mass M
  - minimize height l



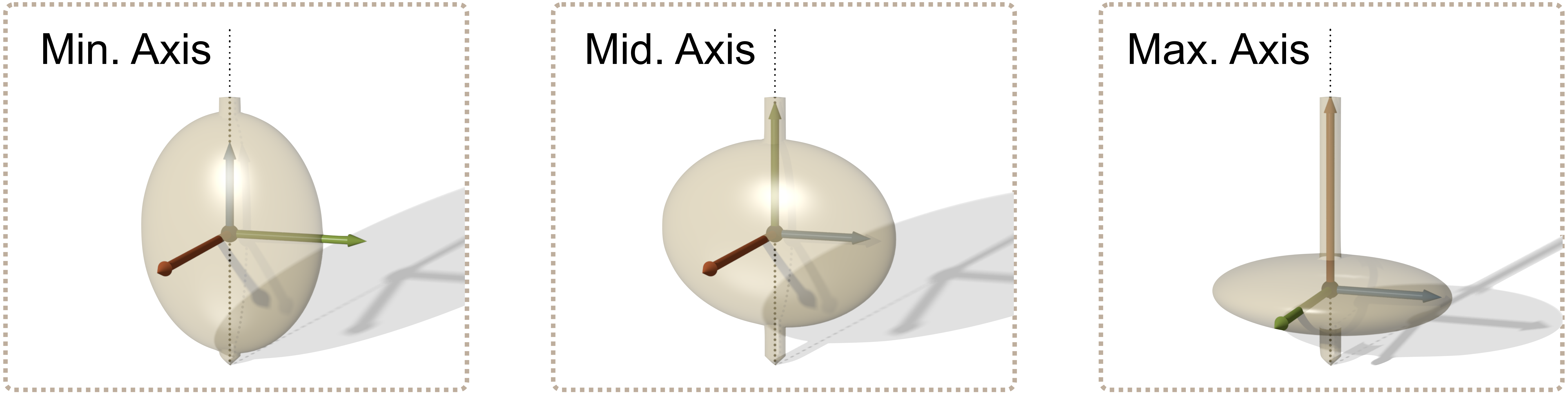
Measuring Spin Quality

- Moment of Inertia
  - spinning axis || principal axis



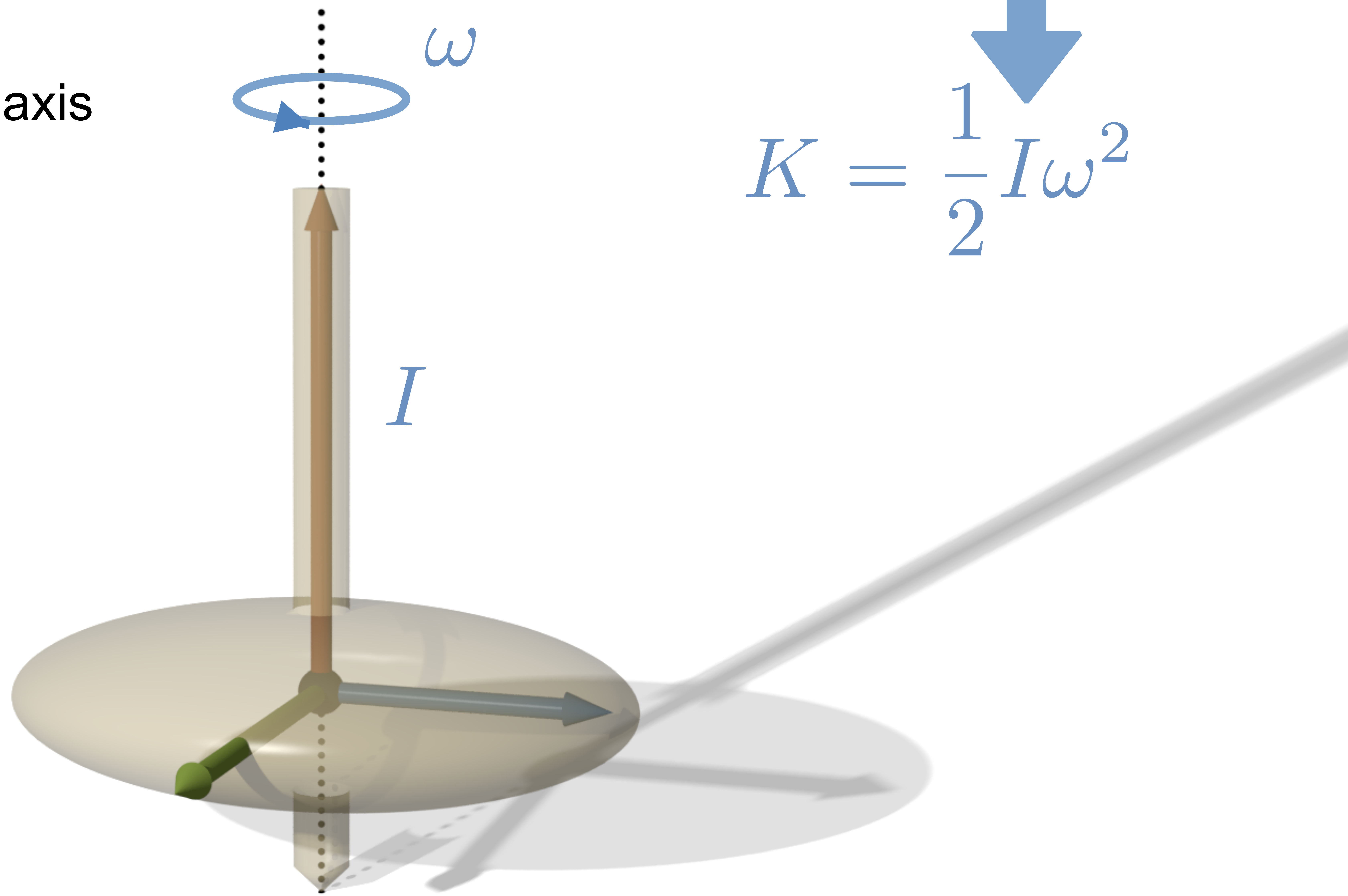
Measuring Spin Quality

- Moment of Inertia
  - spinning axis || principal axis



Measuring Spin Quality

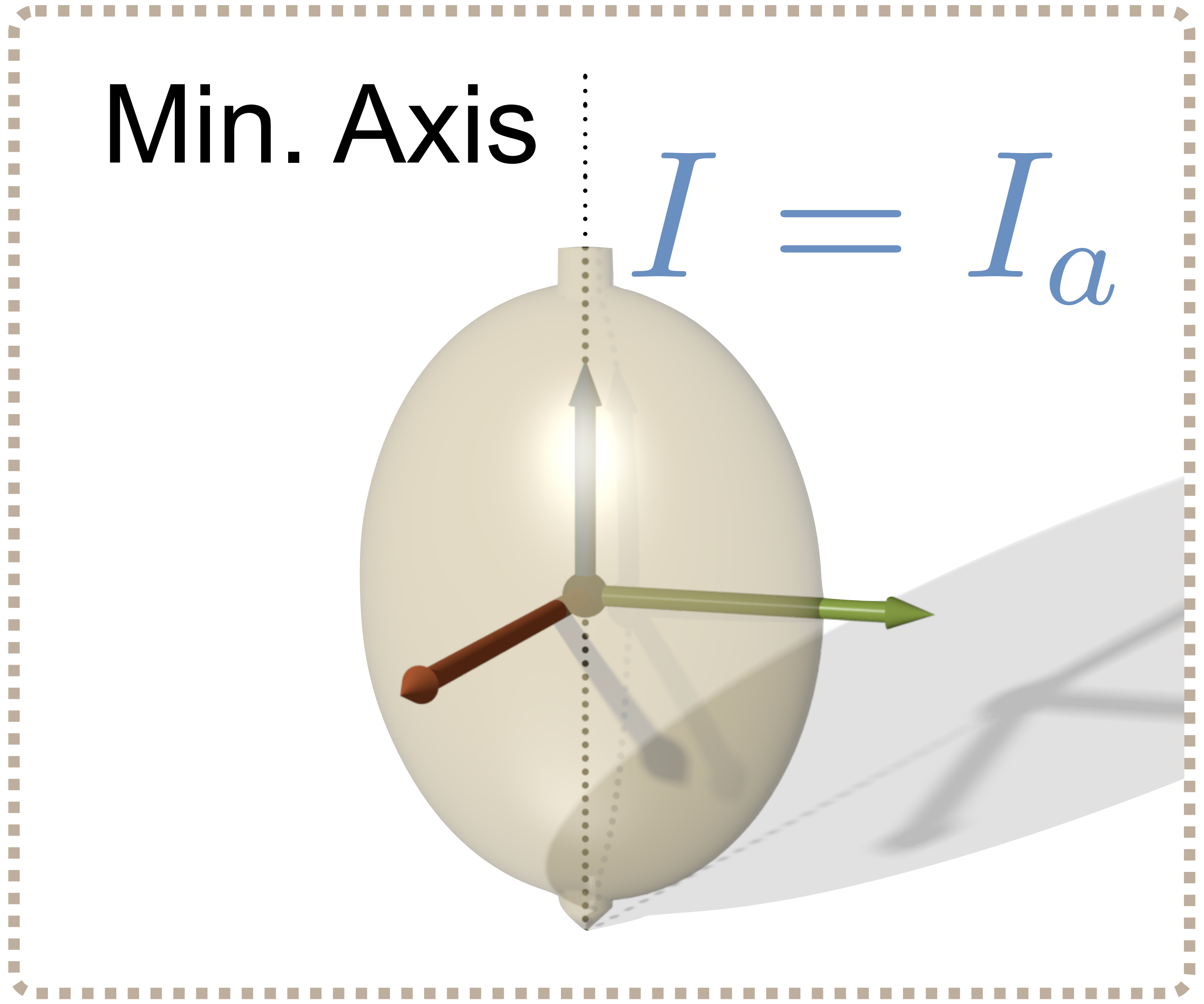
- Moment of Inertia
  - spinning axis || principal axis

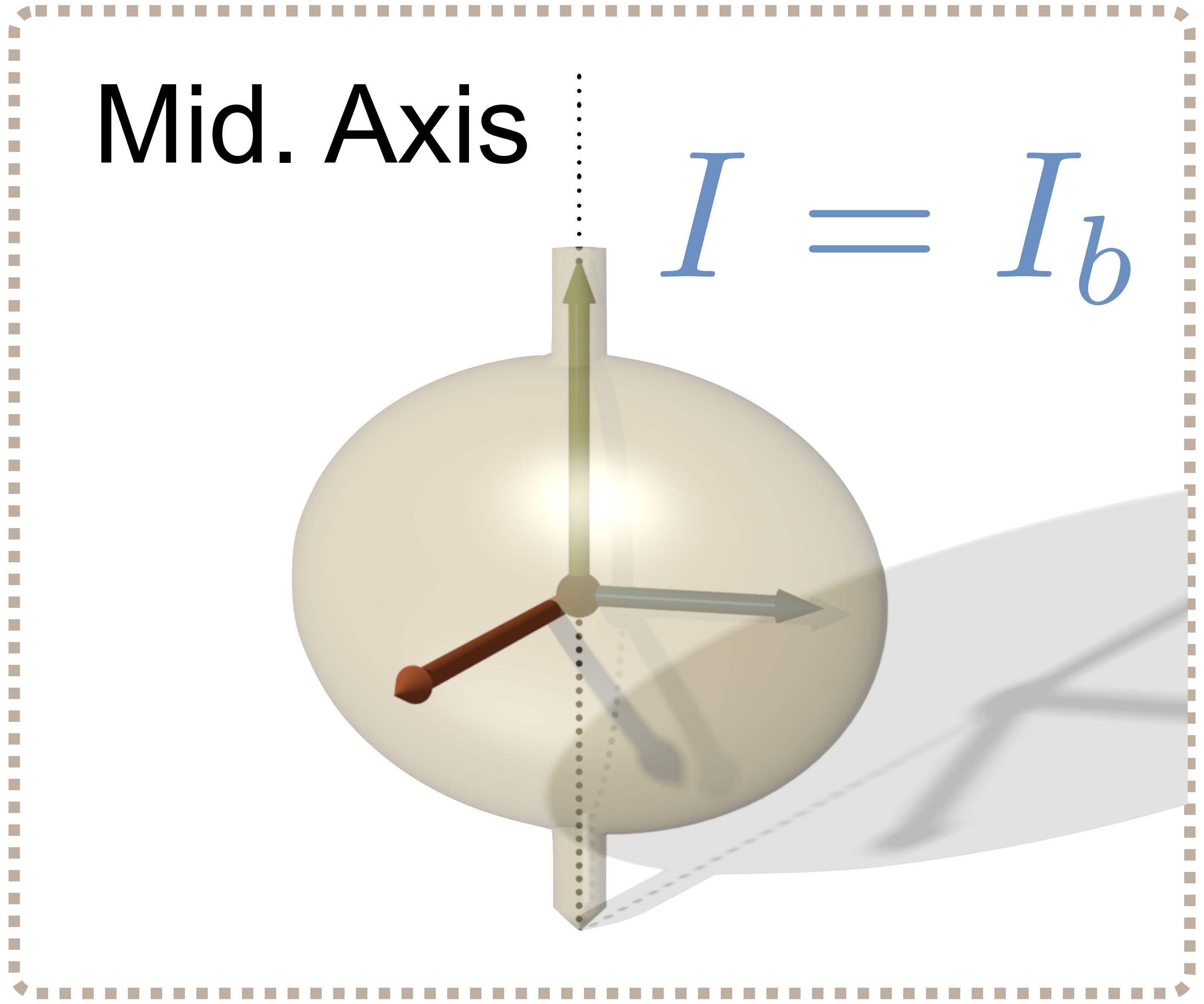


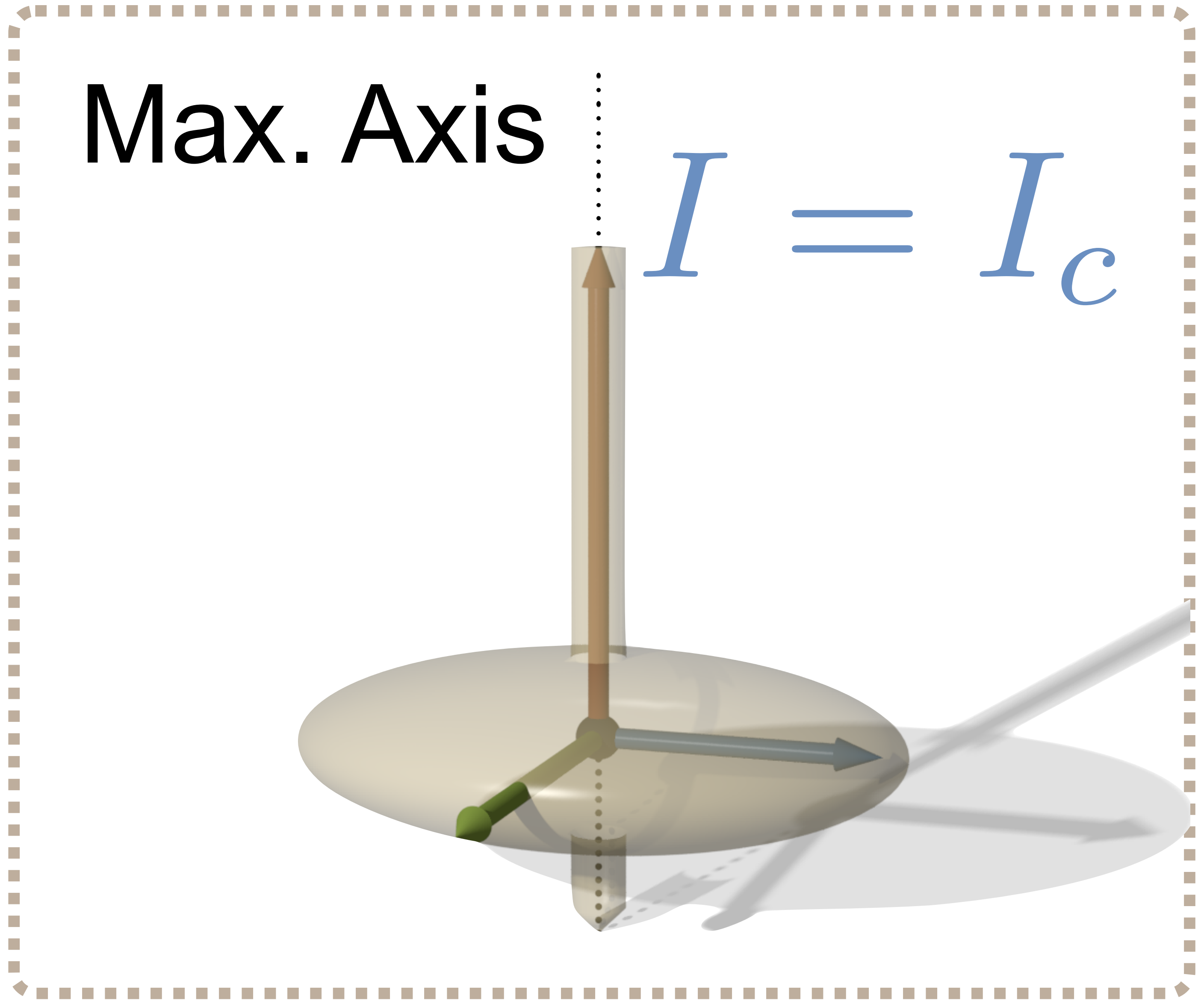
$$K = \frac{1}{2} I \omega^2$$

Measuring Spin Quality

- Moment of Inertia
  - spinning axis || principal axis



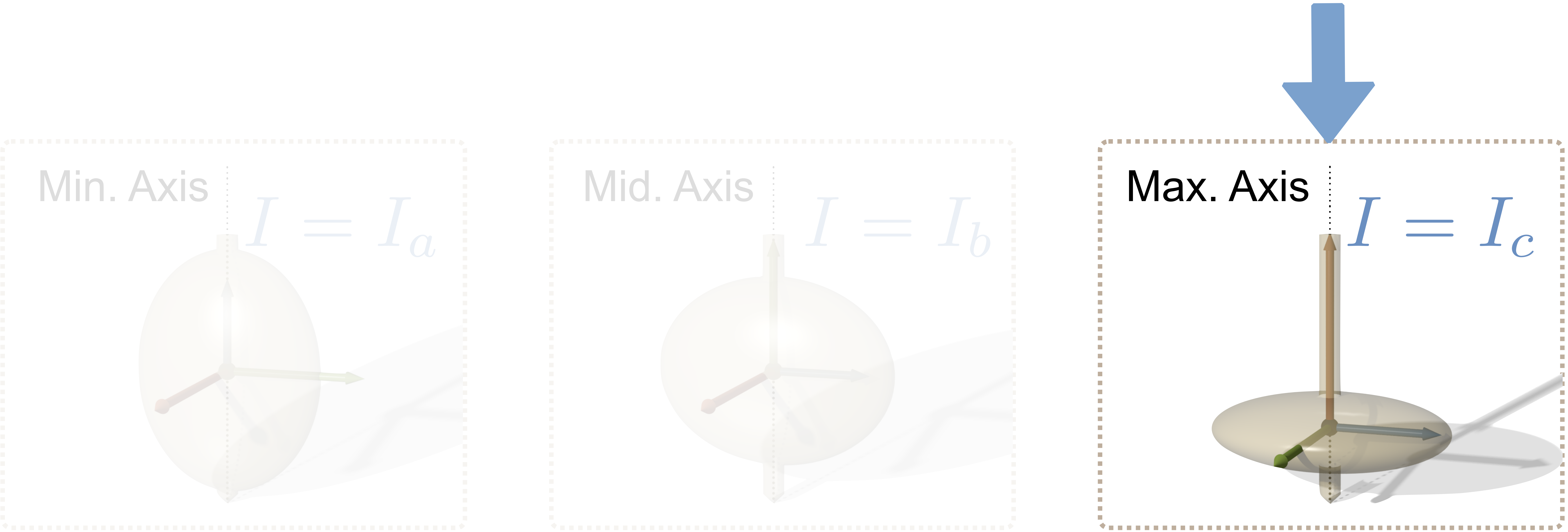




Measuring Spin Quality

- Moment of Inertia
  - spinning axis || principal axis

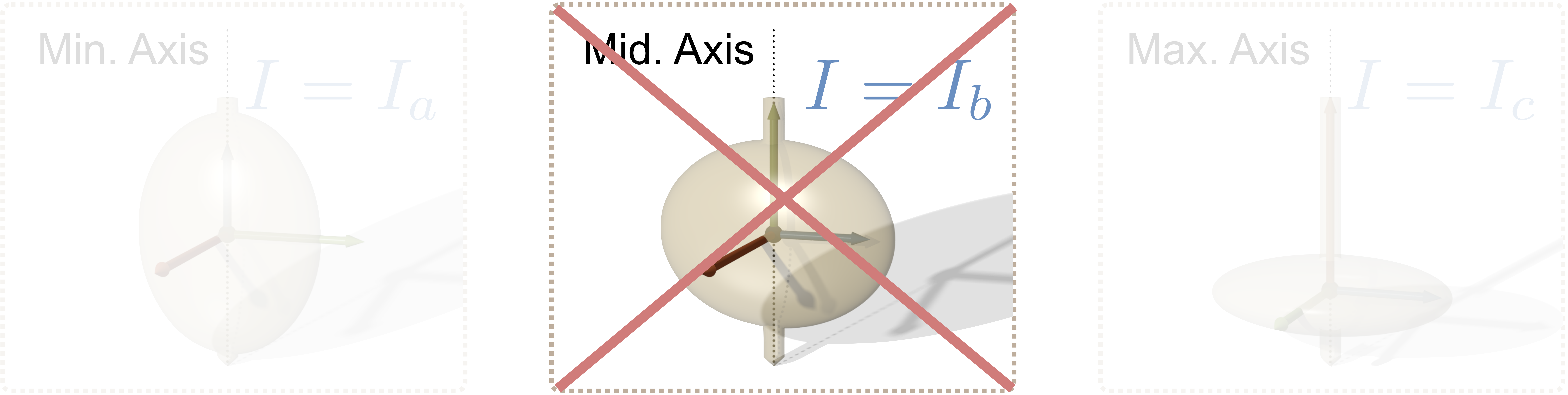
$$K = \frac{1}{2} I \omega^2$$



Measuring Spin Quality

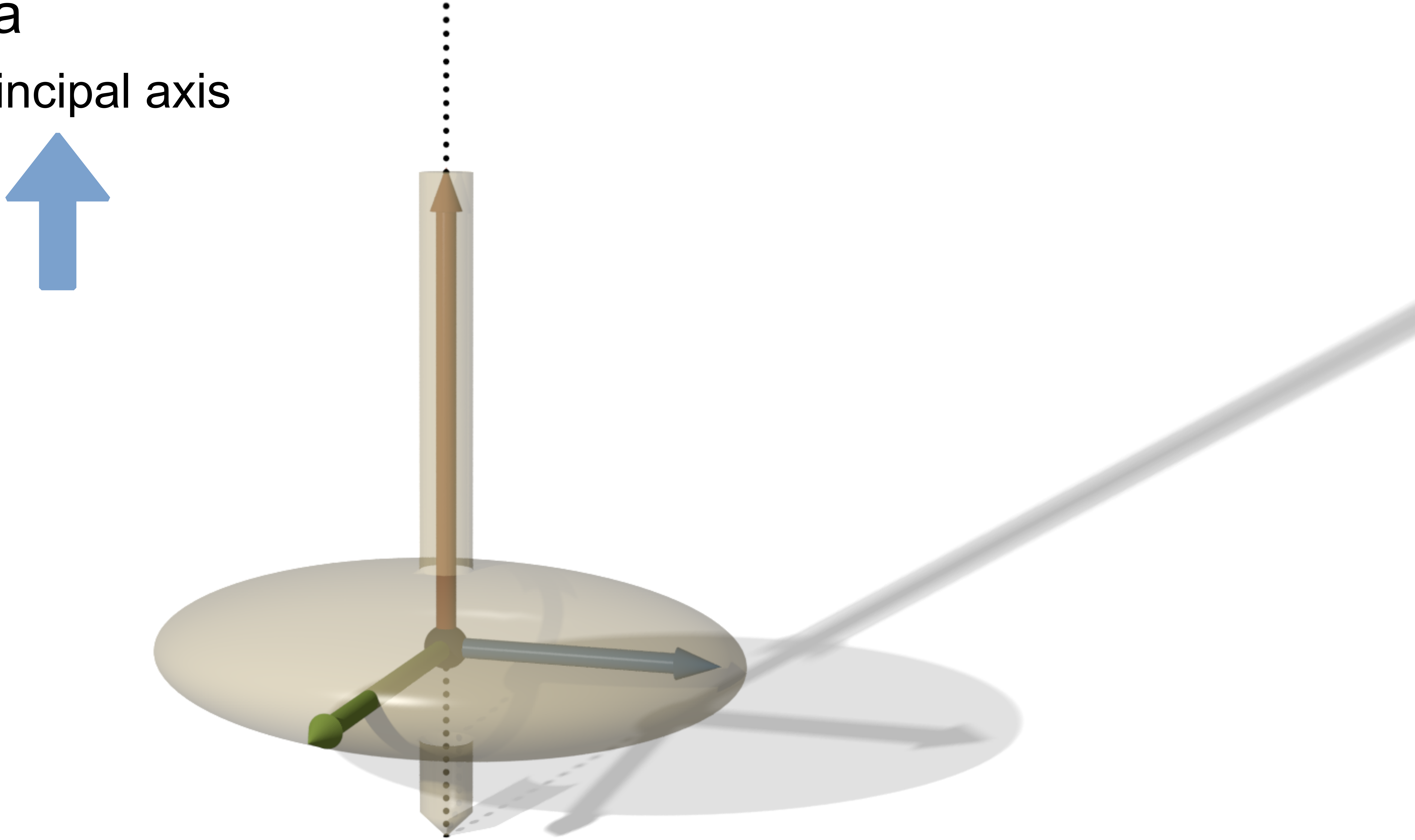
- Moment of Inertia
  - spinning axis || principal axis

$$K = \frac{1}{2} I \omega^2$$



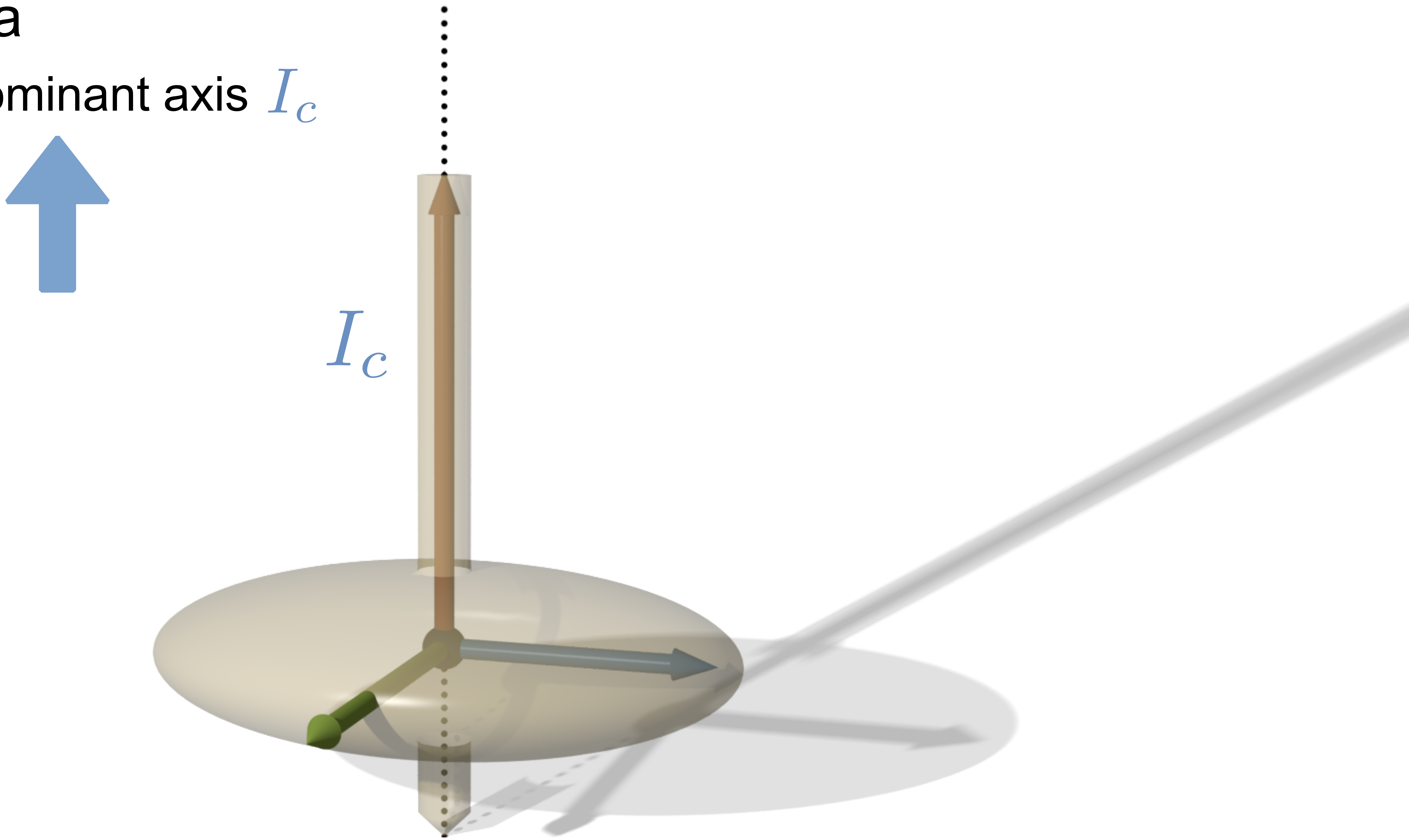
Measuring Spin Quality

- Moment of Inertia
  - spinning axis || principal axis



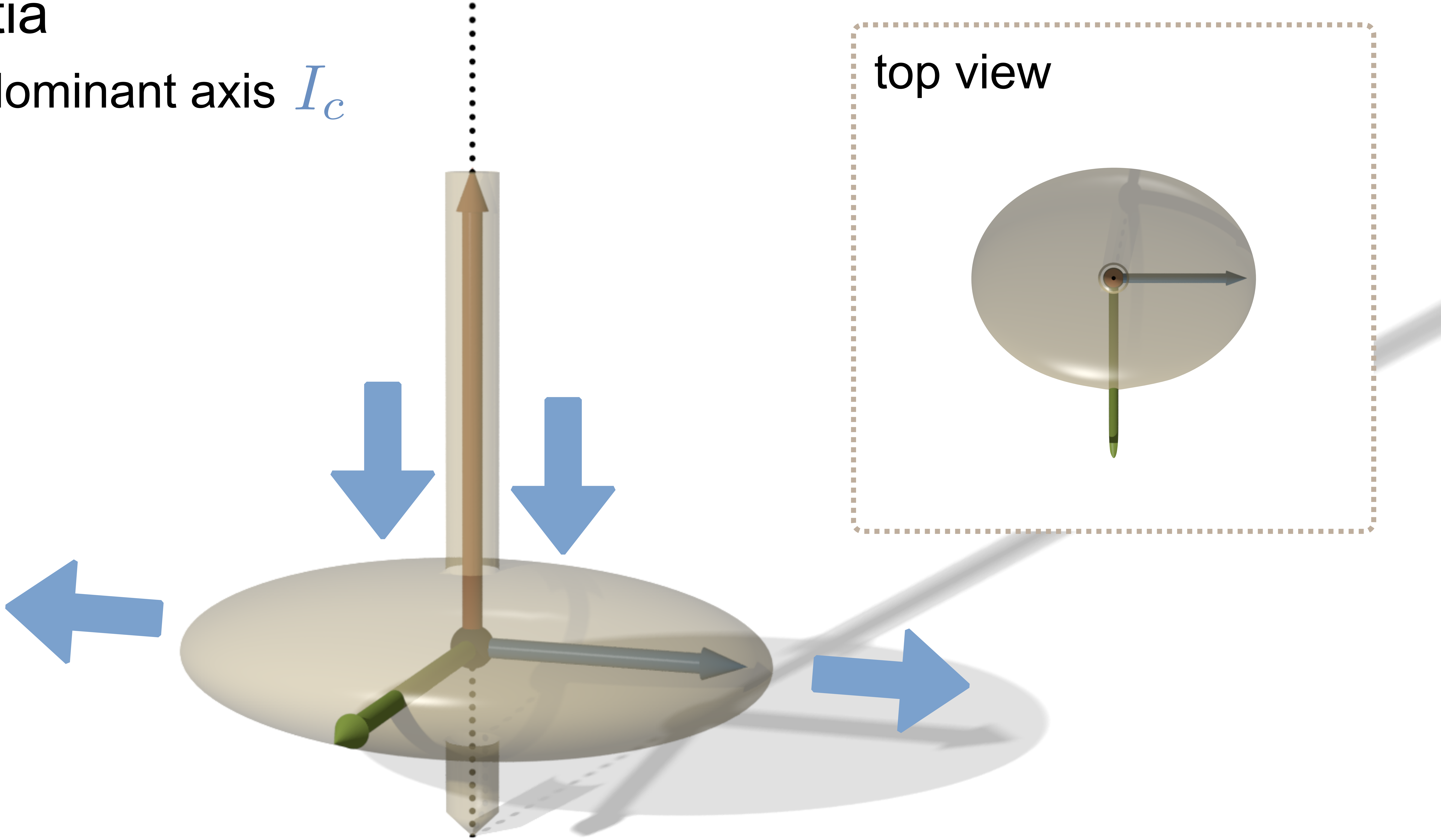
Measuring Spin Quality

- Moment of Inertia
  - spinning axis || dominant axis I_c



Measuring Spin Quality

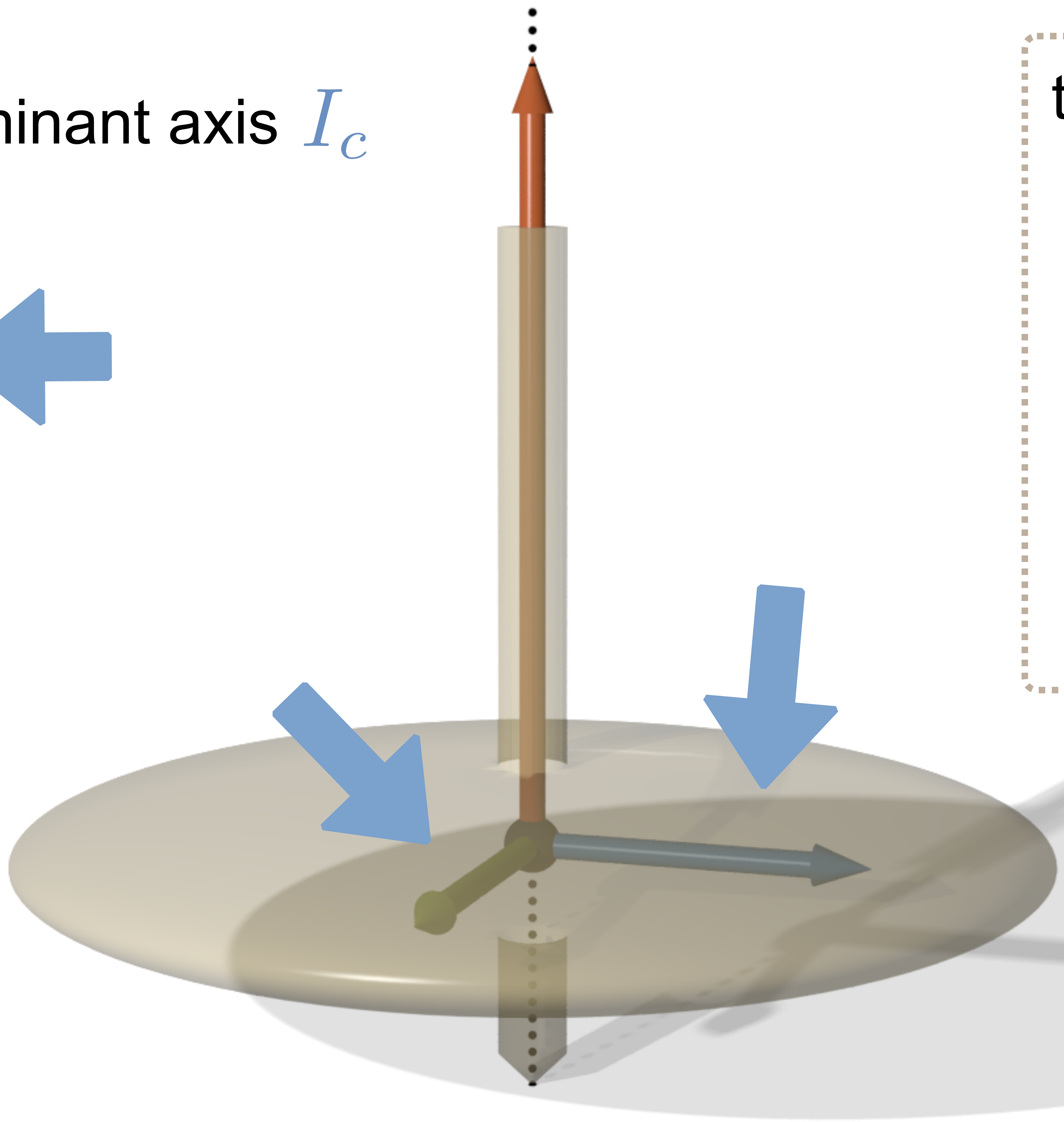
- Moment of Inertia
  - spinning axis || dominant axis I_c



Measuring Spin Quality

- Moment of Inertia
  - spinning axis || dominant axis I_c
  - minimize ratios

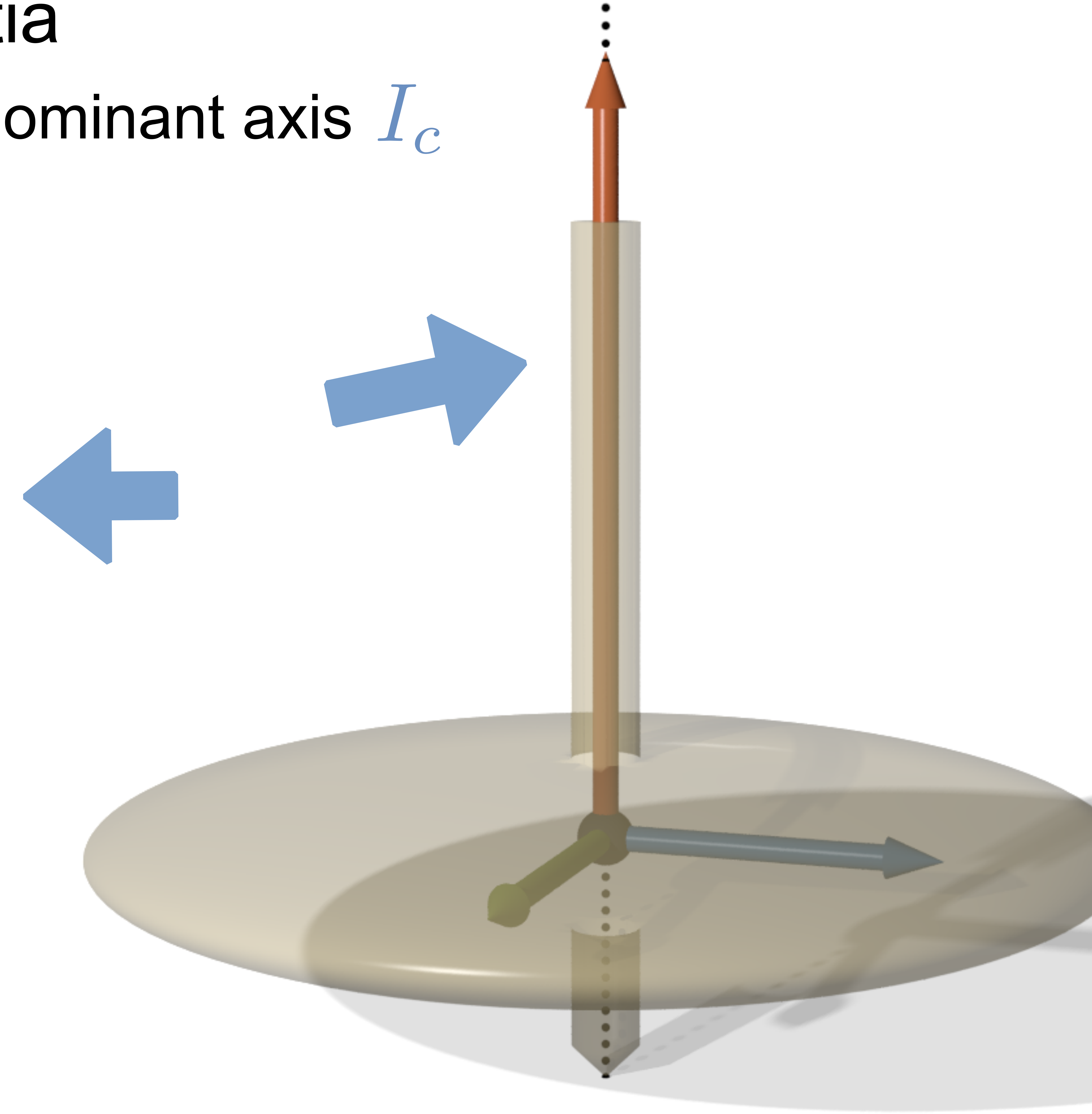
$$\frac{I_a}{I_c} \quad \frac{I_b}{I_c}$$



Measuring Spin Quality

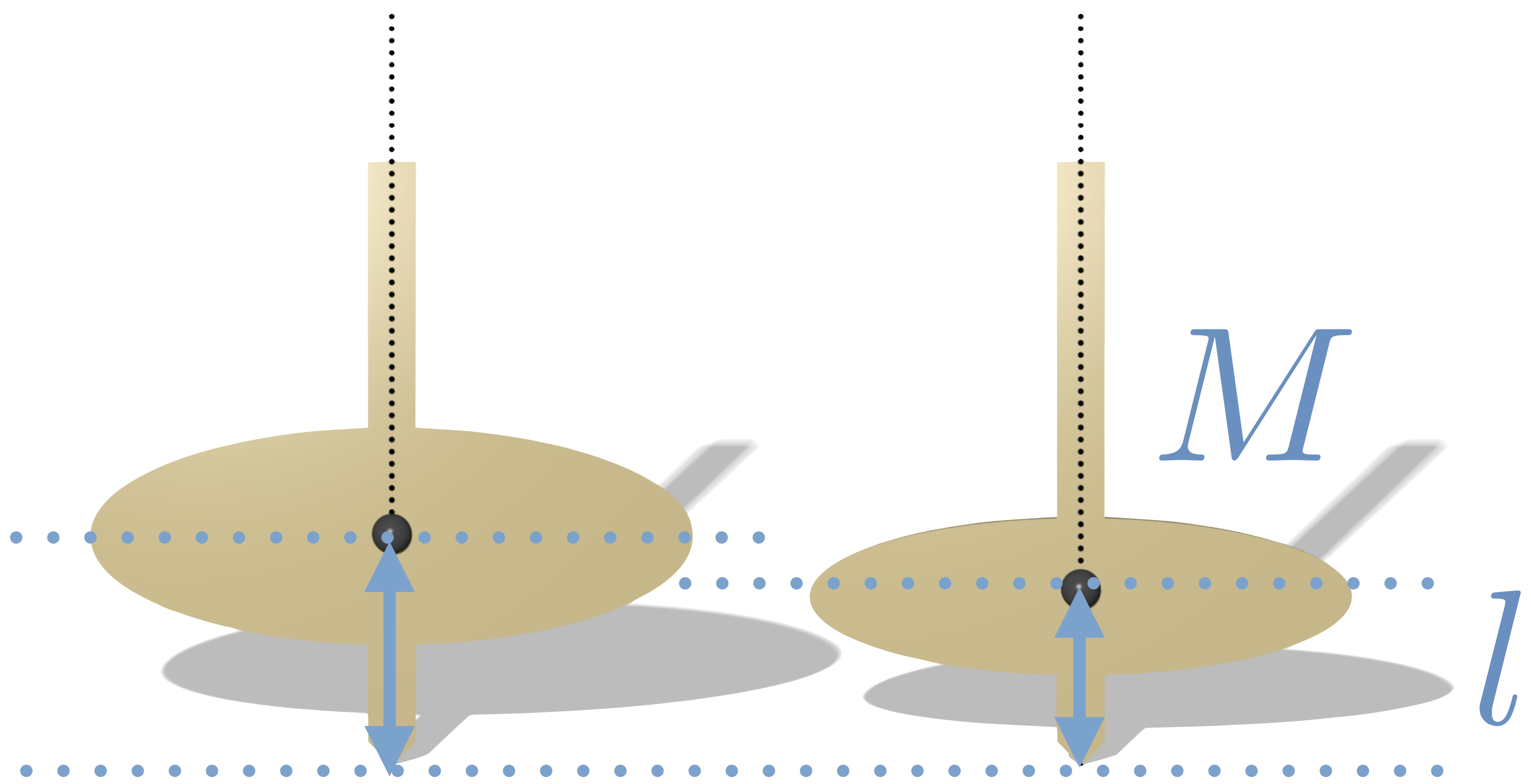
- Moment of Inertia
  - spinning axis || dominant axis I_c
  - minimize ratios

$$\frac{I_a}{I_c} \quad \frac{I_b}{I_c}$$

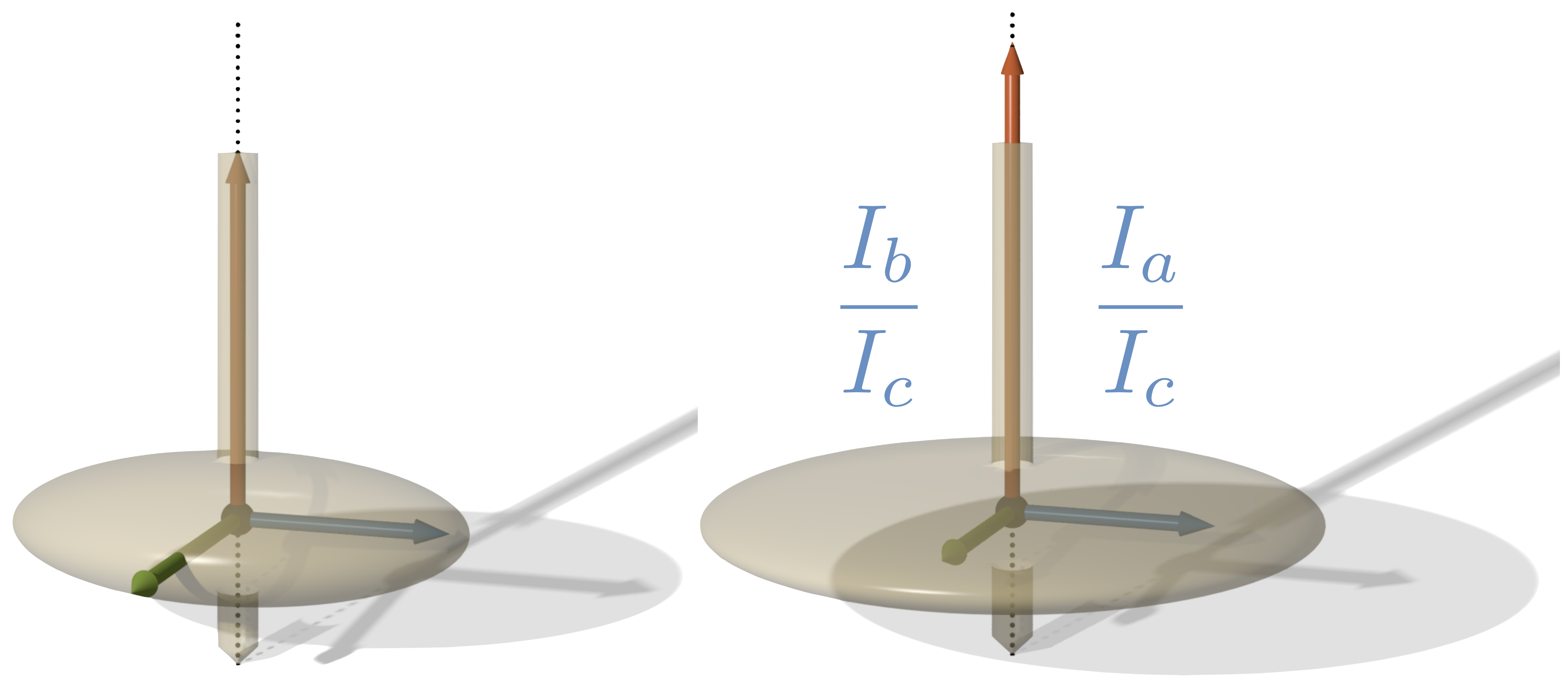


Measuring Spin Quality

Center of Mass $\mathbf{c}$



Moment of Inertia $\mathbf{I}$



Measuring Spin Quality

Center of Mass $\mathbf{c}$

$$(lM)^2$$

Moment of Inertia $\mathbf{I}$

$$\left(\frac{I_a}{I_c}\right)^2 + \left(\frac{I_b}{I_c}\right)^2$$

Measuring Spin Quality

Center of Mass $\mathbf{c}$

Moment of Inertia $\mathbf{I}$

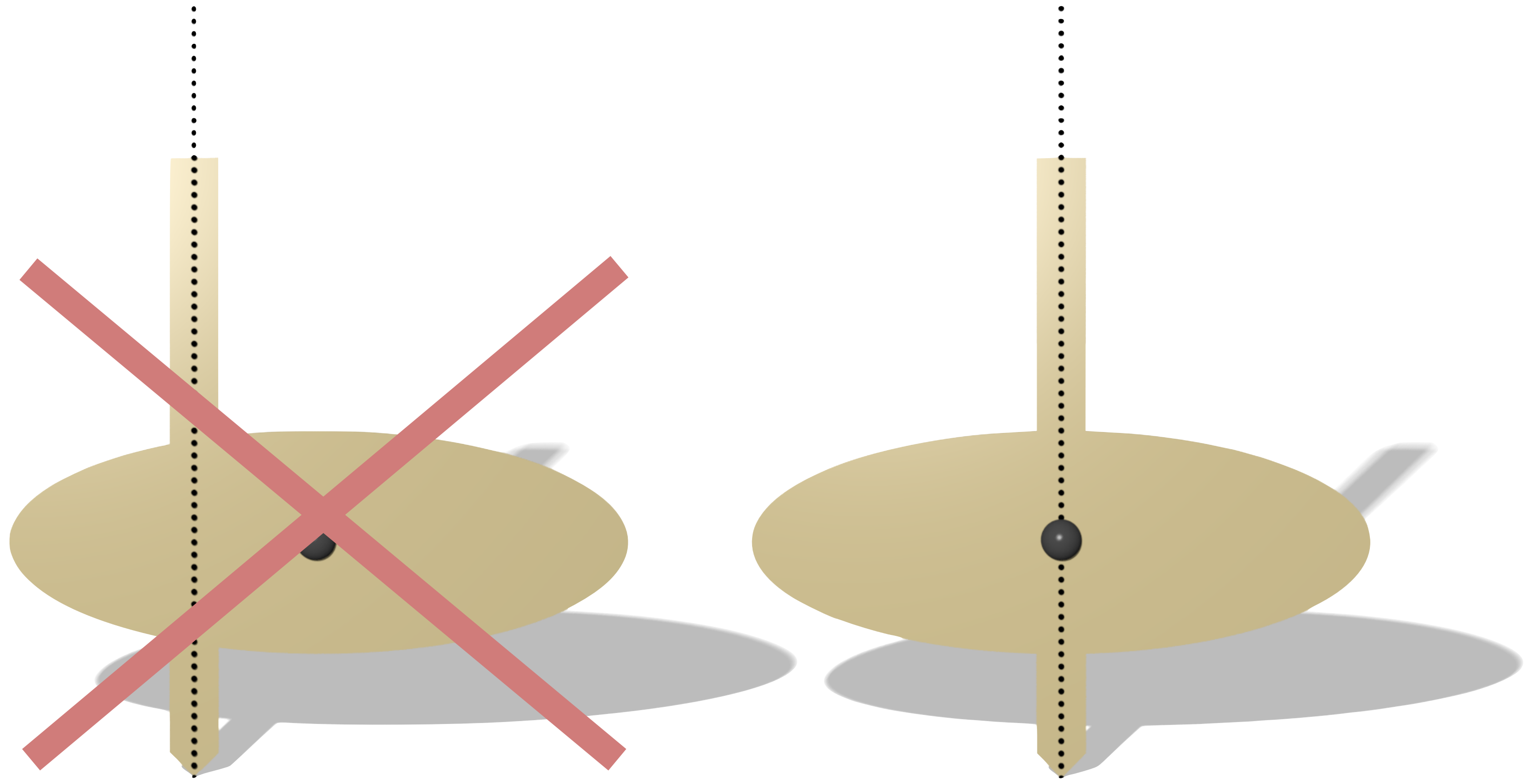
$$\text{minimize} \quad \gamma_{\mathbf{c}} \left(lM\right)^2 \quad + \quad \gamma_{\mathbf{I}} \left[\left(\frac{I_a}{I_c} \right)^2 + \left(\frac{I_b}{I_c} \right)^2 \right]$$

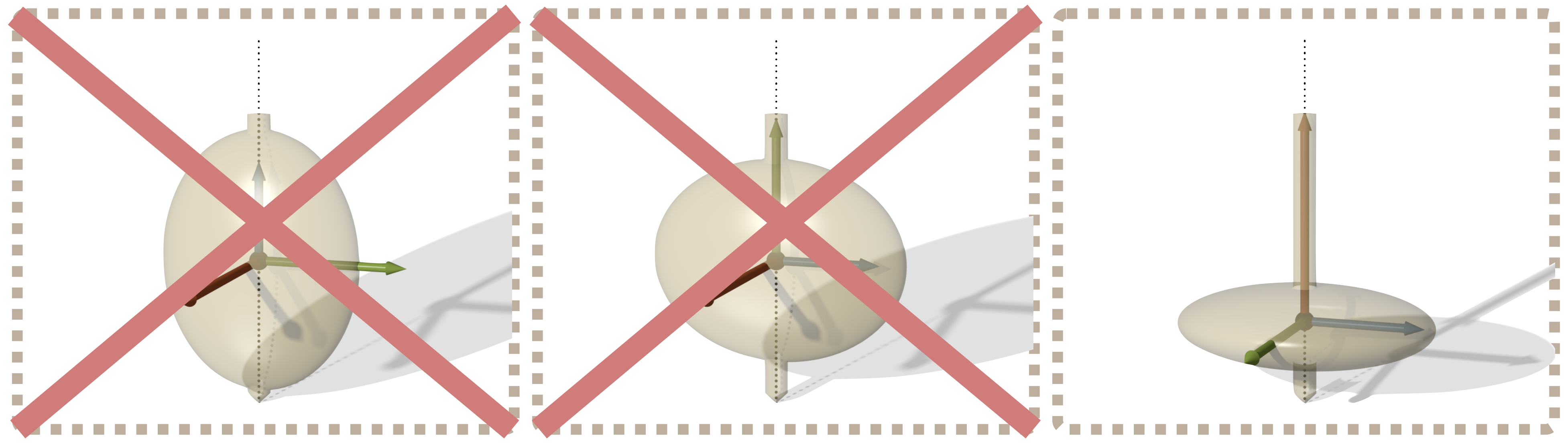
Measuring Spin Quality

Center of Mass $\mathbf{c}$

Moment of Inertia $\mathbf{I}$

$$\text{minimize} \quad \gamma_{\mathbf{c}} \left(lM\right)^2 + \gamma_{\mathbf{I}} \left[\left(\frac{I_a}{I_c}\right)^2 + \left(\frac{I_b}{I_c}\right)^2 \right]$$





Top Energy

Center of Mass $\mathbf{c}$ Moment of Inertia $\mathbf{I}$

$$\text{minimize} \quad \gamma_{\mathbf{c}} \left(l M \right)^2 \quad + \quad \gamma_{\mathbf{I}} \left[\quad \left(\frac{I_a^2 + I_b^2}{I_c^2} \right) \quad \right]$$

$$\text{subject to} \quad \mathbf{c} \in \text{spin. axis} \qquad I_c \; || \; \text{spin. axis}$$

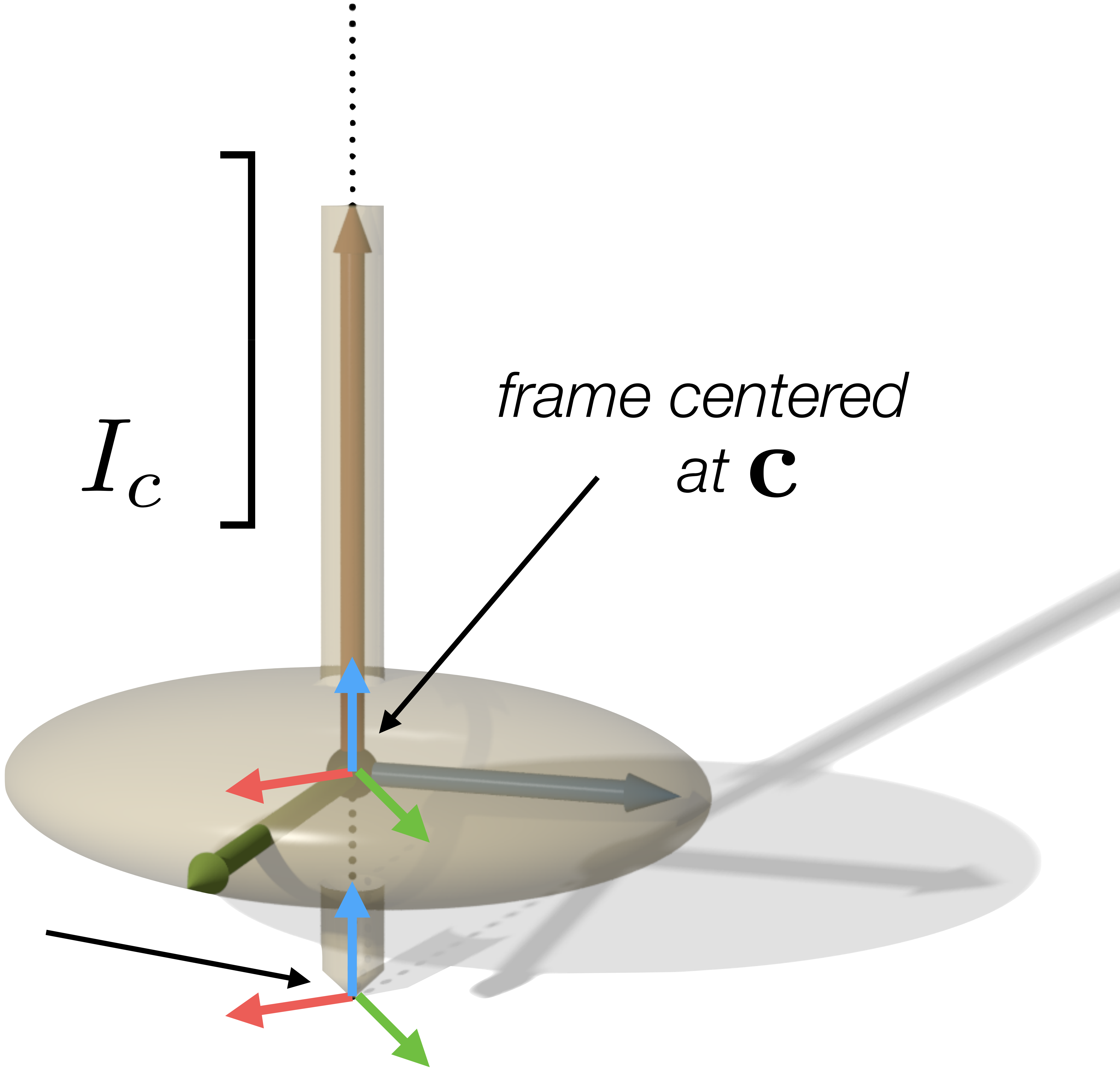
Recap: Moment of Inertia

world frame

$$\mathbf{I_c} = \begin{bmatrix} s_{y^2} + s_{z^2} & -s_{xy} & \\ -s_{xy} & s_{x^2} + s_{z^2} & \\ & & I_c \end{bmatrix}$$

$$+M\left(\mathbf{c}\mathbf{c}^T - \mathbf{c}^T\mathbf{c}\mathbf{E}_{3\times 3}\right)$$

$$\mathbf{c} = \frac{1}{M}[0, 0, s_z]^T$$



frame centered at $\mathbf{c}$

world frame

Recap: Moment of Inertia

world frame

$$\begin{bmatrix} \begin{matrix} s_{y^2} + s_{z^2} & -s_{xy} \\ -s_{xy} & s_{x^2} + s_{z^2} \end{matrix} & \\ & I_c \end{bmatrix}$$

$\bar{\mathbf{I}}$

$$I_a^2 + I_b^2 = \mathrm{tr}(\bar{\mathbf{I}}^2) = (s_{y^2} + s_{z^2})^2 + 2s_{xy}^2 + (s_{x^2} + s_{z^2})^2$$

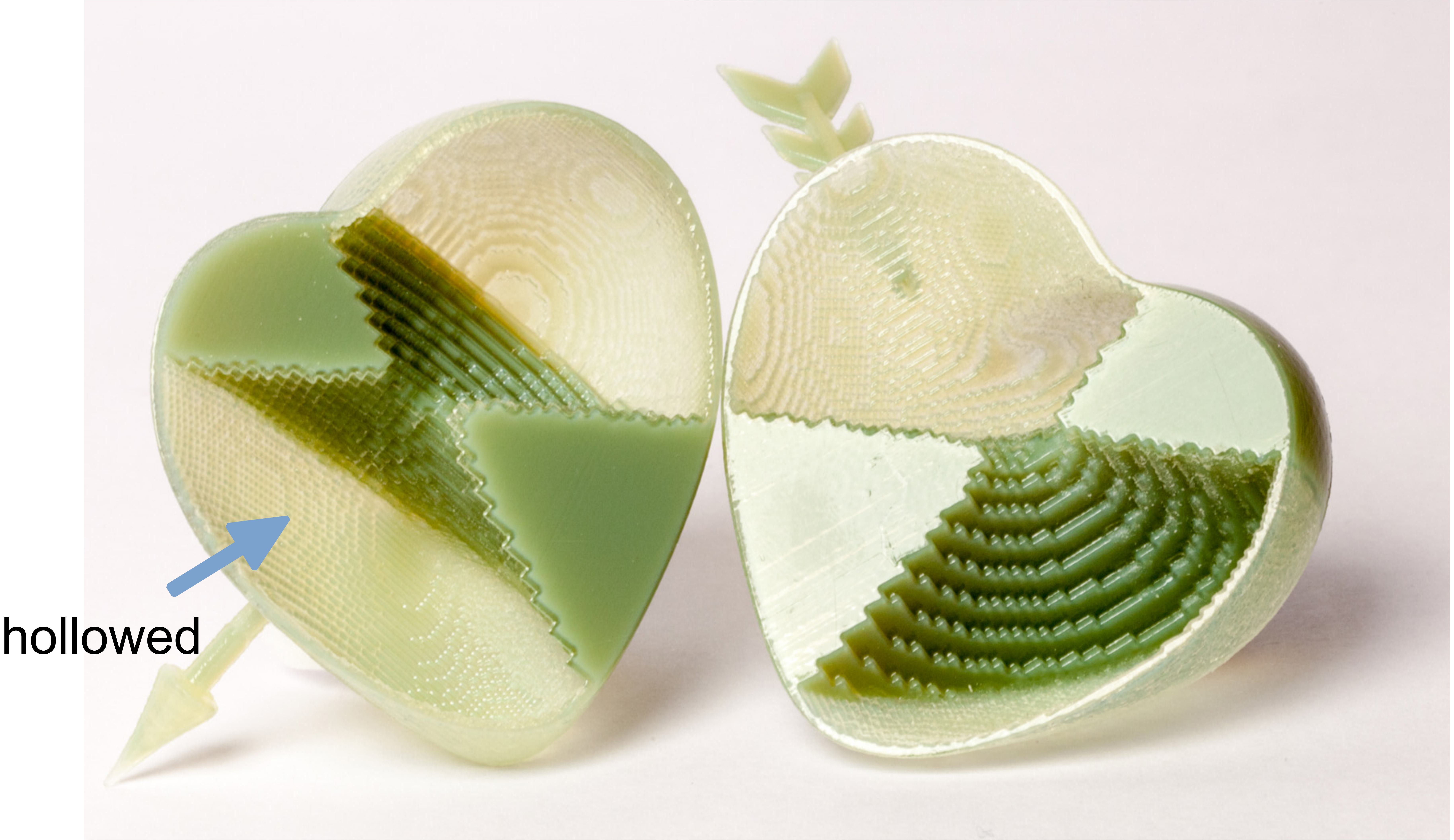
Top Energy

Center of Mass $\mathbf{c}$ 　　　 Moment of Inertia $\mathbf{I}$

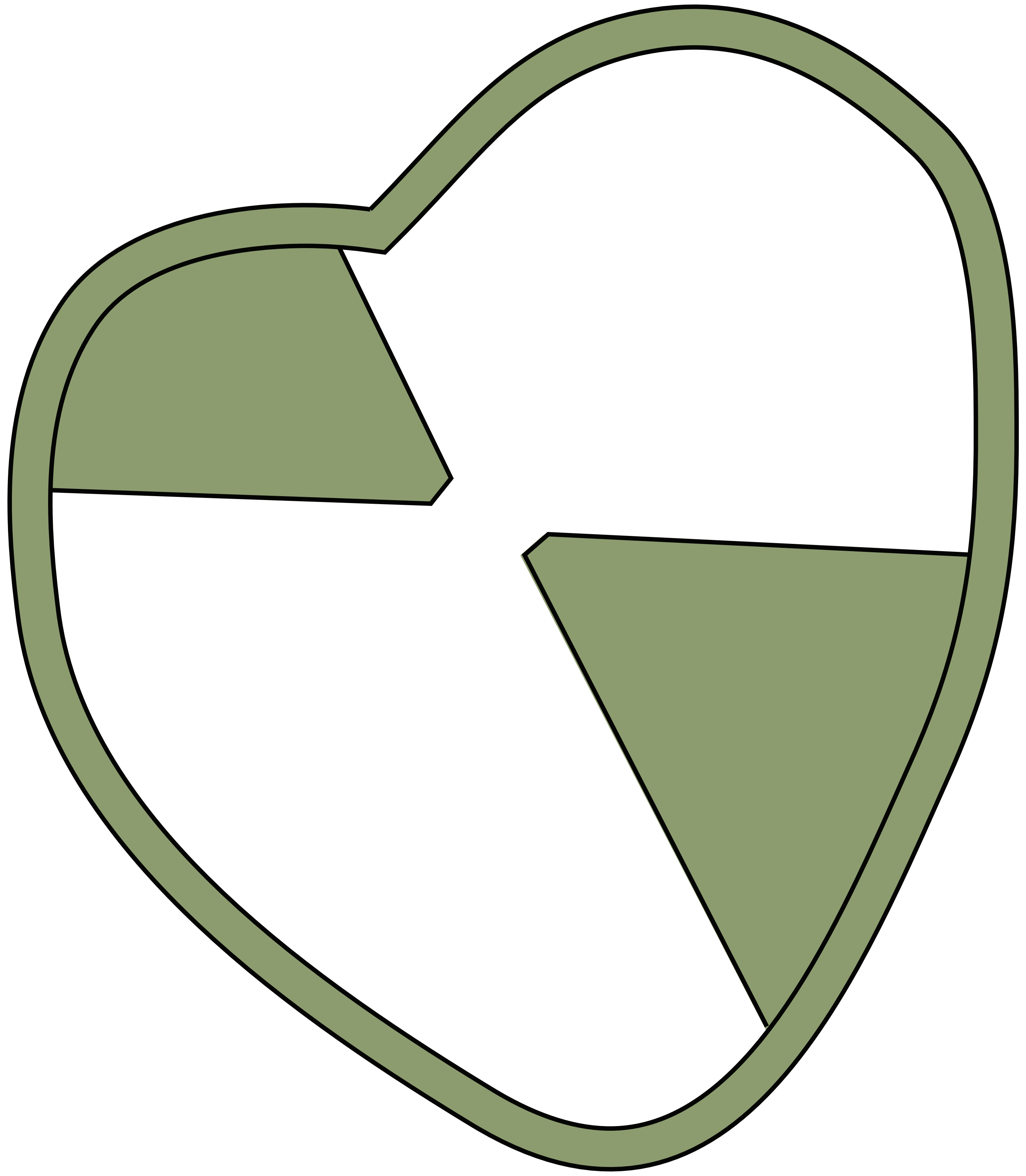
$$\text{minimize} \quad \gamma_{\mathbf{c}} \left(s_z \right)^2 + \gamma_{\mathbf{I}} \left[\frac{(s_{y^2} + s_{z^2})^2 + 2s_{xy}^2 + (s_{x^2} + s_{z^2})^2}{(s_{x^2} + s_{y^2})^2} \right]$$

$$\text{subject to} \quad s_x = 0 \quad s_y = 0 \quad s_{xz} = 0 \quad s_{yz} = 0$$

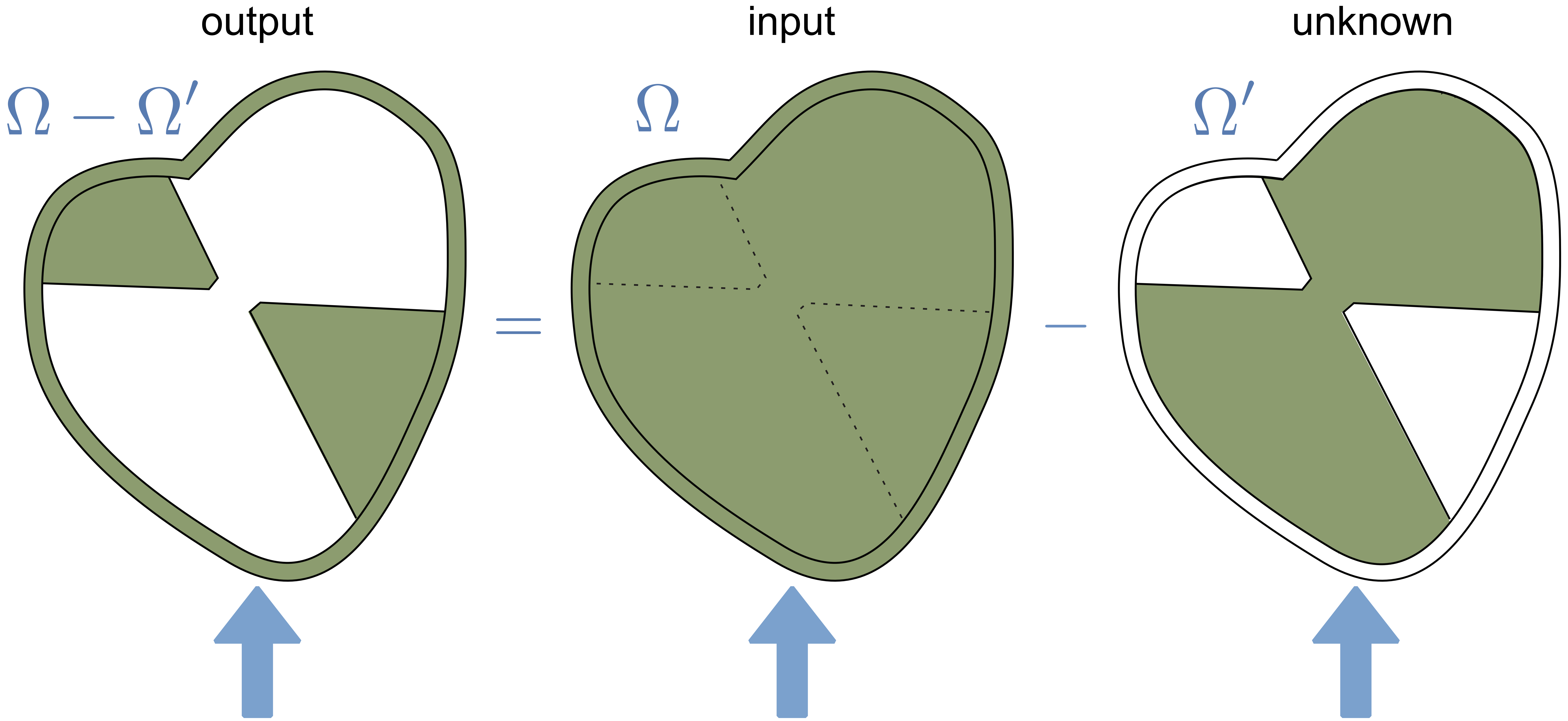
Optimizing Top Energy



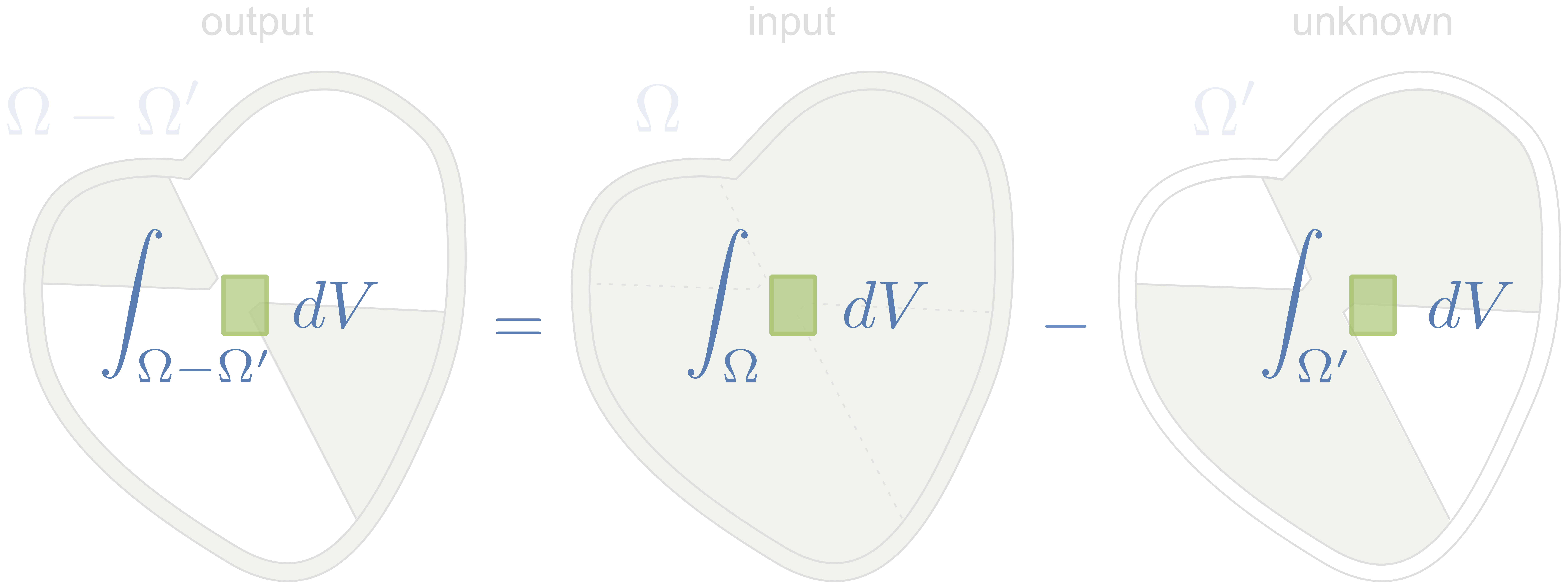
Optimizing Top Energy



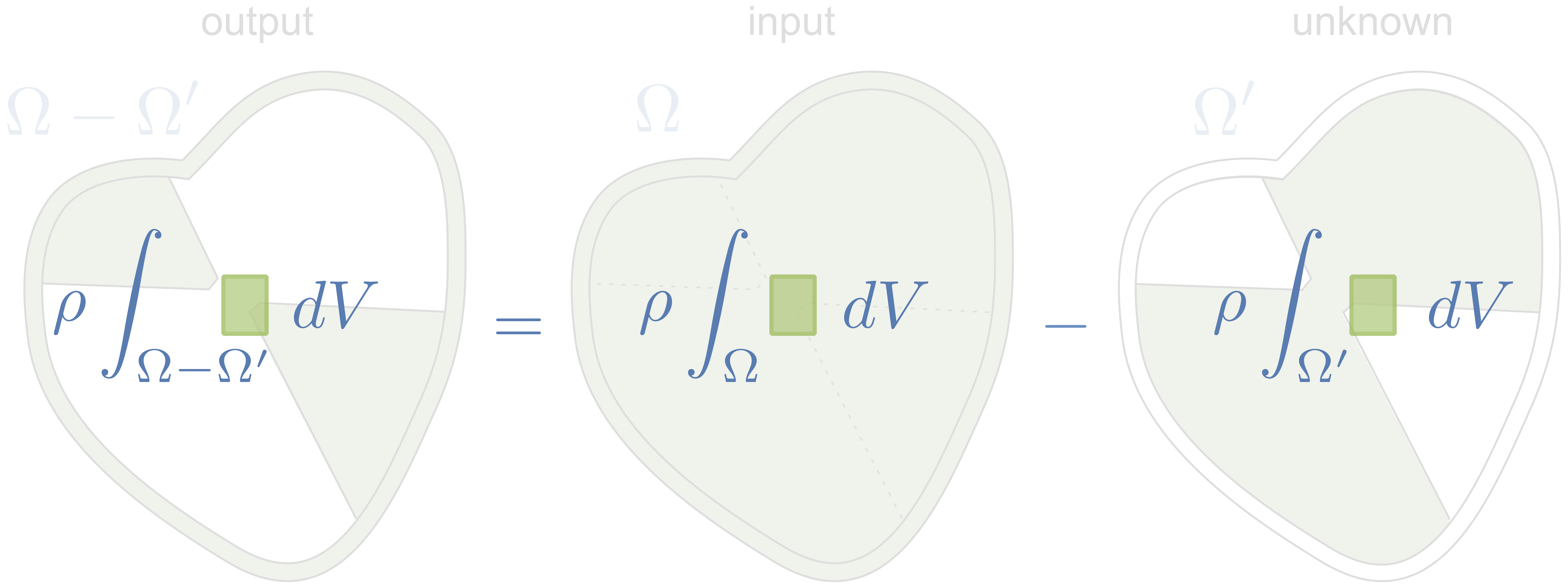
Optimizing Top Energy



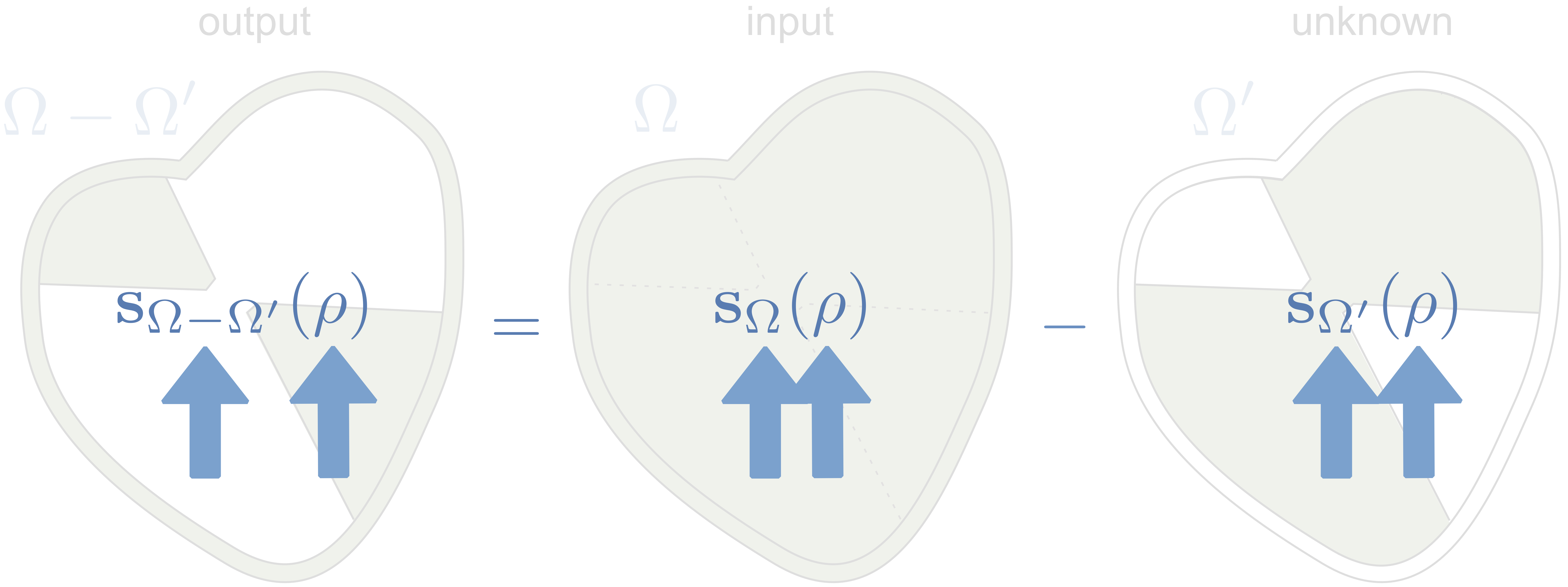
Optimizing Top Energy



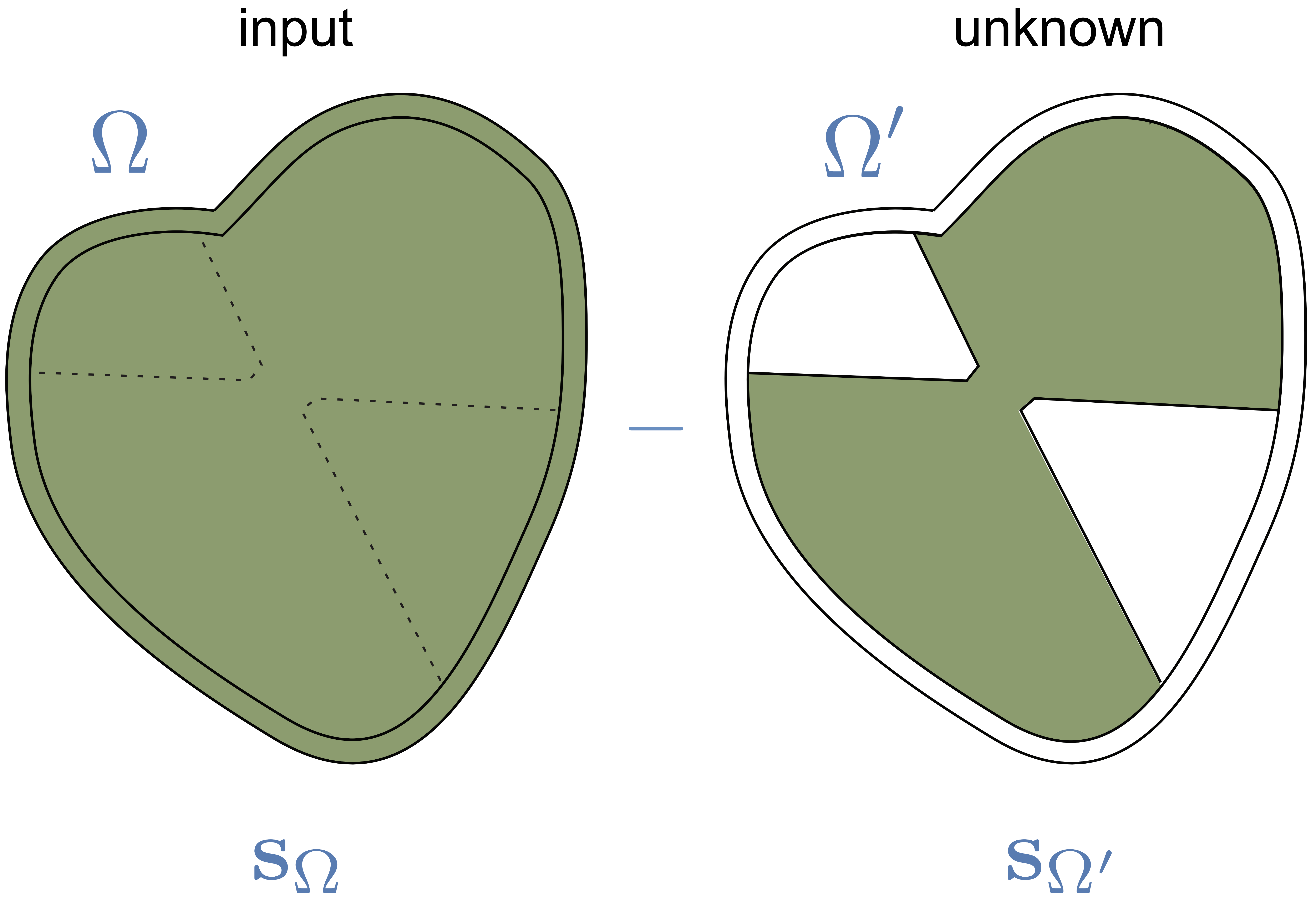
Optimizing Top Energy



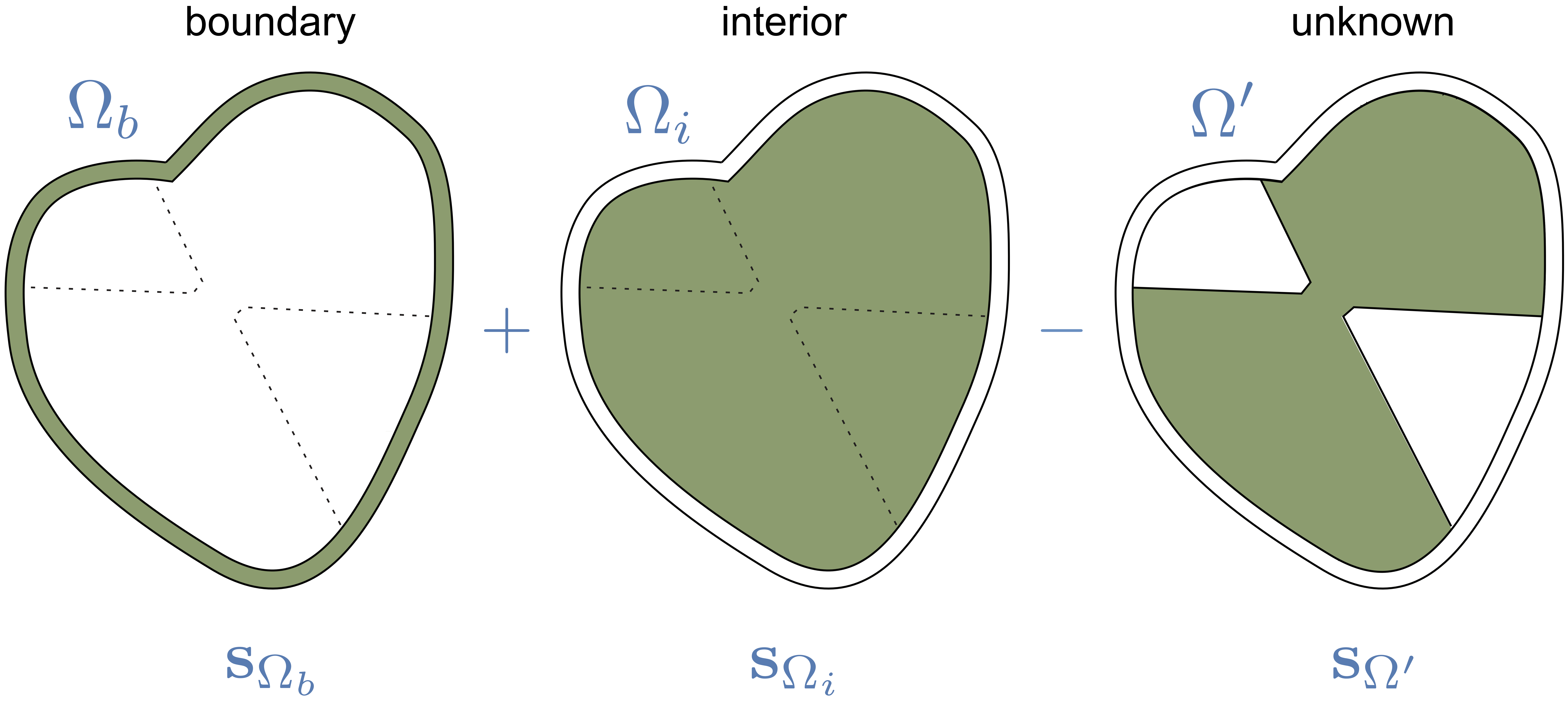
Optimizing Top Energy



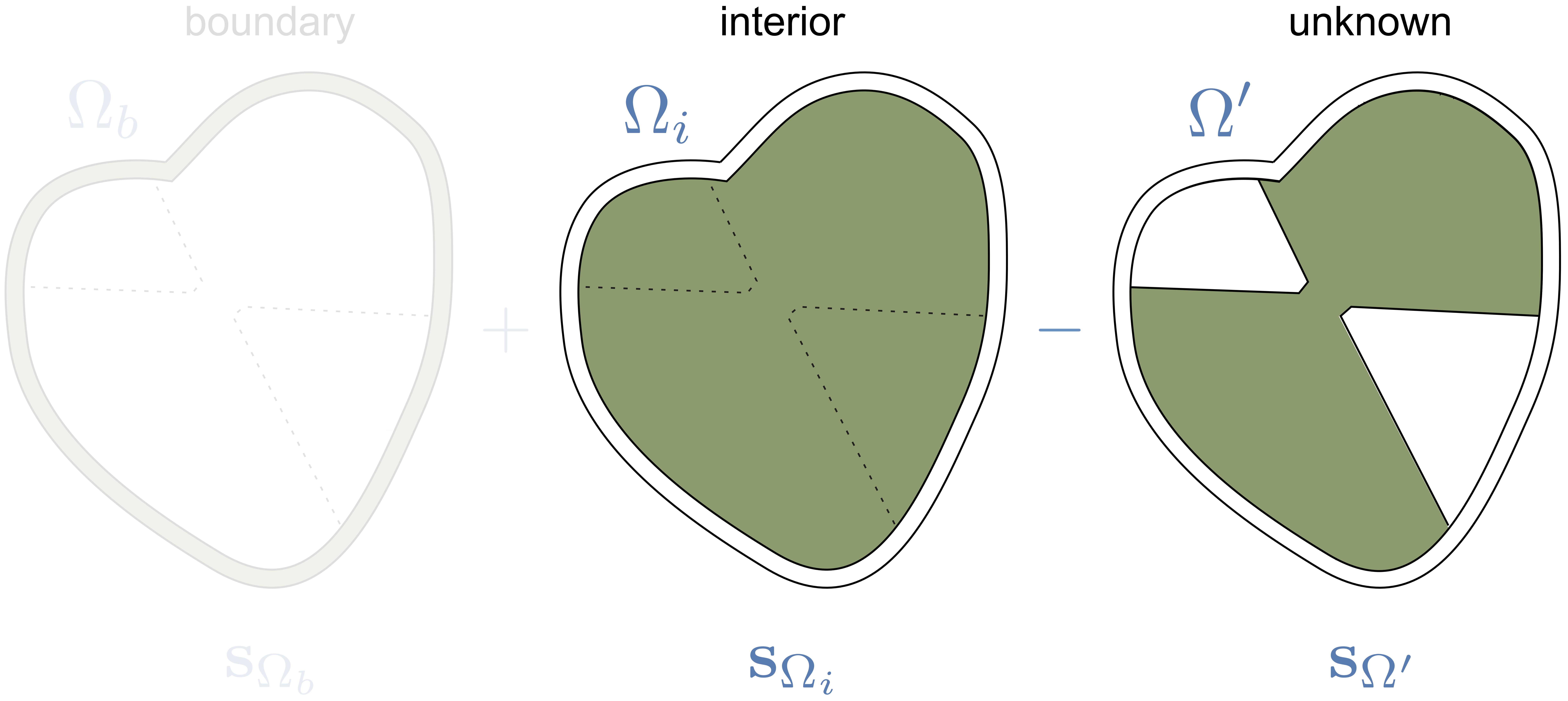
Optimizing Top Energy



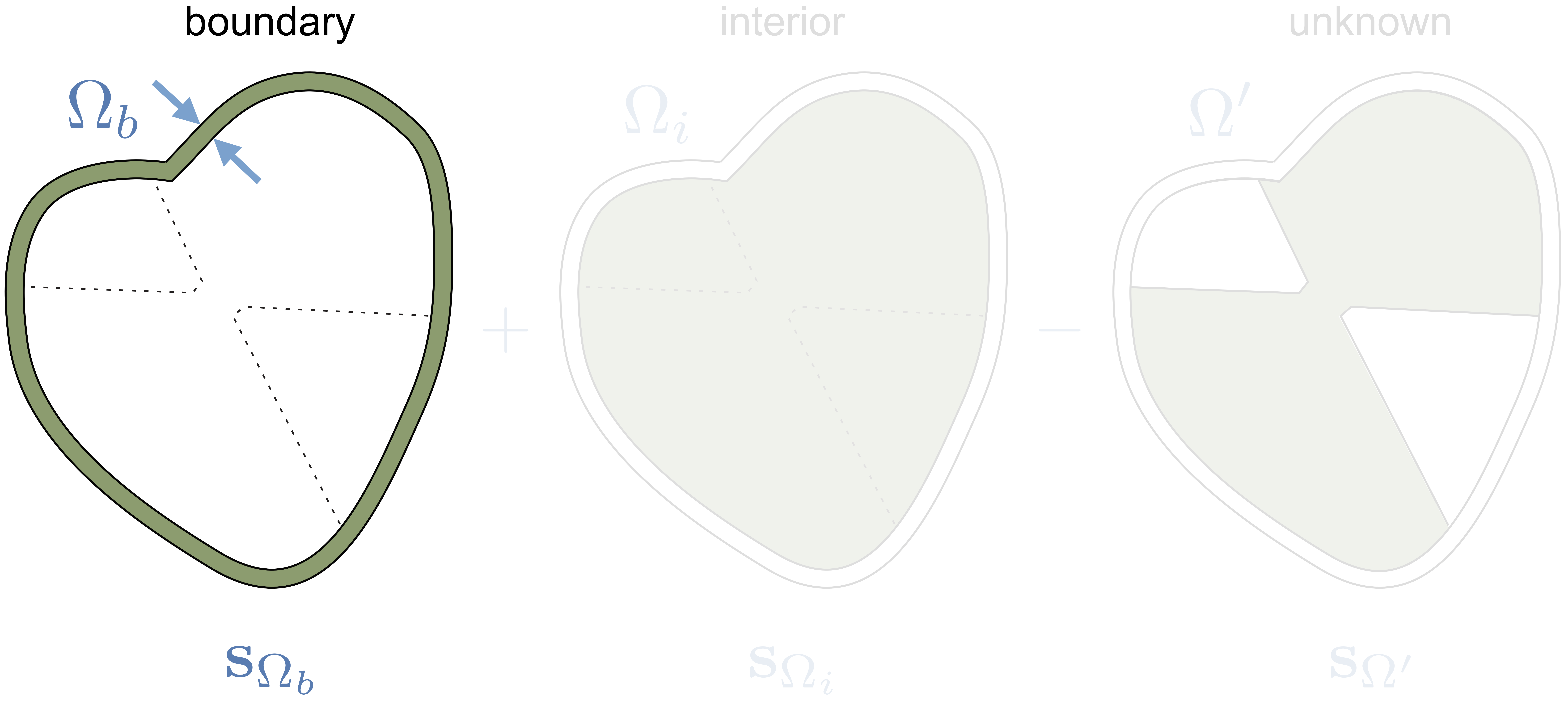
Optimizing Top Energy



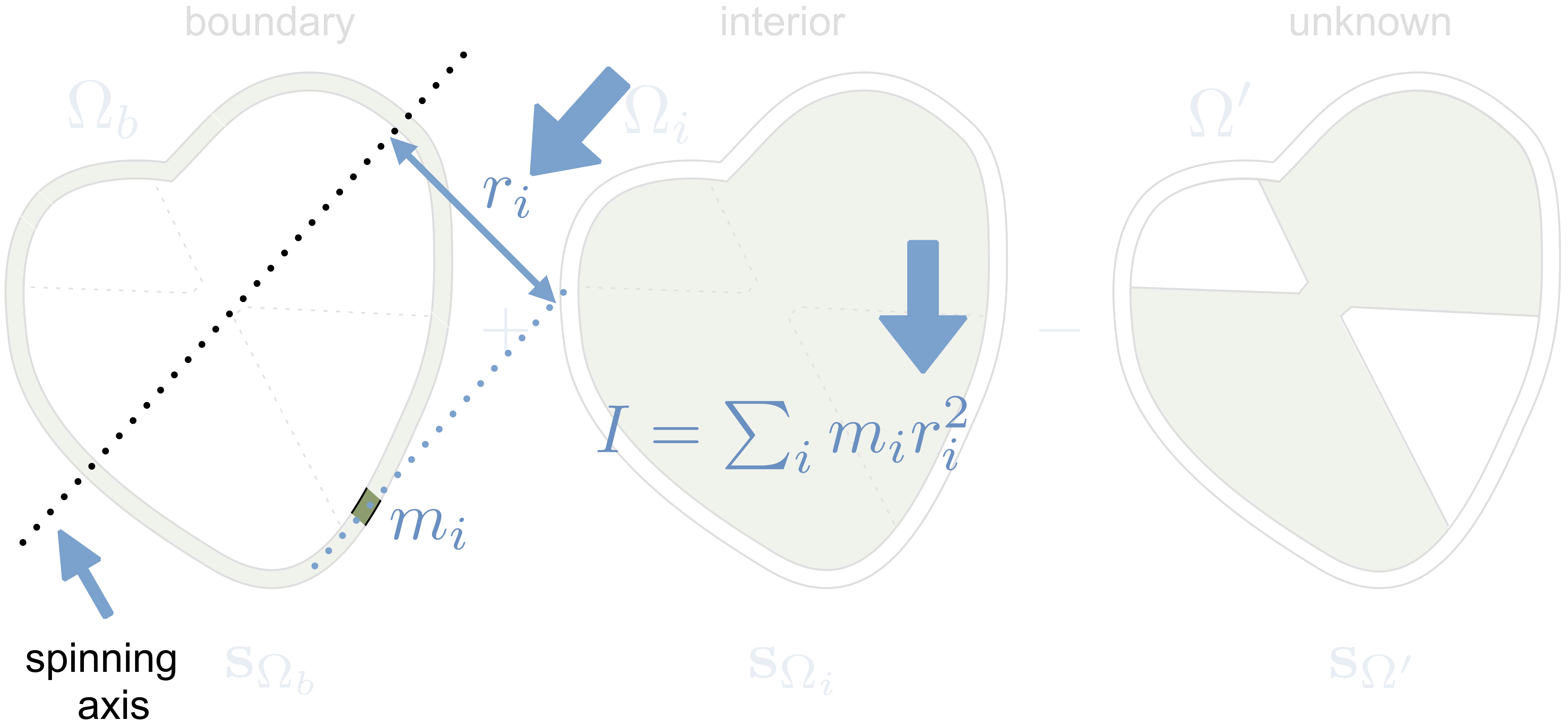
Optimizing Top Energy



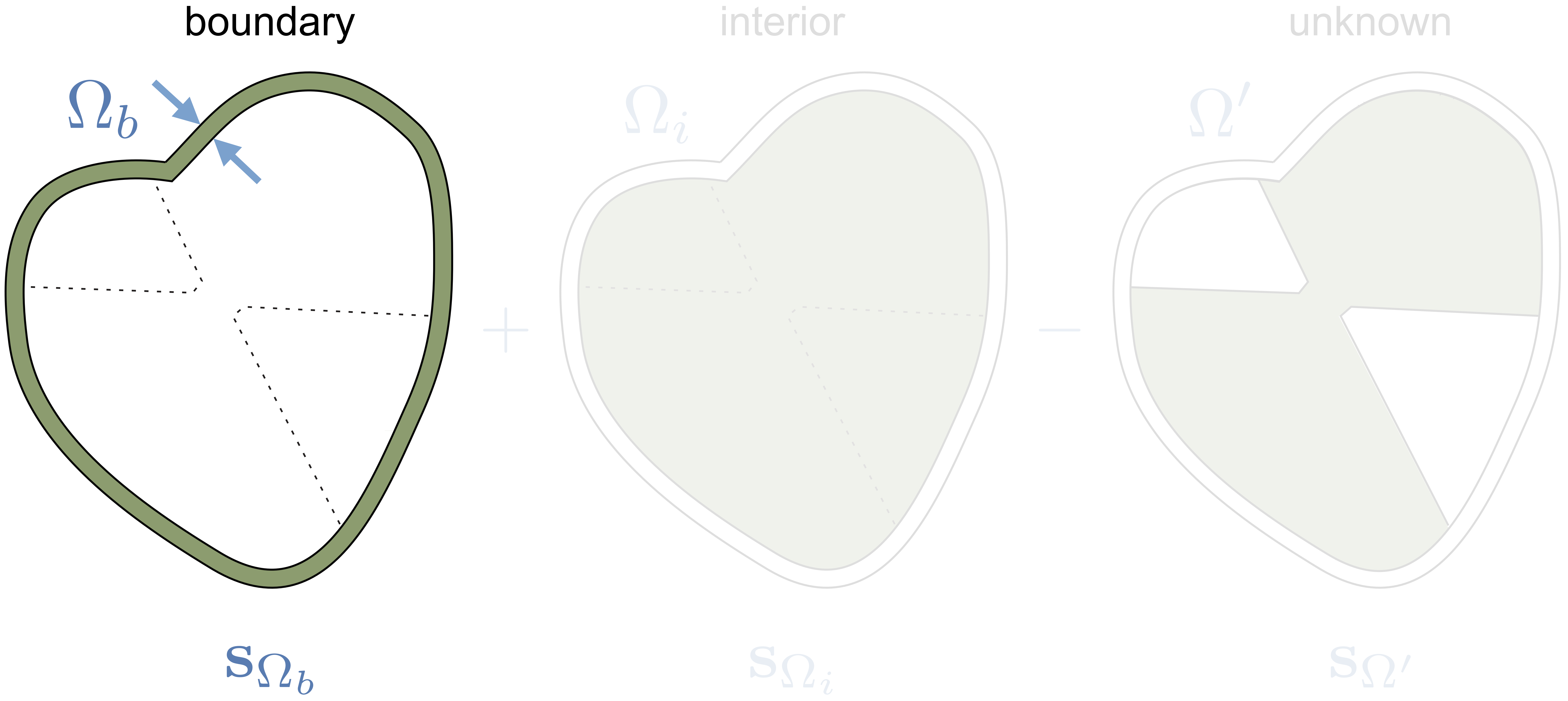
Optimizing Top Energy



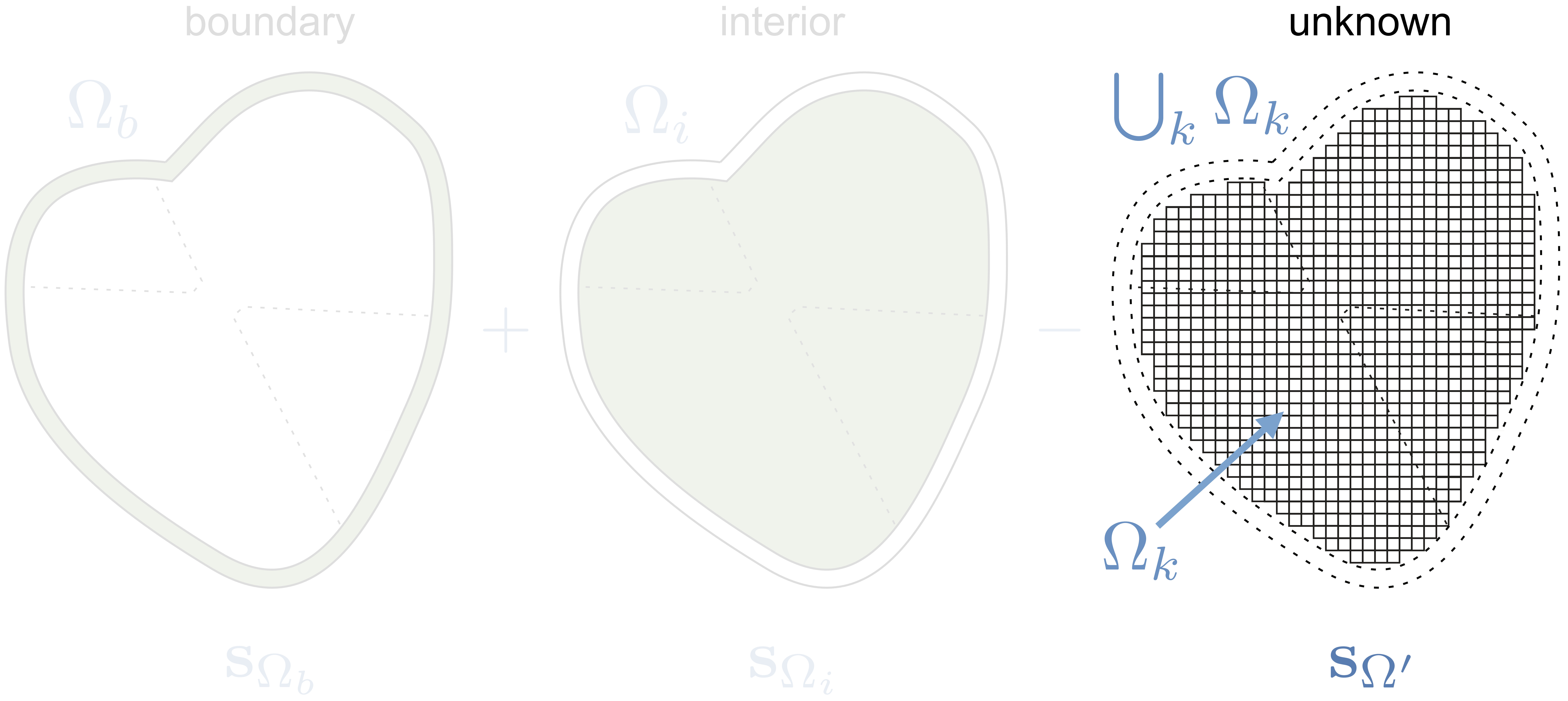
Optimizing Top Energy



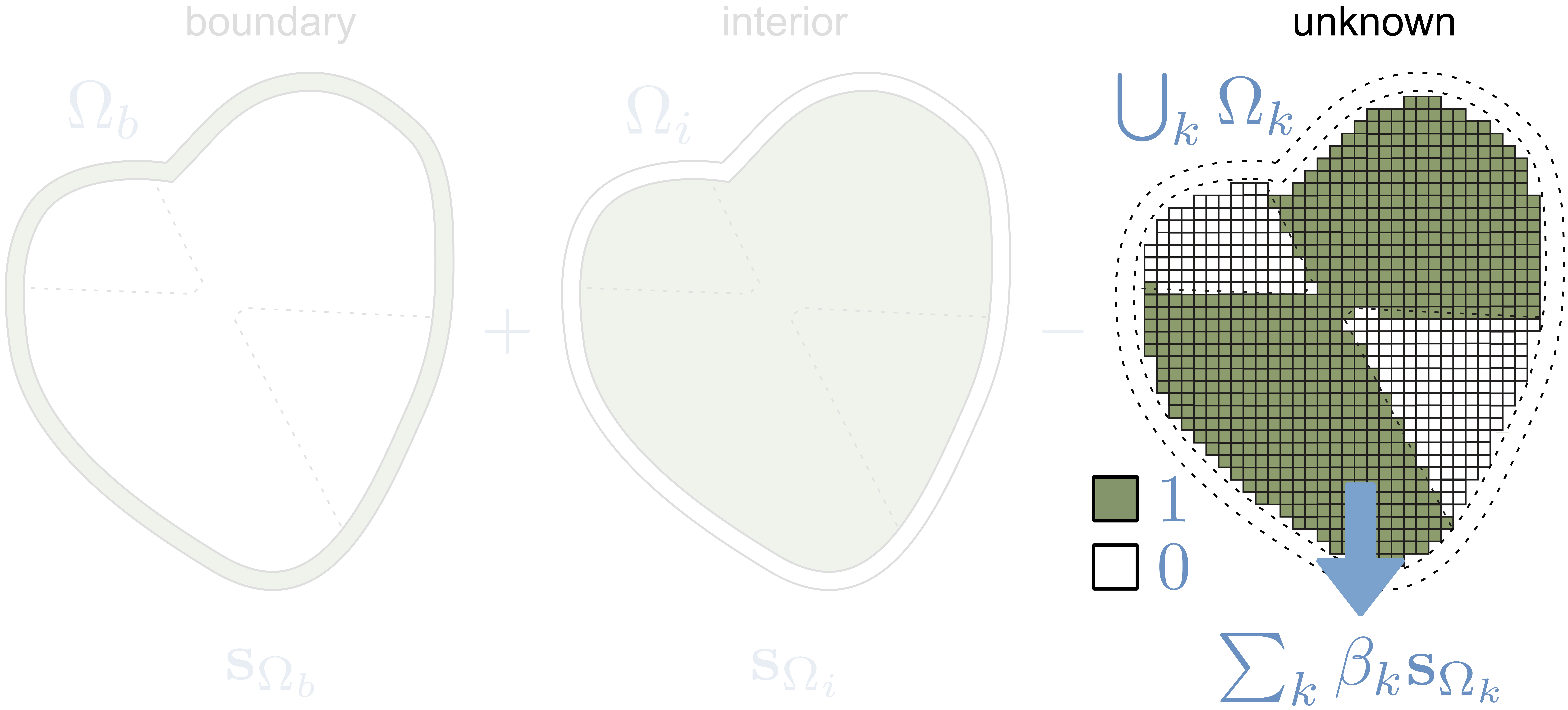
Optimizing Top Energy



Optimizing Top Energy



Optimizing Top Energy



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