252-0538-00L, Spring 2018

Shape Modeling and Geometry Processing

Discrete Differential Geometry

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Differential Geometry – Motivation

Formalize geometric properties of shapes

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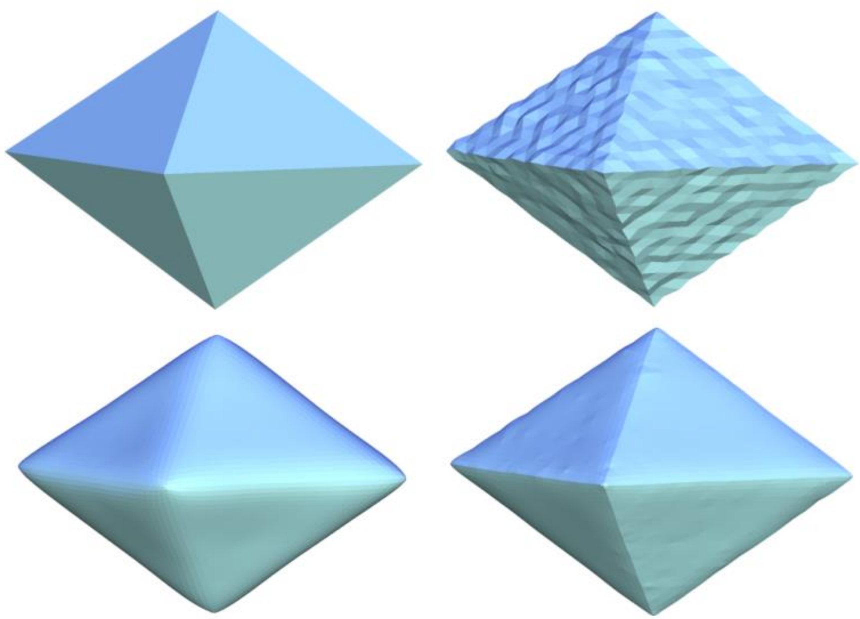
Formalize geometric properties of shapes

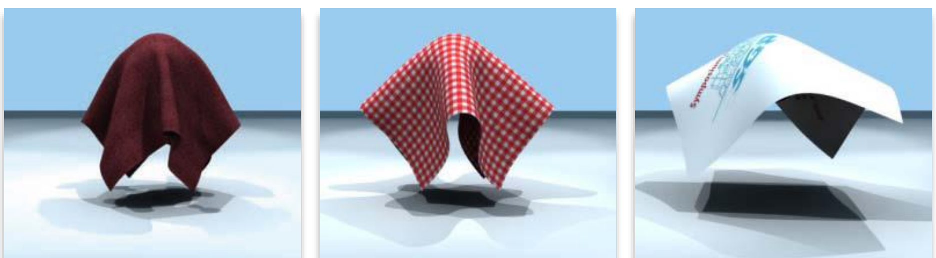
- Smoothness

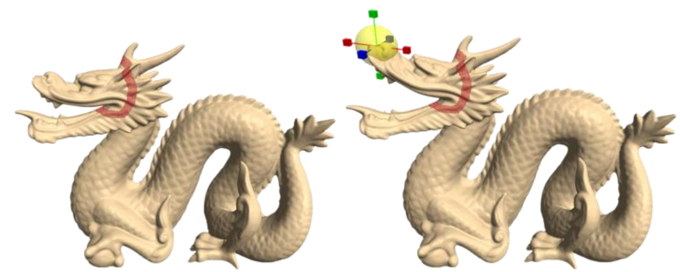


Differential Geometry – Motivation

Formalize geometric properties of shapes

- Smoothness
- Deformation





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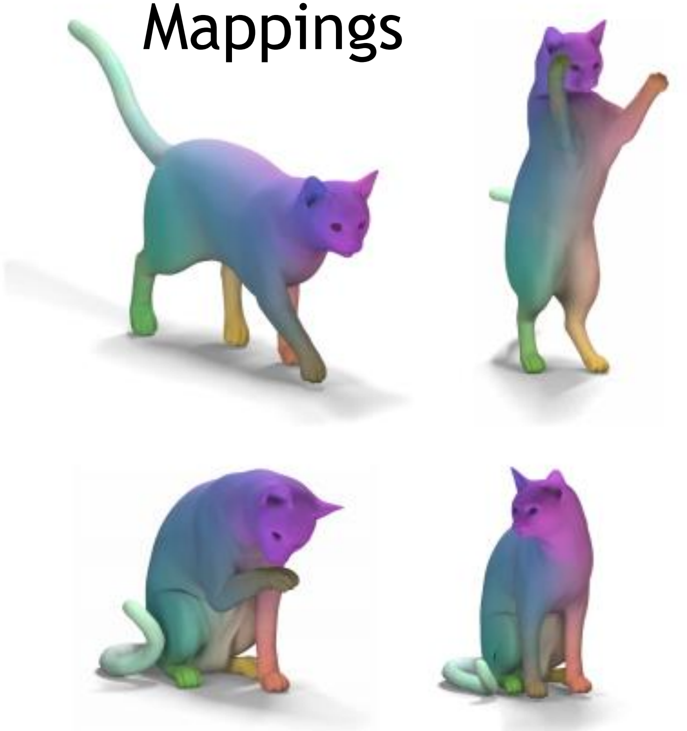
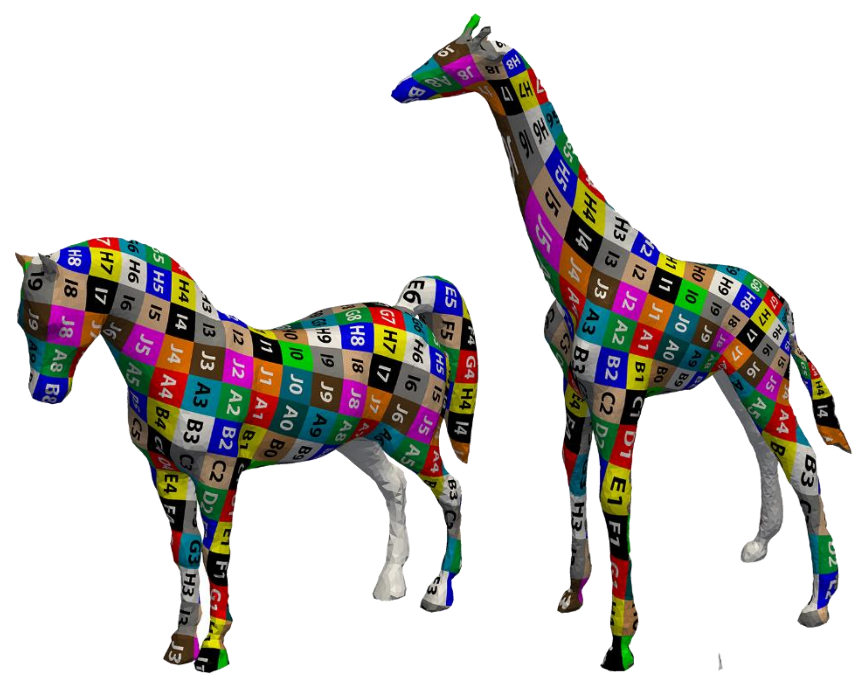
Differential Geometry – Motivation

Formalize geometric properties of shapes

- Smoothness
- Deformation
- Mappings





Differential Geometry Basics

Geometry of manifolds

Things that can be explored locally

point + neighborhood

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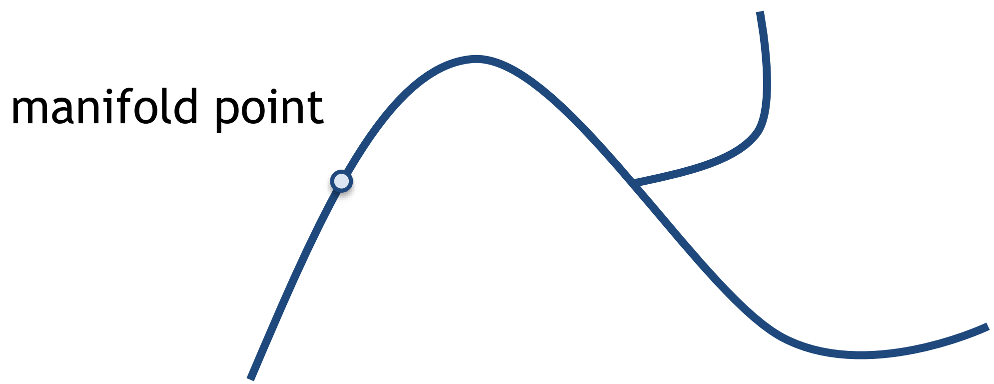

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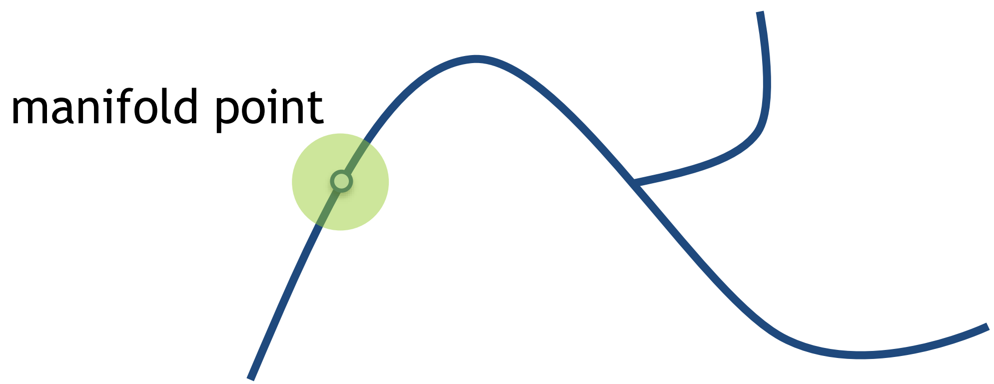

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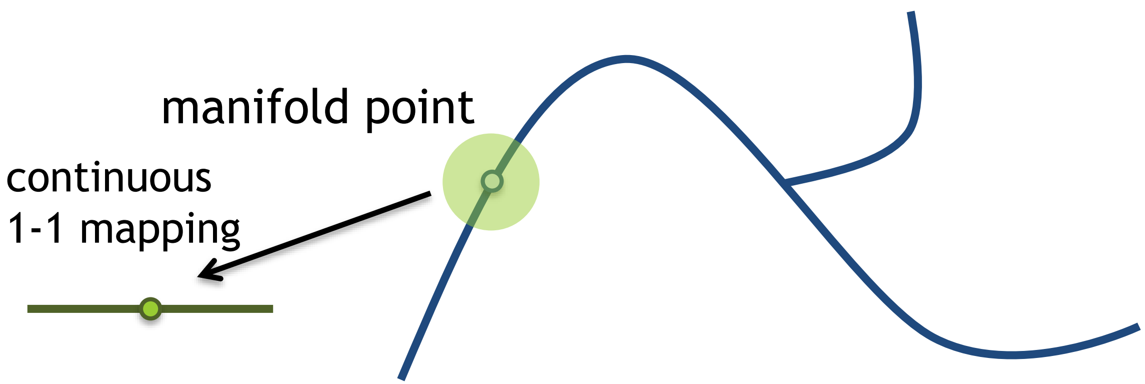

Differential Geometry Basics

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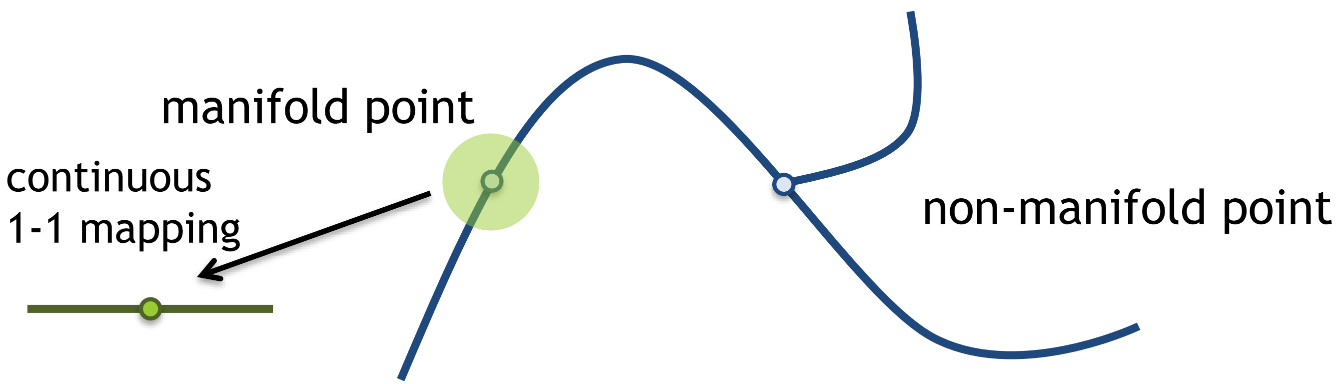

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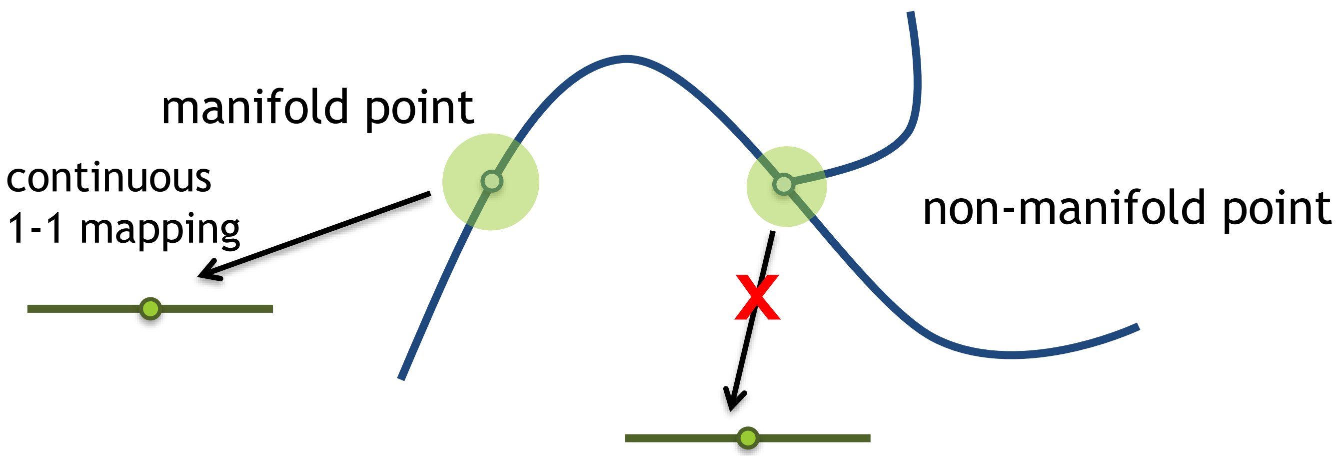

Differential Geometry Basics

Geometry of manifolds

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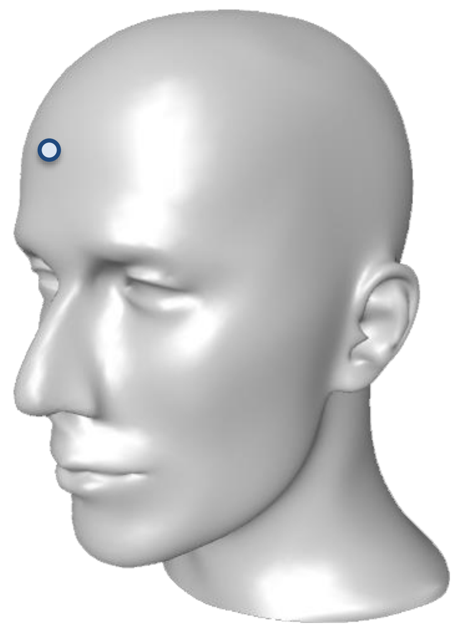
point + neighborhood



Differential Geometry Basics

Geometry of manifolds

Things that can be explored locally

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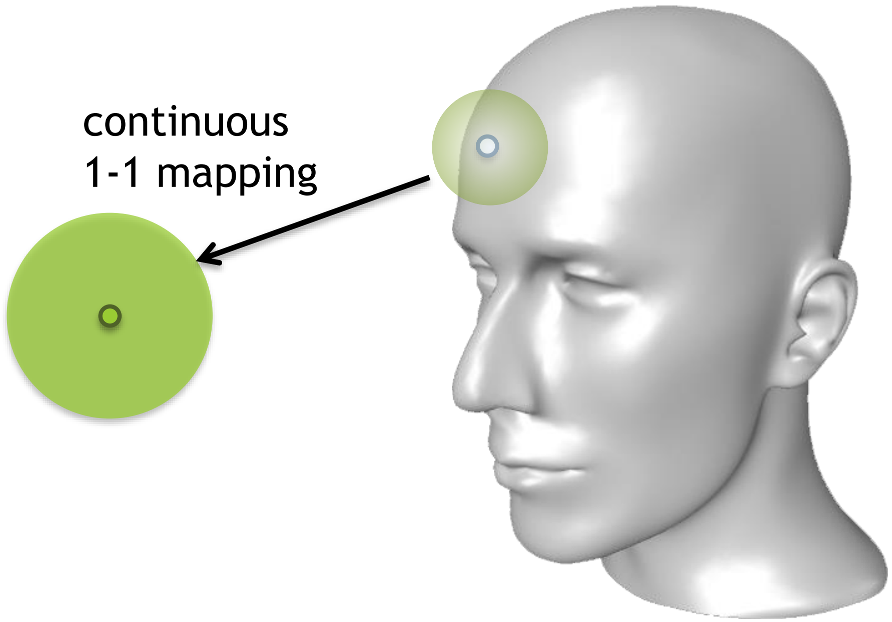
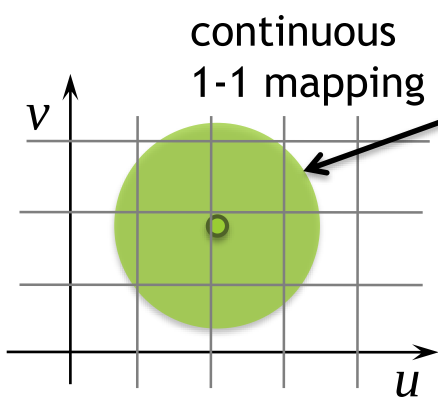


Differential Geometry Basics

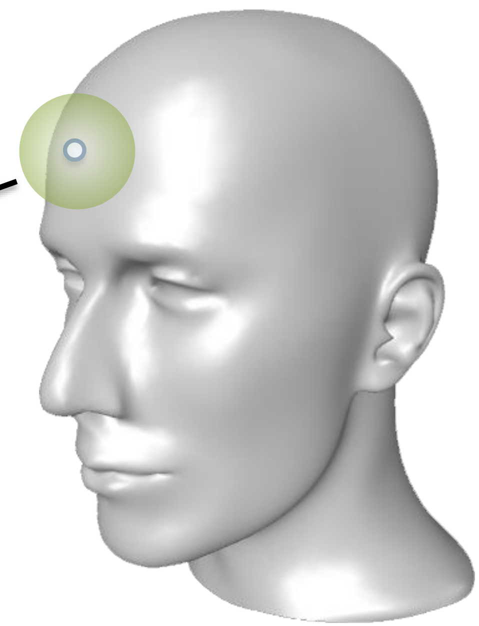
Geometry of manifolds

Things that can be explored locally

point + neighborhood



$$\frac{\partial}{\partial u} \quad \frac{\partial}{\partial v} \quad \frac{\partial^2}{\partial^2 u} \quad \frac{\partial^2}{\partial u \partial v} \quad \dots$$



If a sufficiently smooth mapping can be constructed, we can look at its first and second derivatives

Tangents, normals, curvatures, curve angles

Distances, topology

Differential Geometry of Curves

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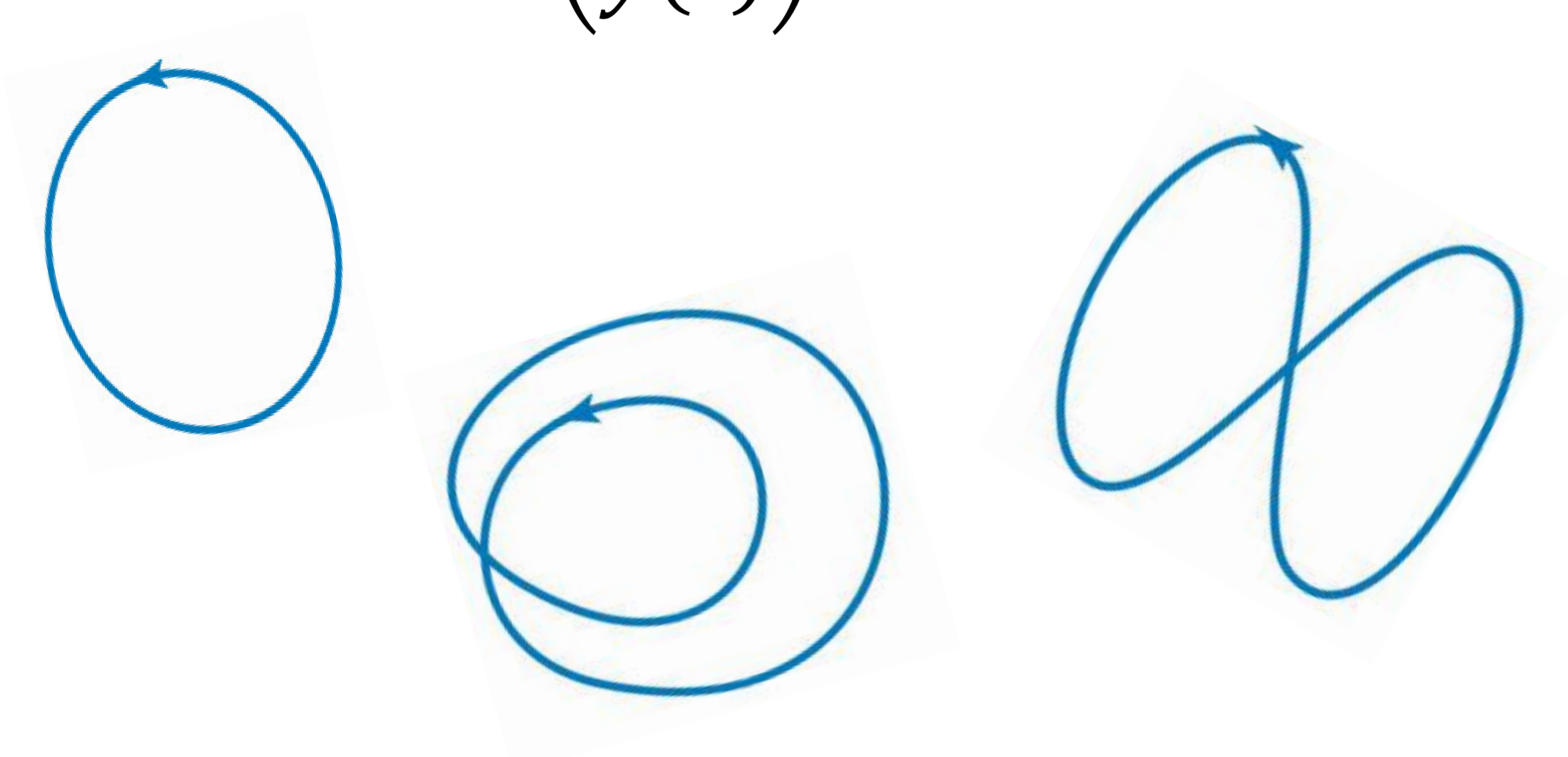
Planar Curves

$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, t \in [t_0, t_1]$$



Planar Curves

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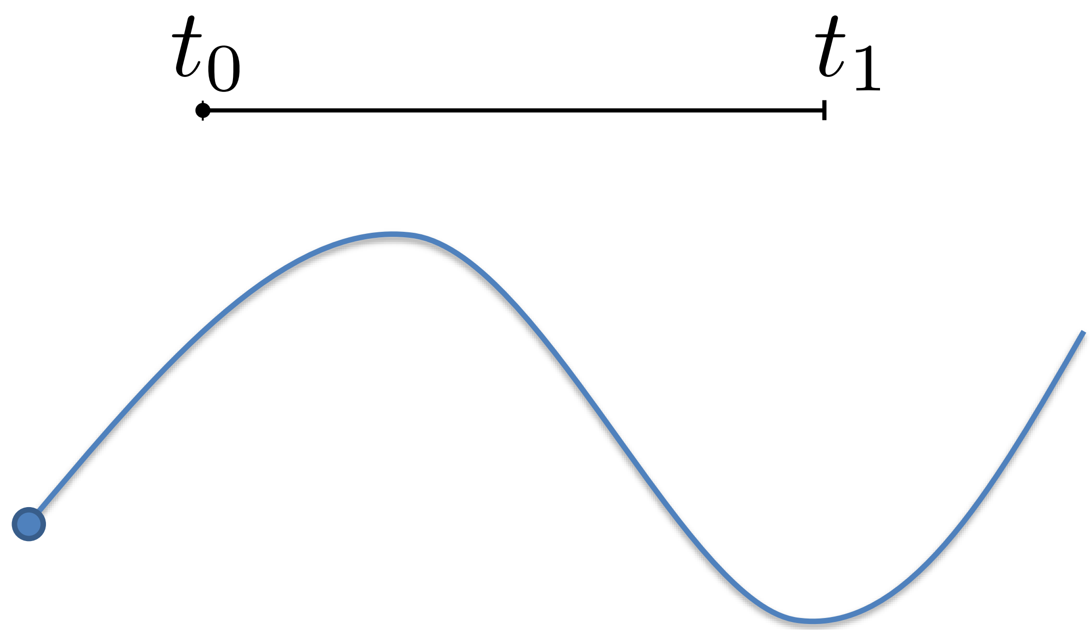
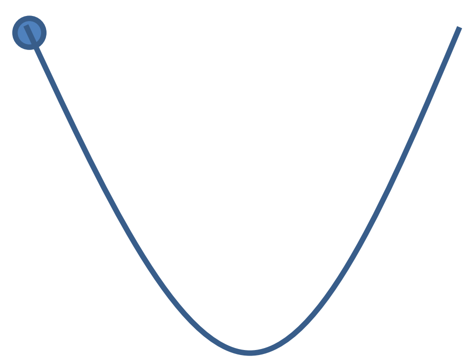


Planar Curves

$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, t \in [t_0, t_1]$$

$$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$



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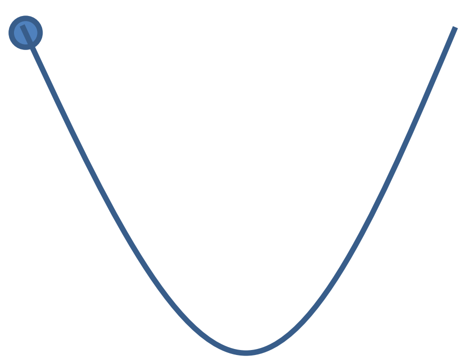
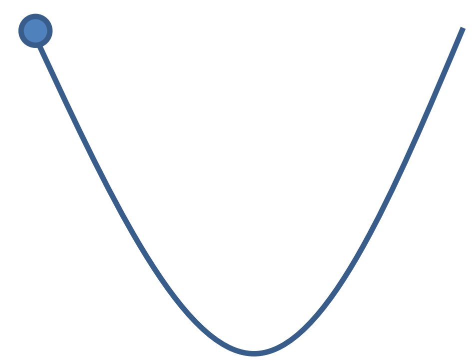


$$\gamma(t) = \begin{pmatrix} t^2 \\ t^4 \end{pmatrix}$$

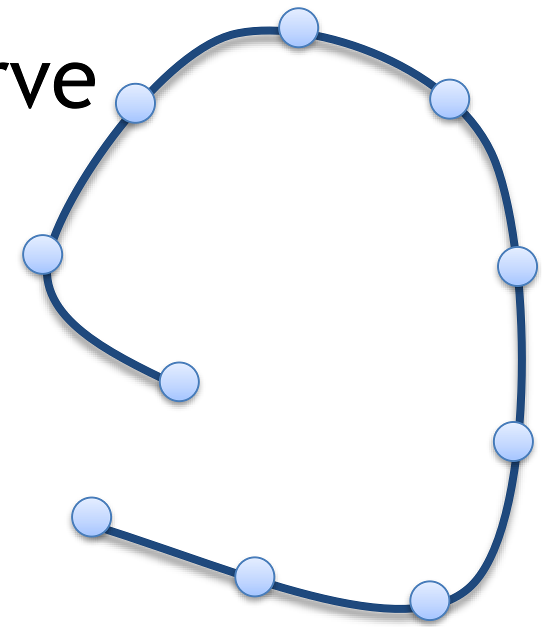


Arc Length Parameterization

Same curve has many parameterizations!

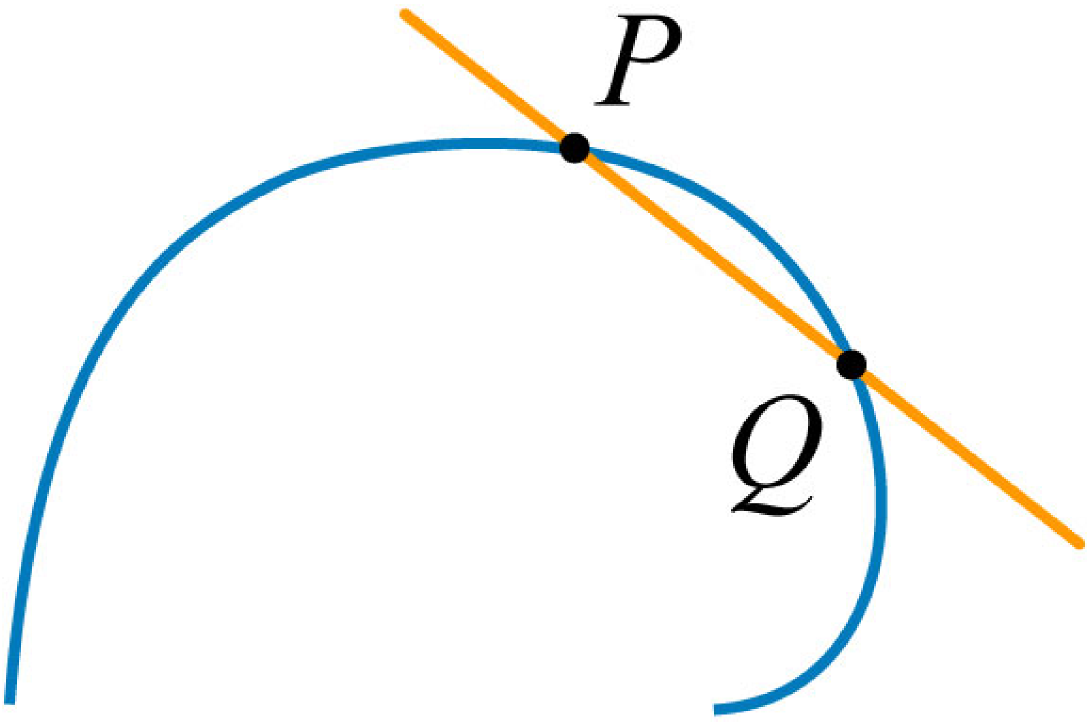
Arc-length: equal speed of the parameter along the curve

$$L(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$$



Secant

A line through two points on the curve.



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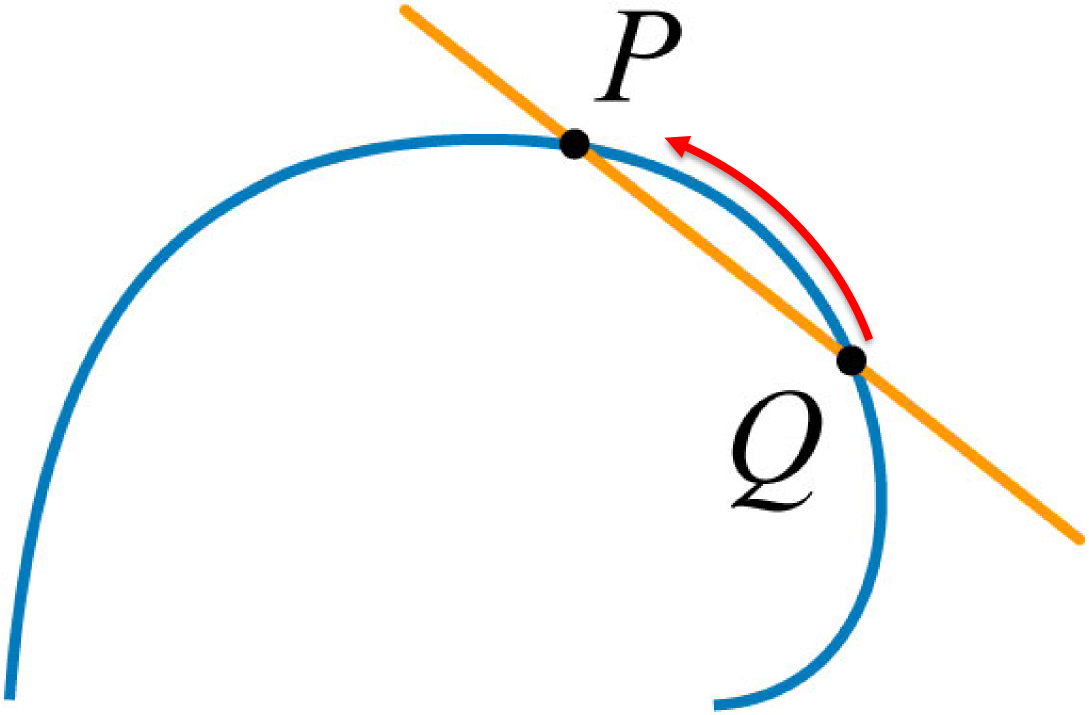

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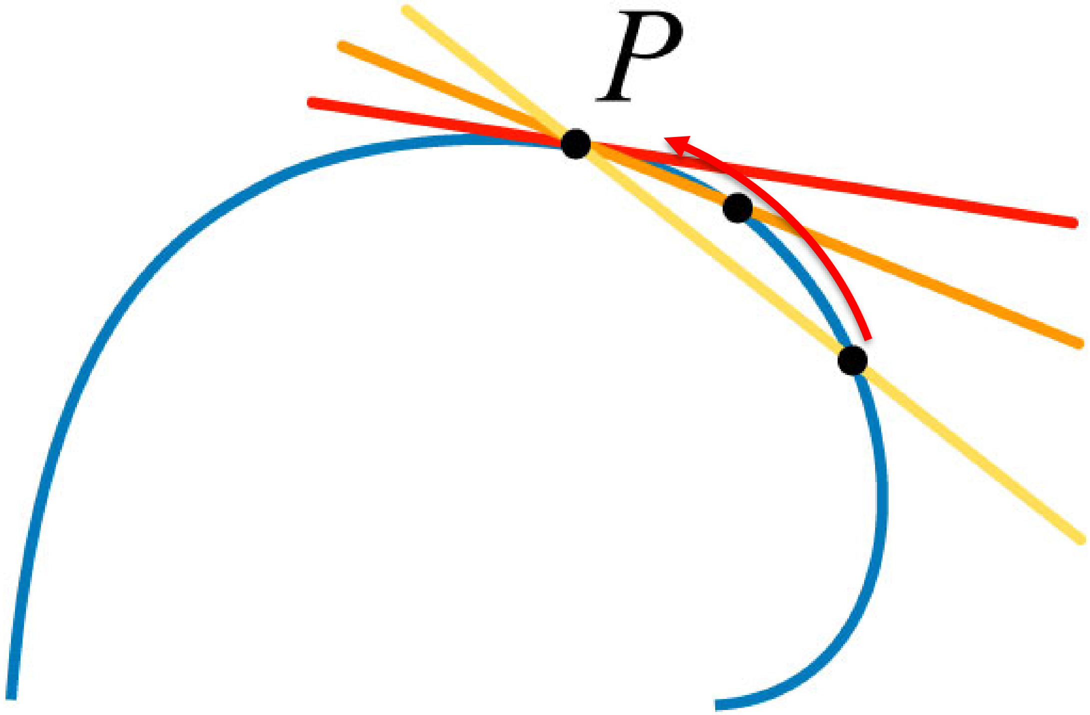

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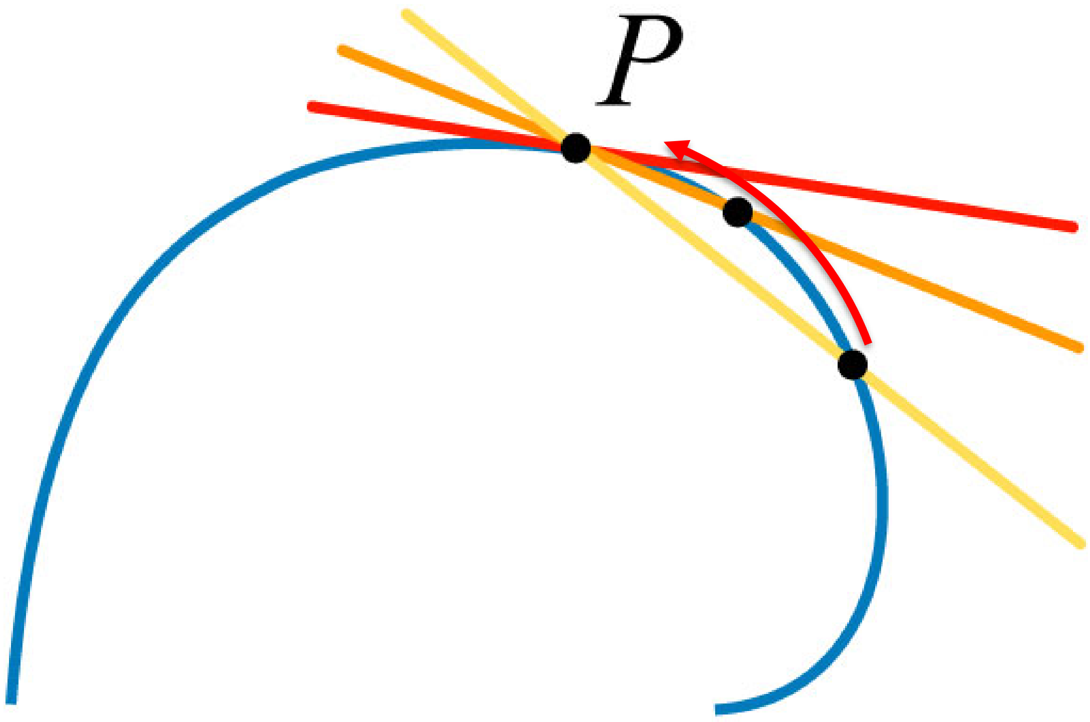
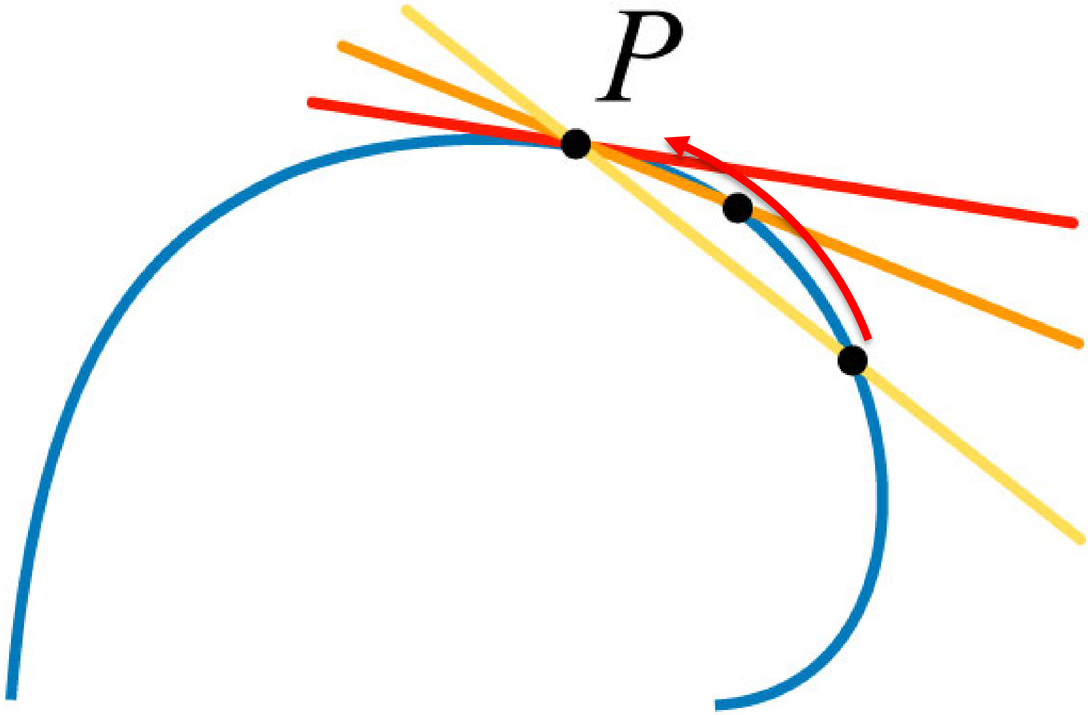
Tangent

A line through two points on the curve.



Tangent

The limit secant as two points come together.



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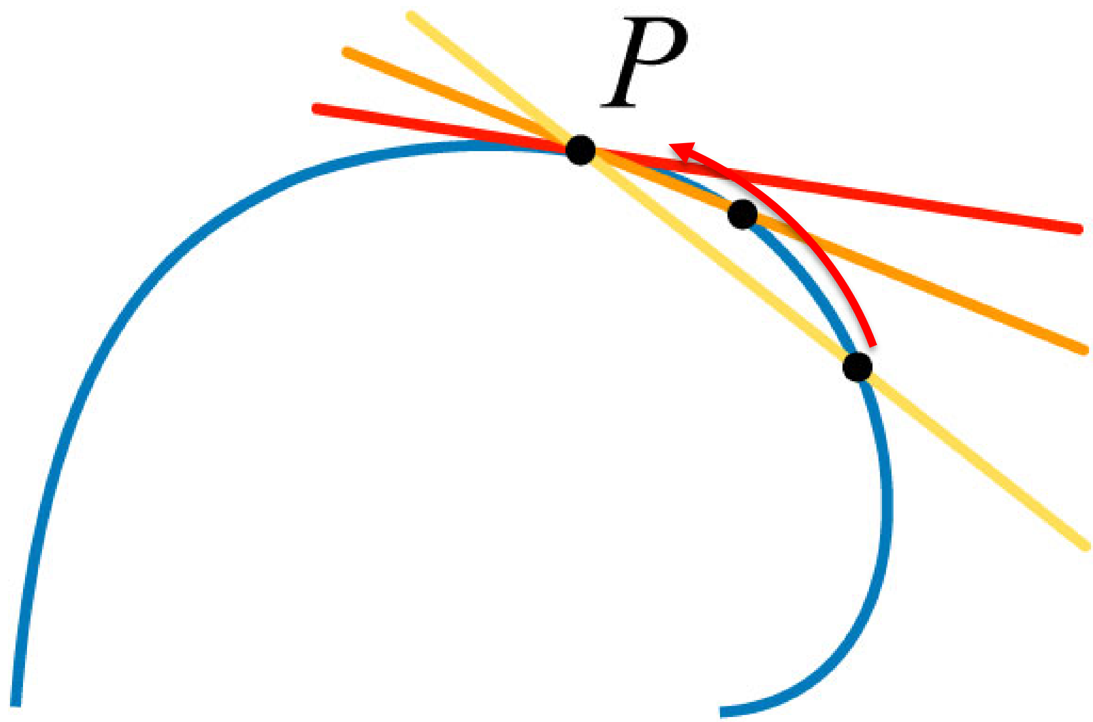
Secant and Tangent

Secant: line through $\boldsymbol{p}(P) - \boldsymbol{p}(Q)$

Tangent: $\gamma'(P) = (x'(P), y'(P), \ldots)^T$



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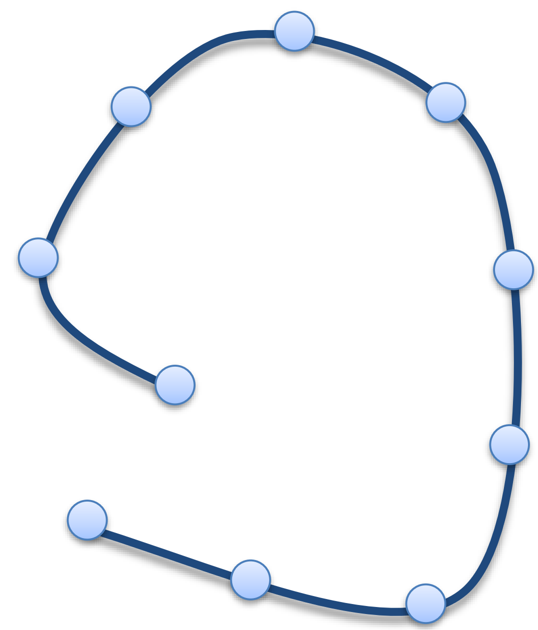
$$\|\gamma'(t)\| =$$



Arc Length Parameterization

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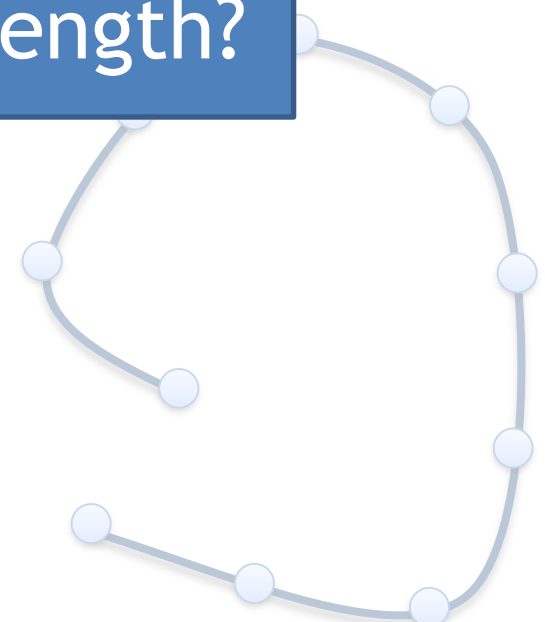
Arc-length: equal speed of the parameter along the curve

$L(\gamma($

$\|\gamma'(t)\| =$

What it $\gamma(t)$ is not arc length?



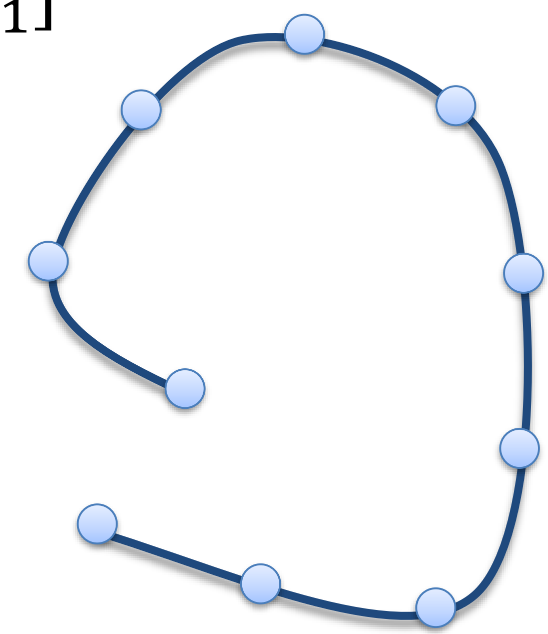
Arc Length Parameterization

Re

Curve Reparamterization

$$\gamma(t) \longrightarrow \gamma(p(t))$$

$$p: [t_0, t_1] \to [t_0, t_1]$$
$$p'(t) \neq 0$$



Arc length reparamterization

$$\left\|\gamma'\big(p(t)\big)\right\| = 1$$

Arc Length Parameterization

Re

Arc length reparamterization

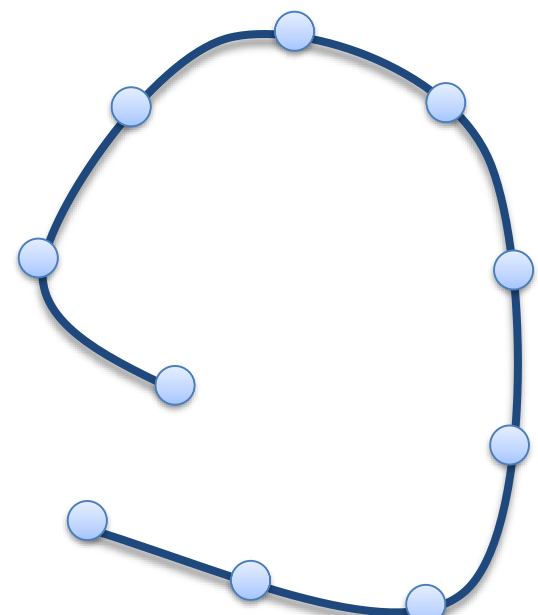
$$\|\gamma'(p(t))\| = 1$$

Let

$$q(t) = \int_{t0}^{t} \|\gamma'(t)\|$$

Then

$$p(t) = q^{-1}(t)$$



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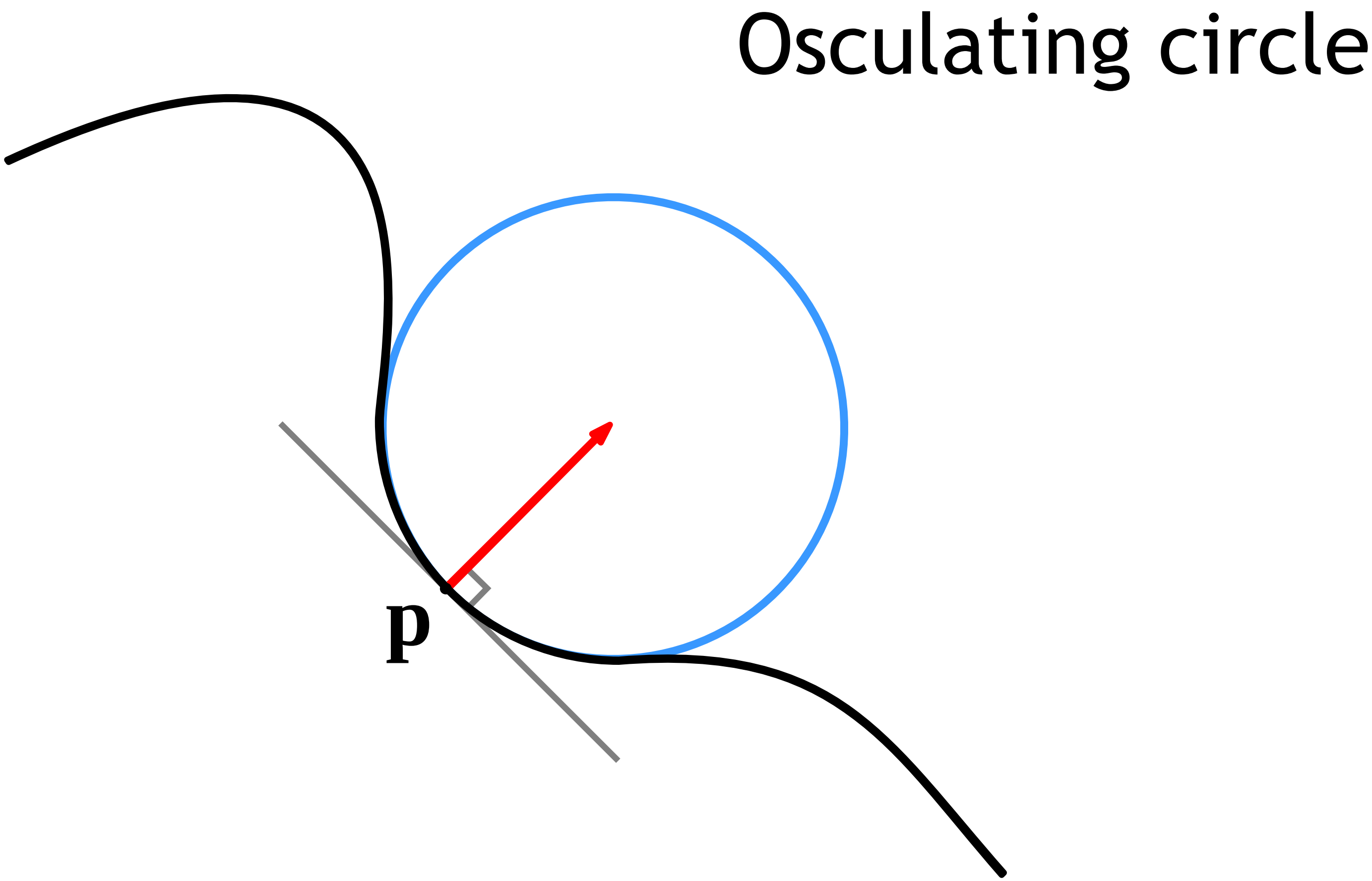
Tangent, normal, curvature



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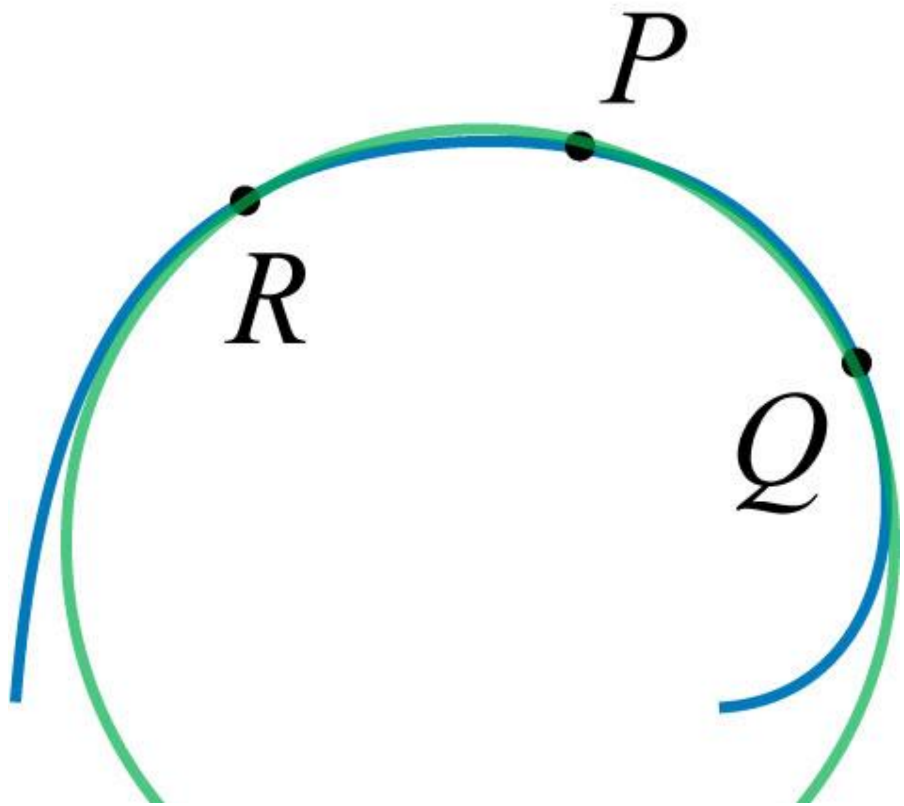
Curvature

Circle through three points on the curve



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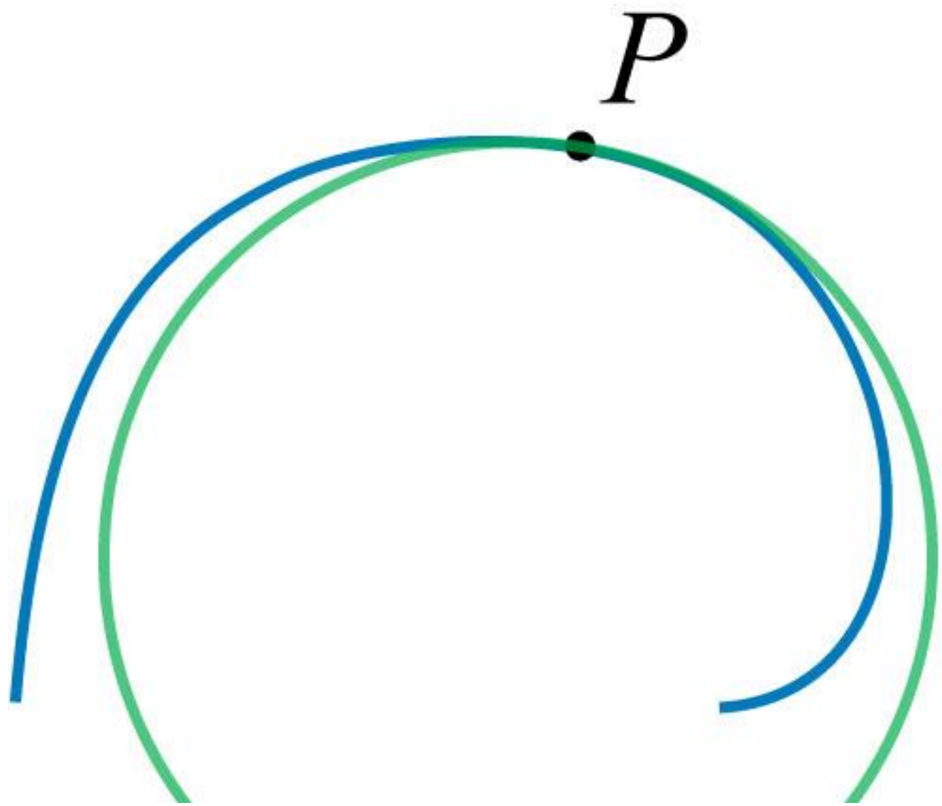
Curvature

The limit circle as points come together.

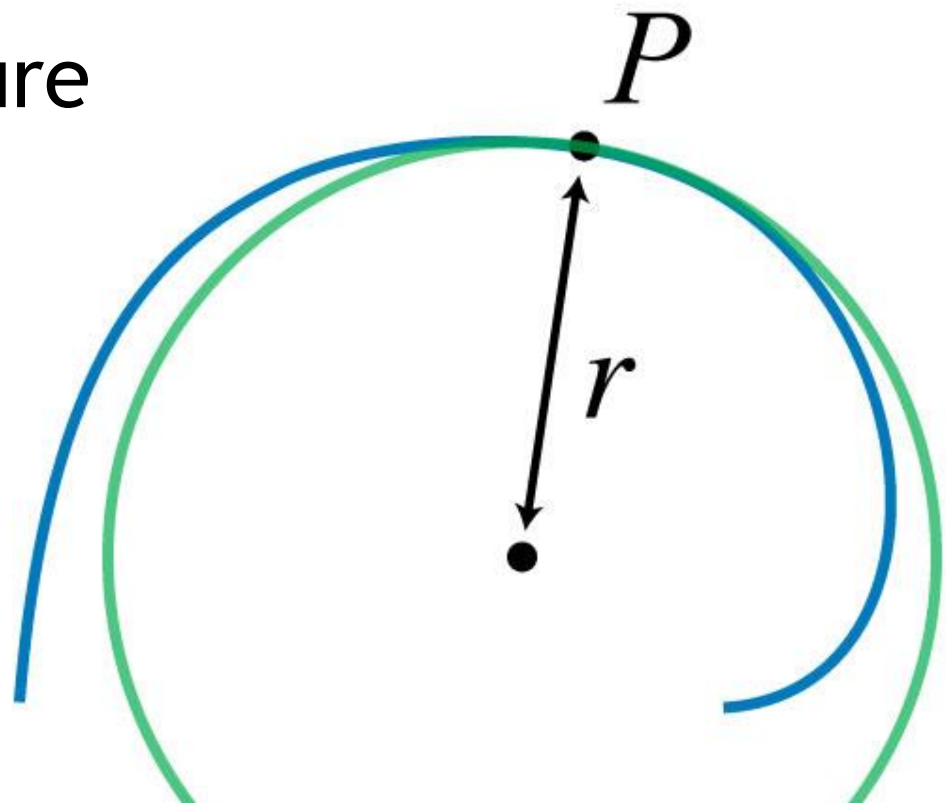


Curvature

The limit circle as points come together.

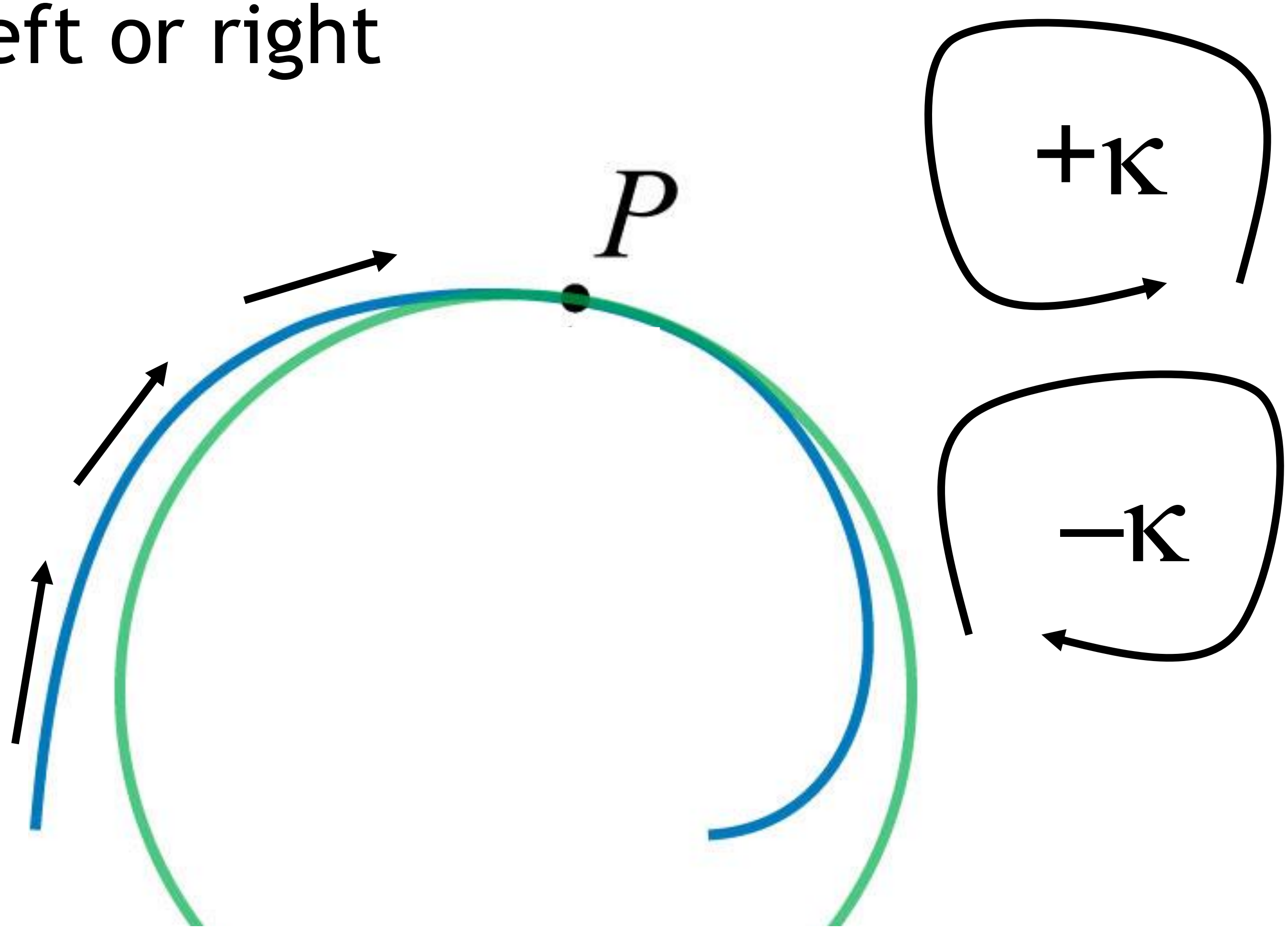
Curvature

$$\kappa = \frac{1}{r}$$



Signed Curvature

Curving left or right



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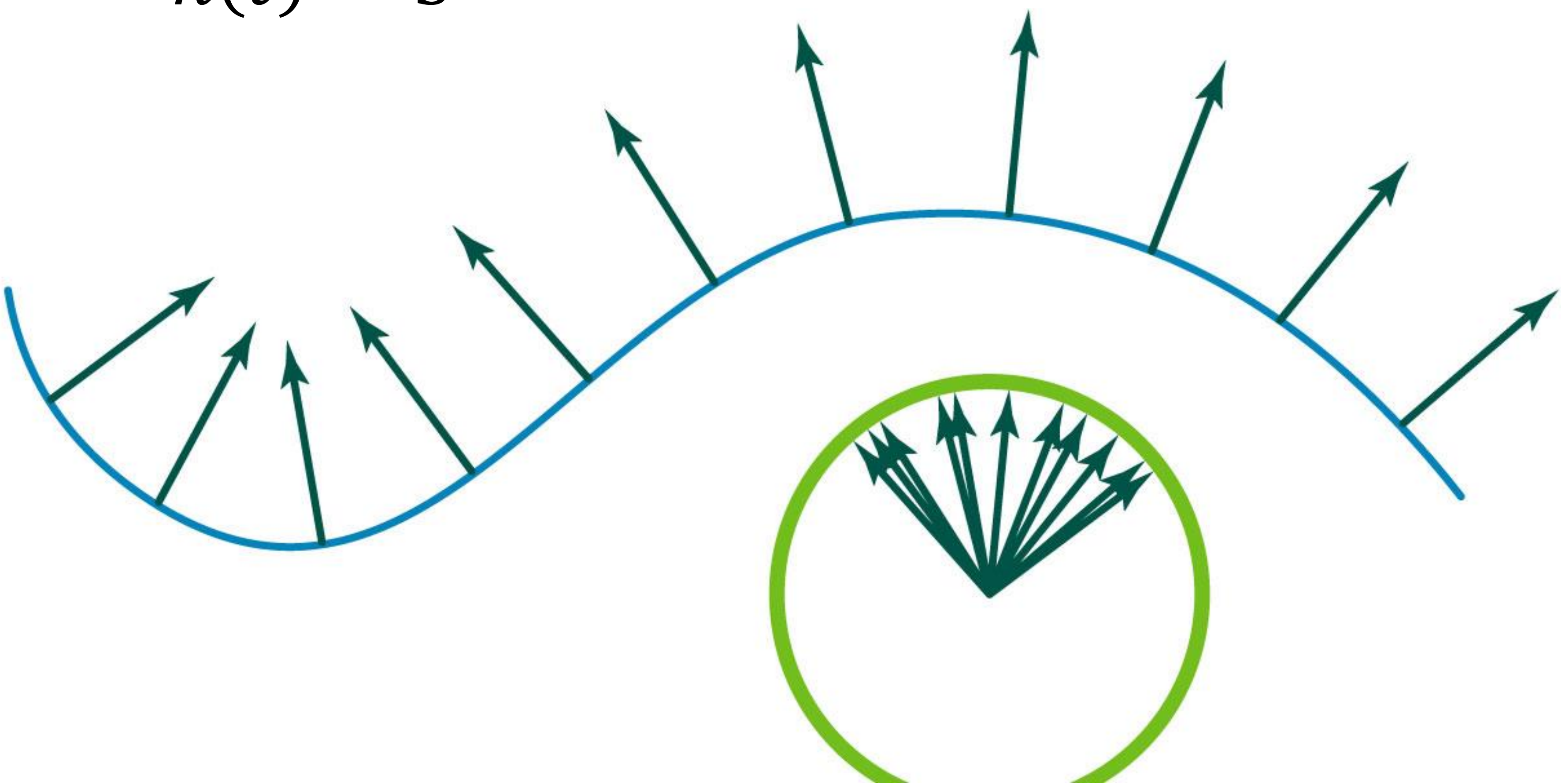
Gauss map $\hat{n}(t)$

Point on curve maps to point on unit circle.

$$\hat{n}(t) \rightarrow S^1$$

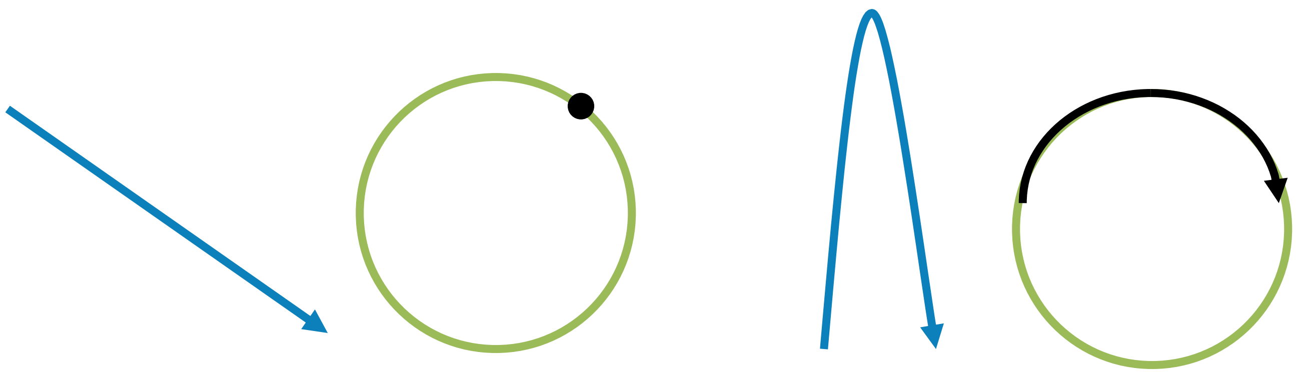


Curvature = change in normal direction

Absolute curvature (assuming arc length)

$$\kappa = \|\hat{\mathbf{n}}'(t)\|$$

via the Gauss map

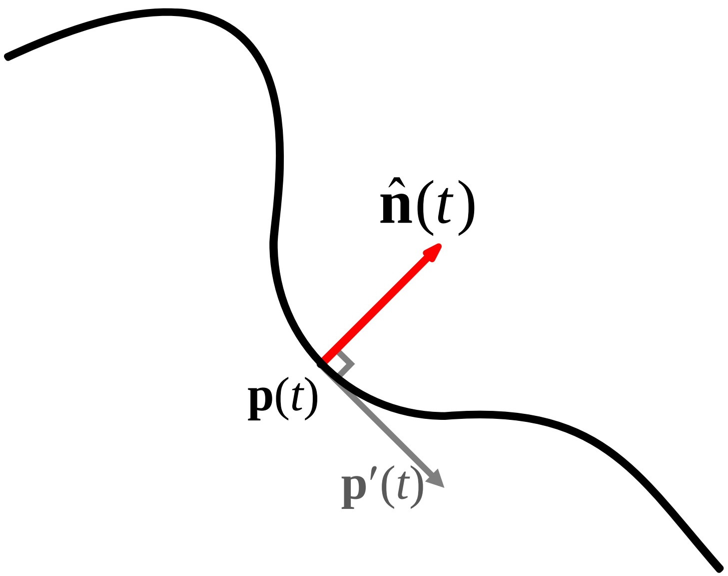


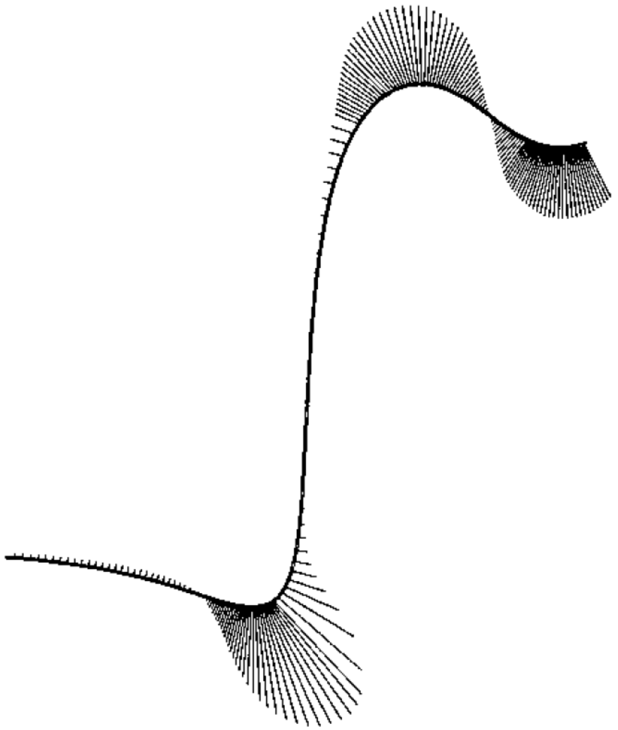
curve Gauss map curve Gauss map

Curvature Normal

Assume t is arc-length parameter

$$\mathbf{p}''(t) = \kappa \hat{\mathbf{n}}(t)$$





"A multiresolution framework for variational subdivision", Kobbelt and Schröder, ACM TOG 17(4), 1998

Curvature Normal

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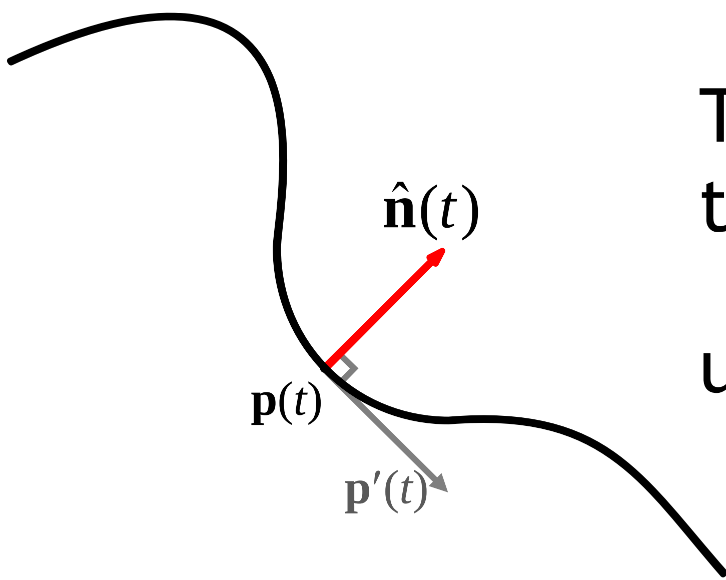
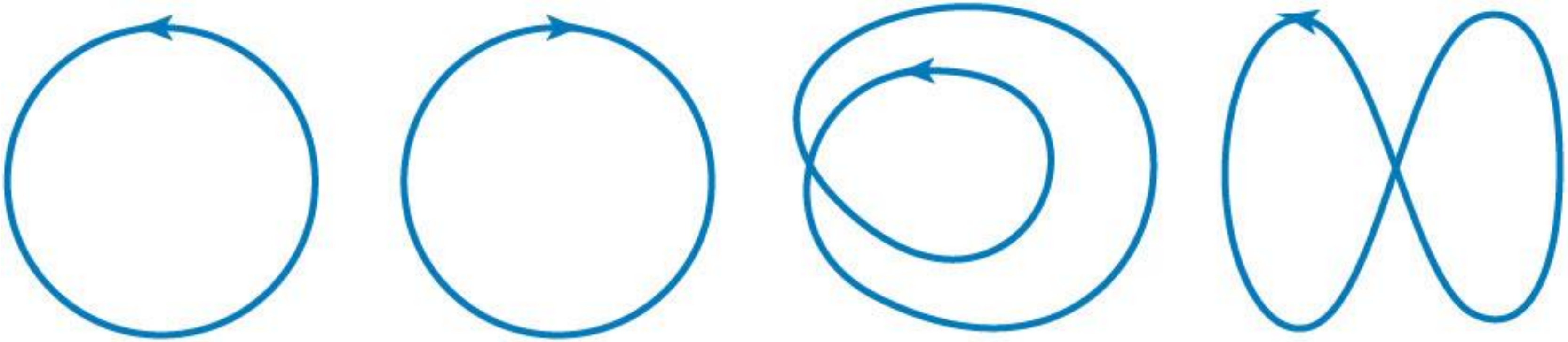


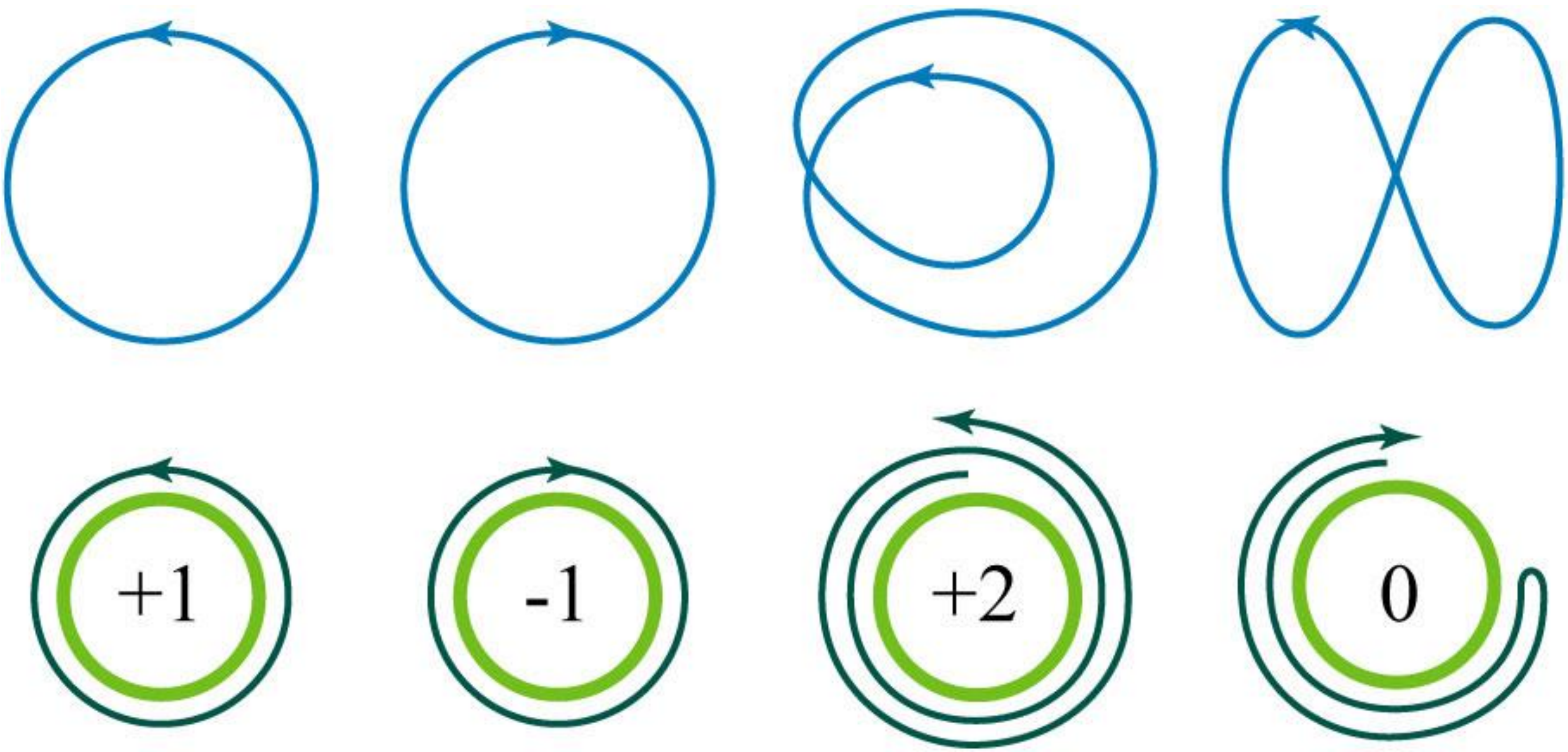
The curvature **defines** the planar curve **shape**

up to rotation and translation!

Turning Number, k



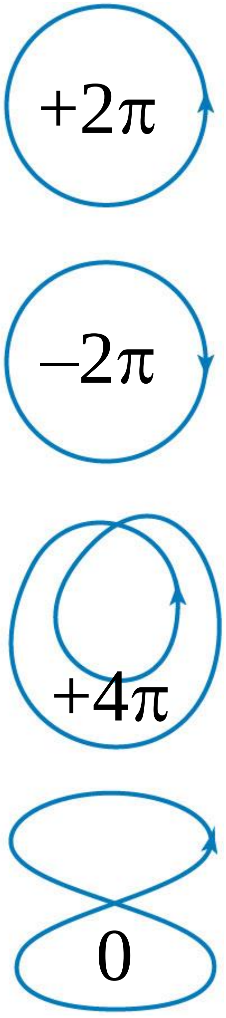
Turning Number, k



Turning Number Theorem

$$\int_\gamma \kappa \, dt = 2\pi k$$

For a closed curve,
the integral of curvature is
an integer multiple of 2π.



Discrete Planar Curves

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Discrete Planar Curves

Piecewise linear curves

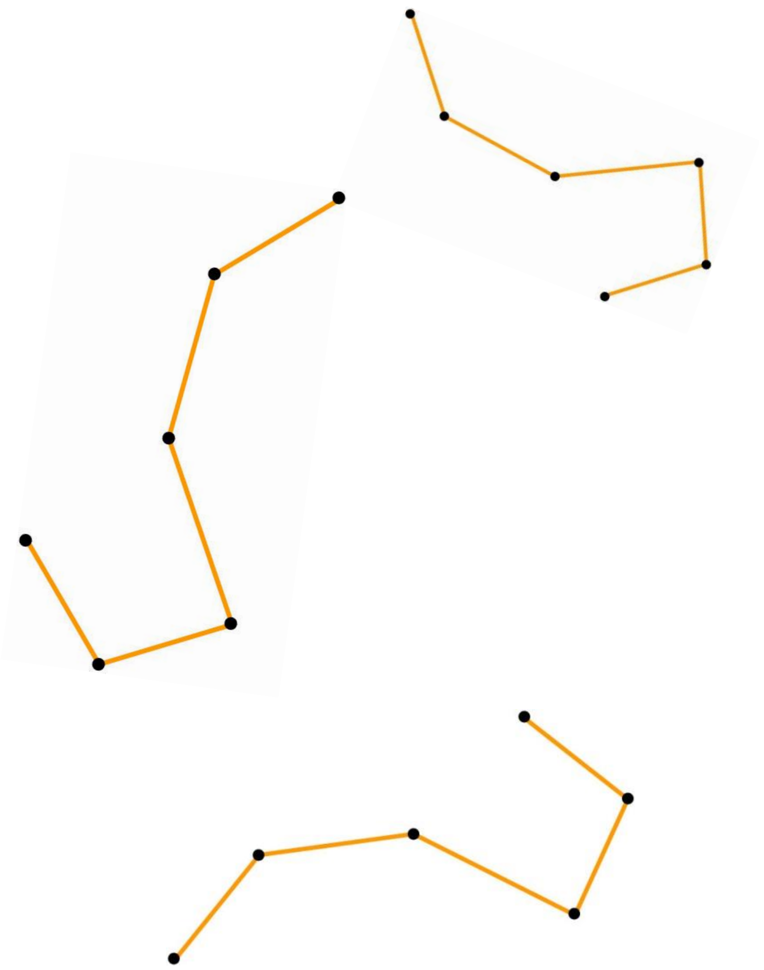
Not smooth at vertices

Can’t take derivatives

Goal :Generalize notions

From the smooth world for the discrete case

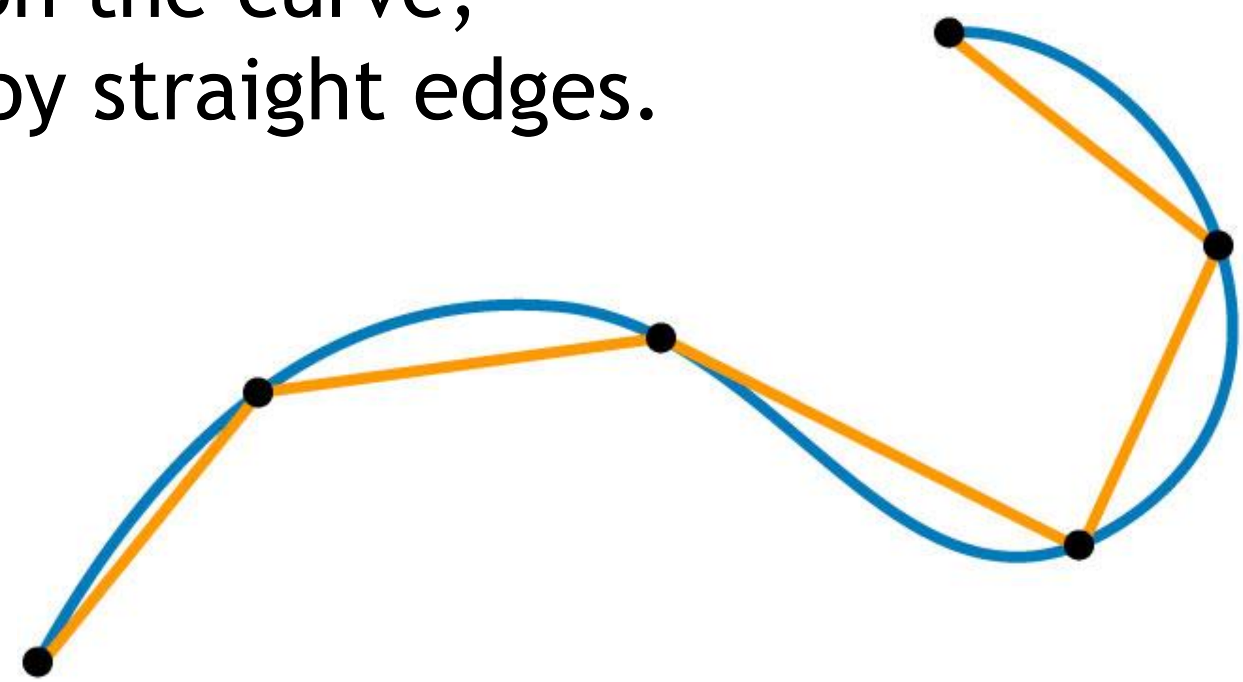
There is no one single way!



Sampling

Connection between discrete and smooth

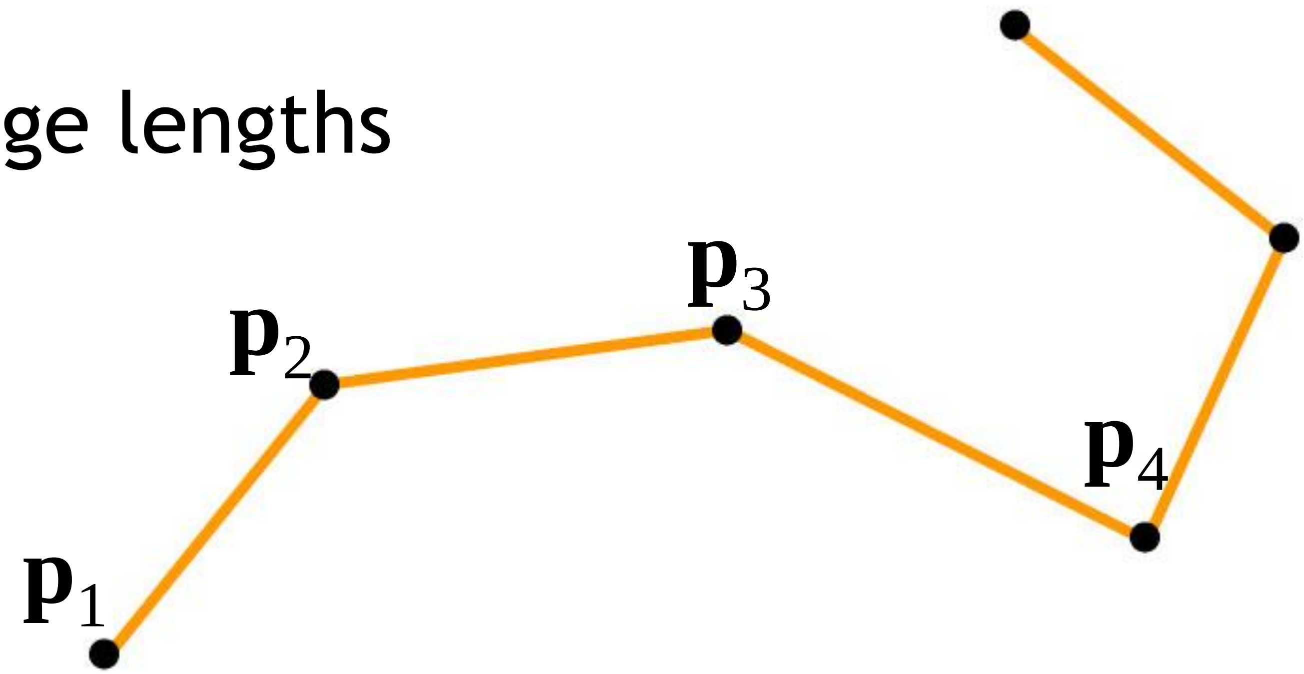
Finite number of vertices
each lying on the curve,
connected by straight edges.



The Length of a Discrete Curve

$$\mathrm{len}(p) = \sum_{i=1}^{n-1} \|\mathbf{p}_{i+1} - \mathbf{p}_i\|$$

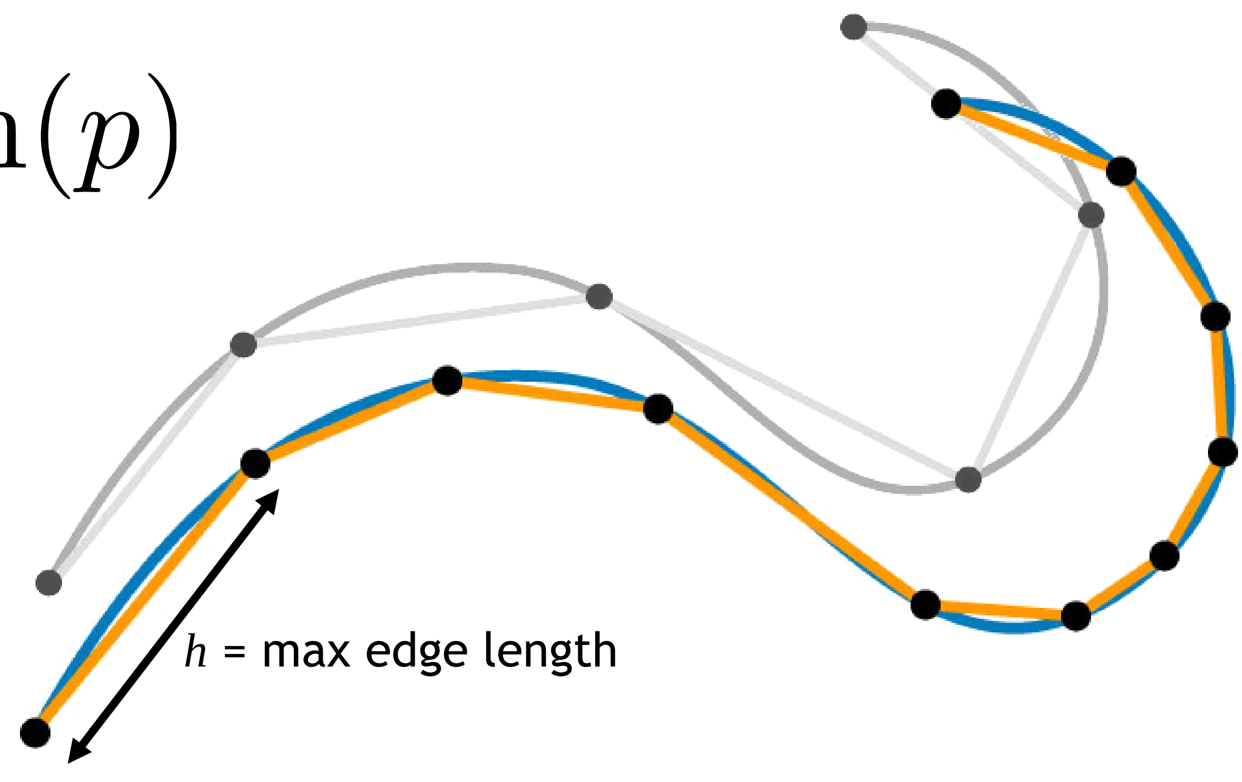
Sum of edge lengths



The Length of a Continuous Curve

limit over a refinement sequence

$$\lim_{h \to 0} \text{len}(p)$$

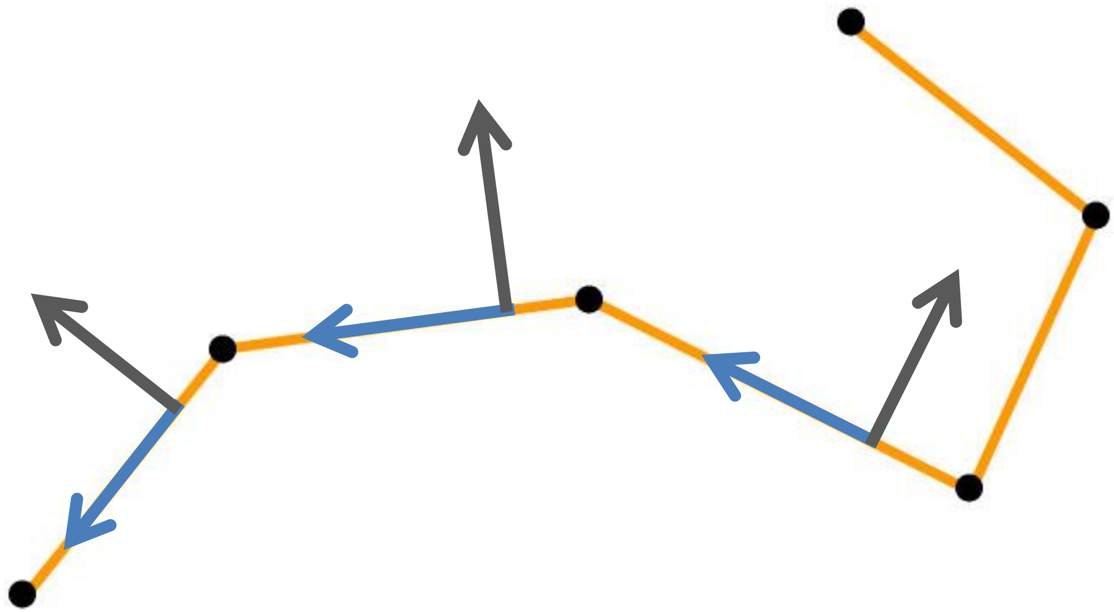


Tangents, Normals

On edges

tangent is the unit vector along edge

normal is the perpendicular vector



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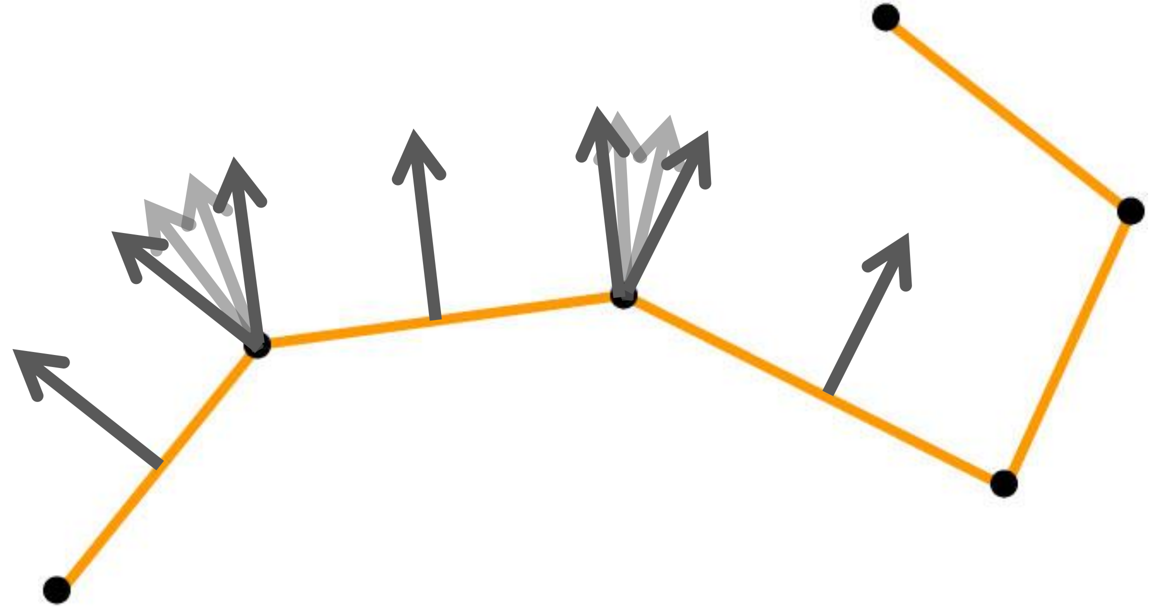
Tangents, Normals

On vertices

Many options…



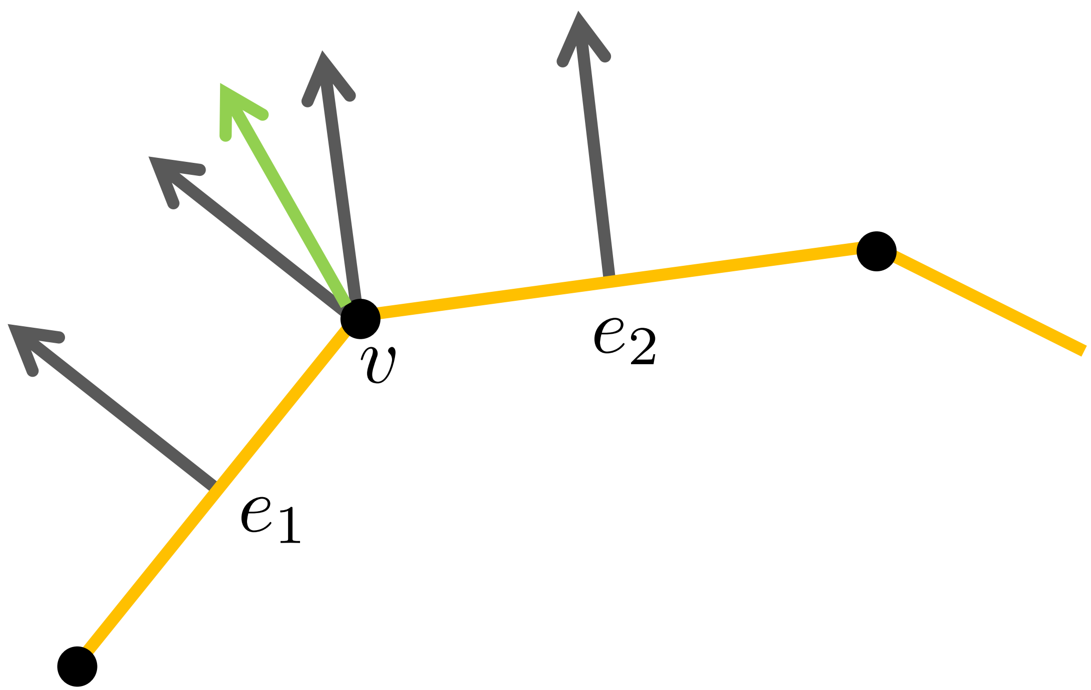
Tangents, Normals

On vertices

Many options…

Average the adjacent edge normals

$$\hat{\mathbf{n}}_v = \frac{\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}}{\|\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}\|}$$



Tangents, Normals

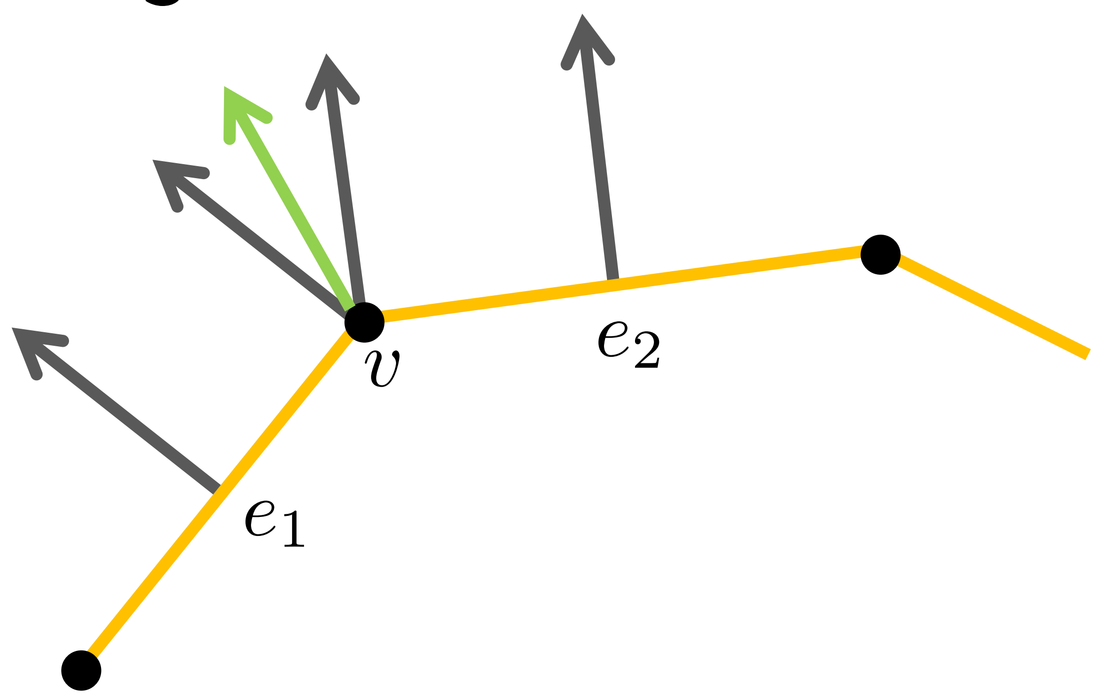
On vertices

Many options…

Average the adjacent edge normals

Weighting by edge lengths

$$\hat{\mathbf{n}}_v = \frac{\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}}{\|\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}\|}$$



Tangents, Normals

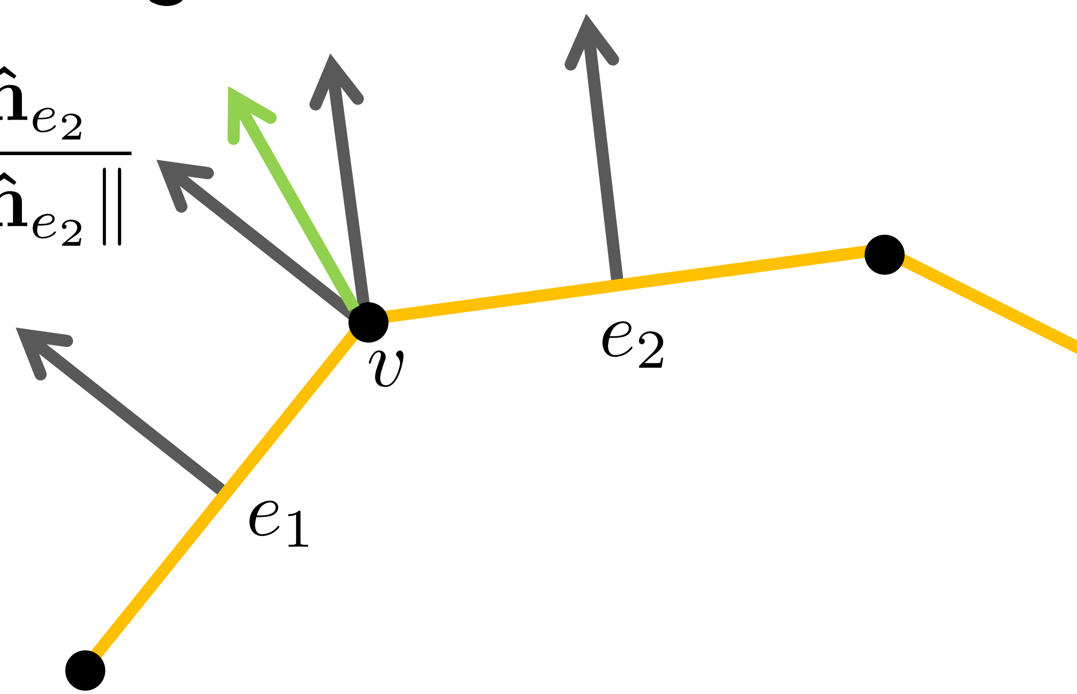
On vertices

Many options…

Average the adjacent edge normals

Weighting by edge lengths

$$\hat{\mathbf{n}}_v = \frac{|e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}}{\||e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}\|}$$



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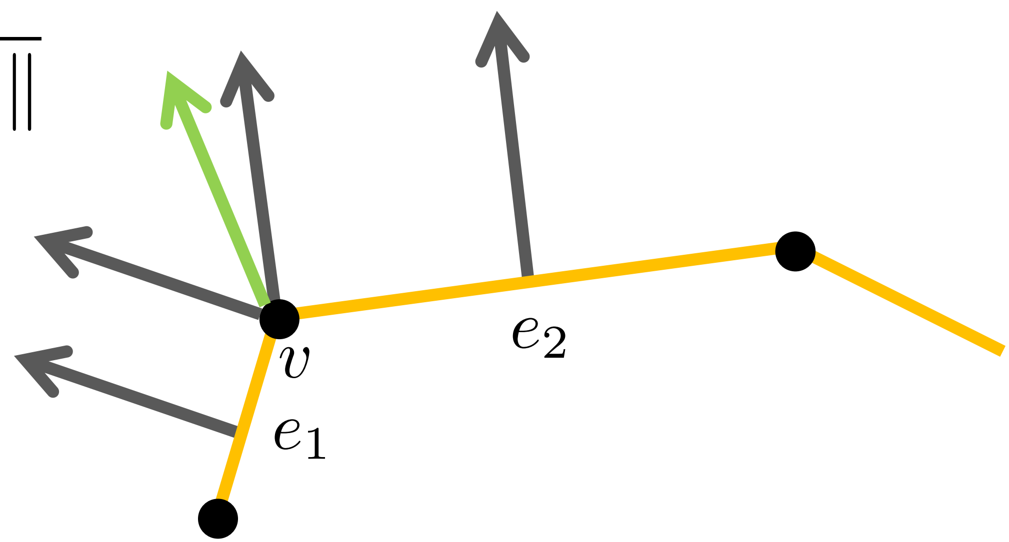
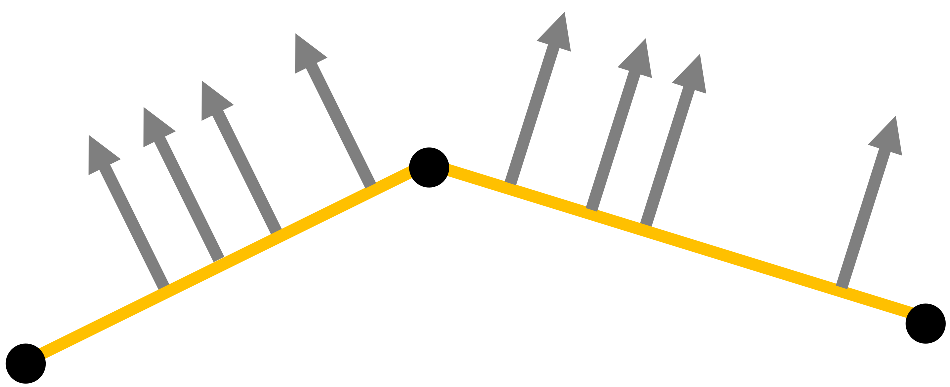


Curvature of a Discrete Curve

Again: change in normal direction



no change along each edge –
curvature is zero along edges

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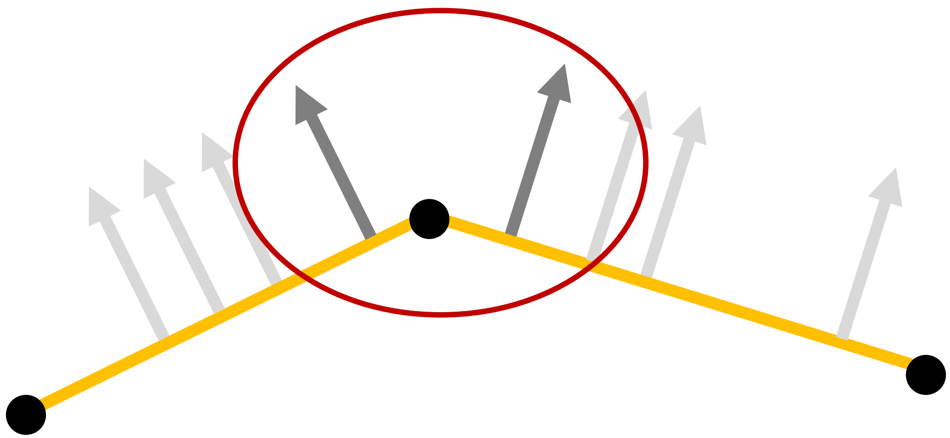
Curvature of a Discrete Curve

Again: change in normal direction



no change along each edge –
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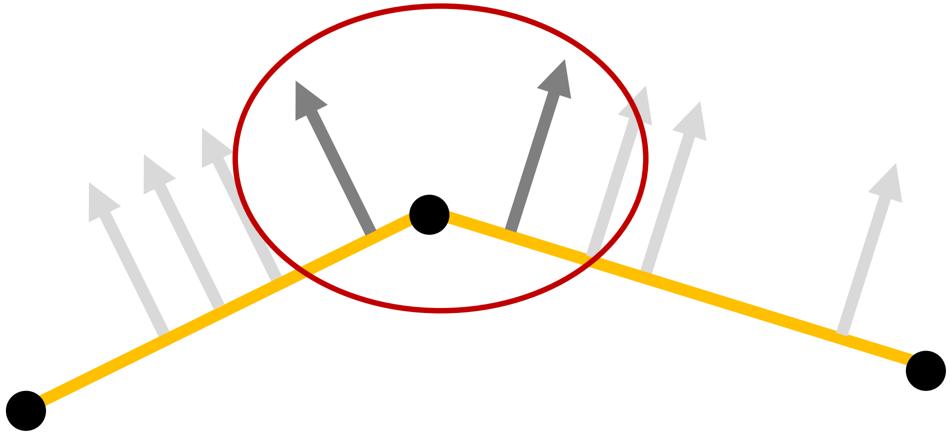
Curvature of a Discrete Curve

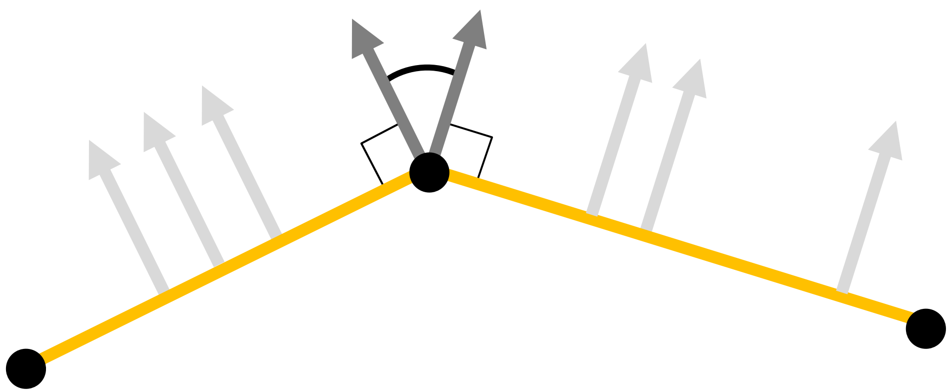
Again: change in normal direction



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

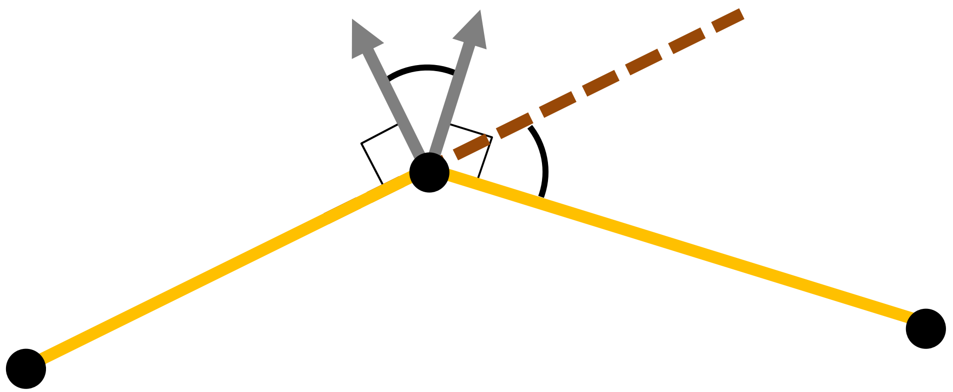
Again: change in normal direction



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

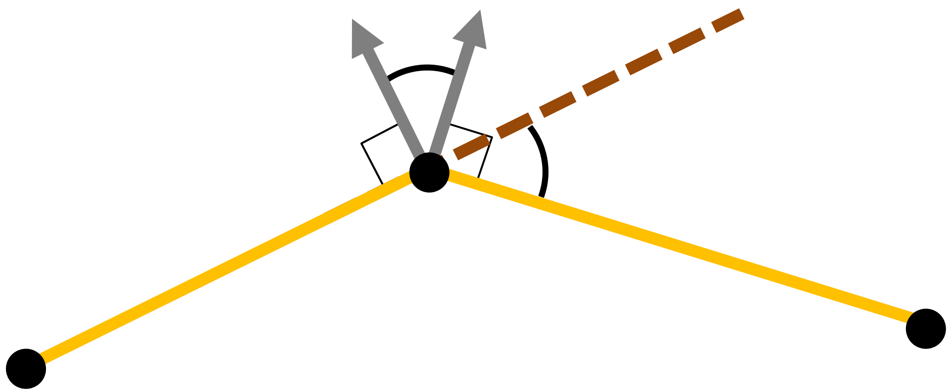
Again: change in normal direction



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

Again: change in normal direction

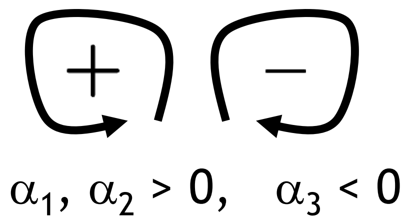


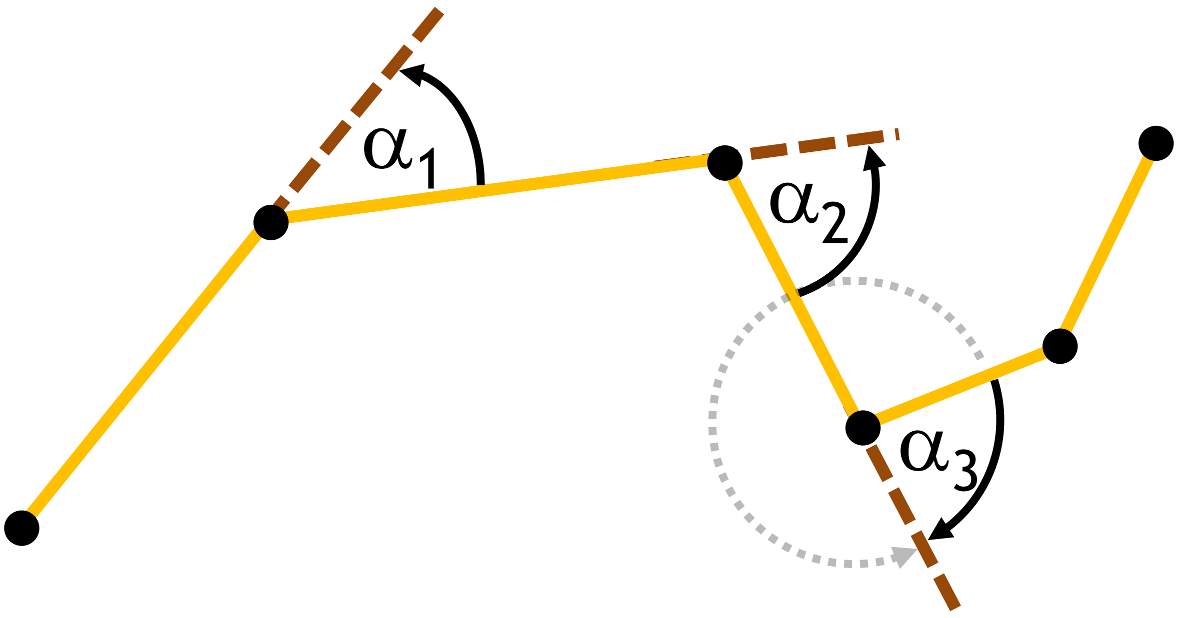
same as the turning angle
between the edges

Curvature of a Discrete Curve

Zero along the edges

Turning angle at the vertices
= the change in normal direction





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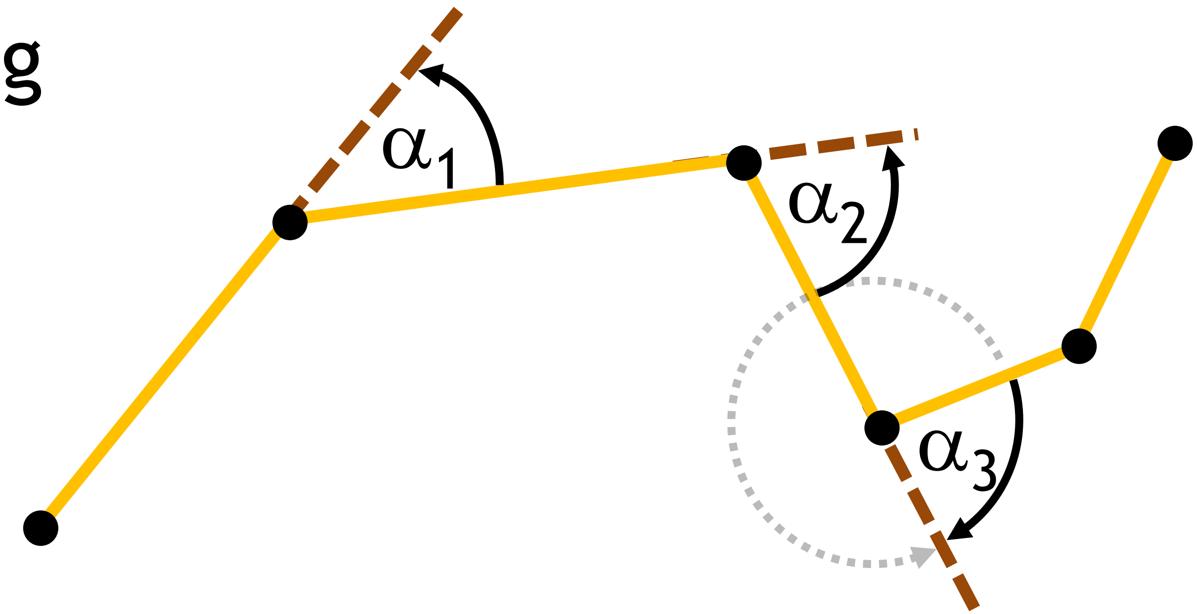
Total Signed Curvature

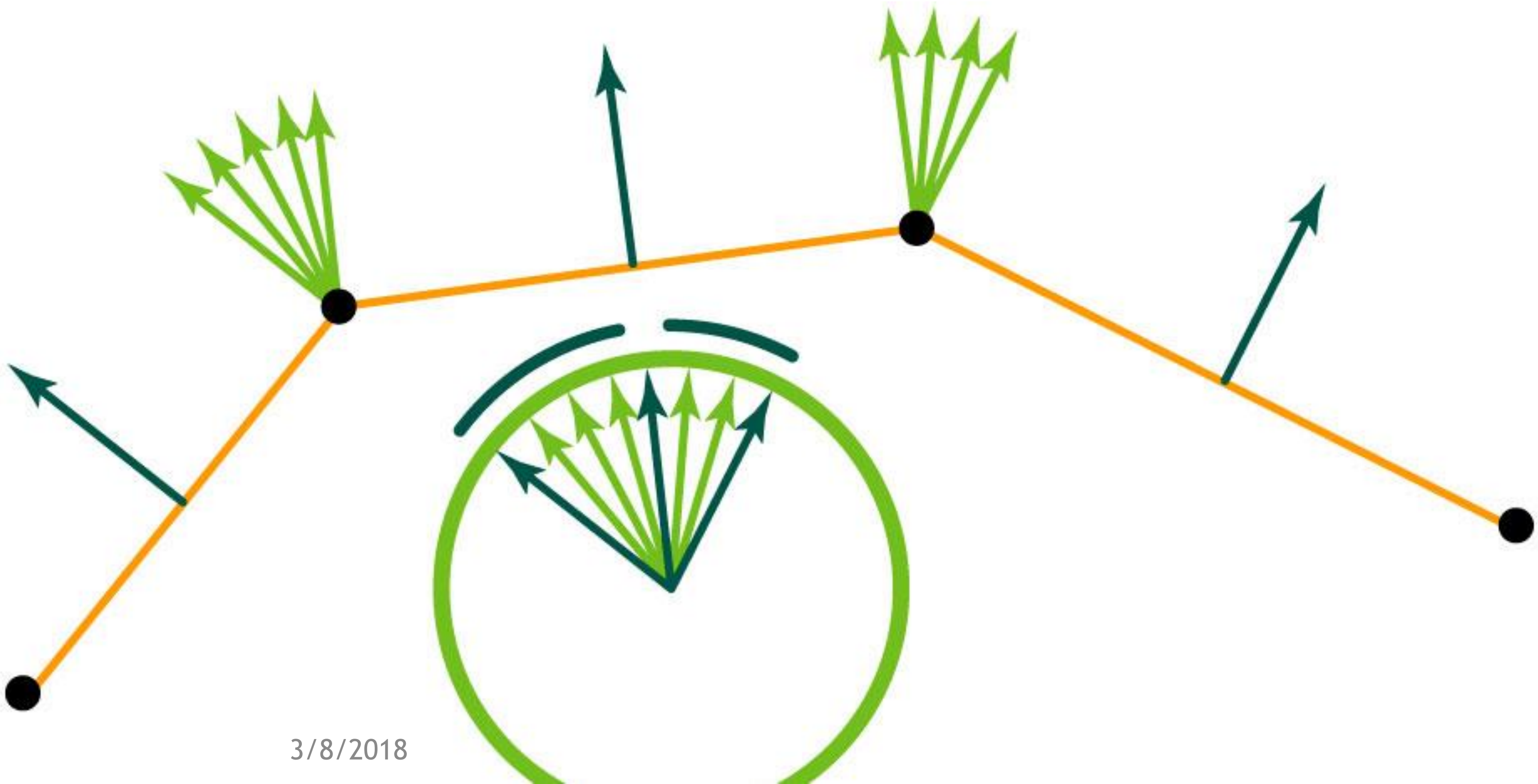
$$\mathrm{tsc}(p) = \sum_{i=1}^{n} \alpha_i$$

Sum of turning angles



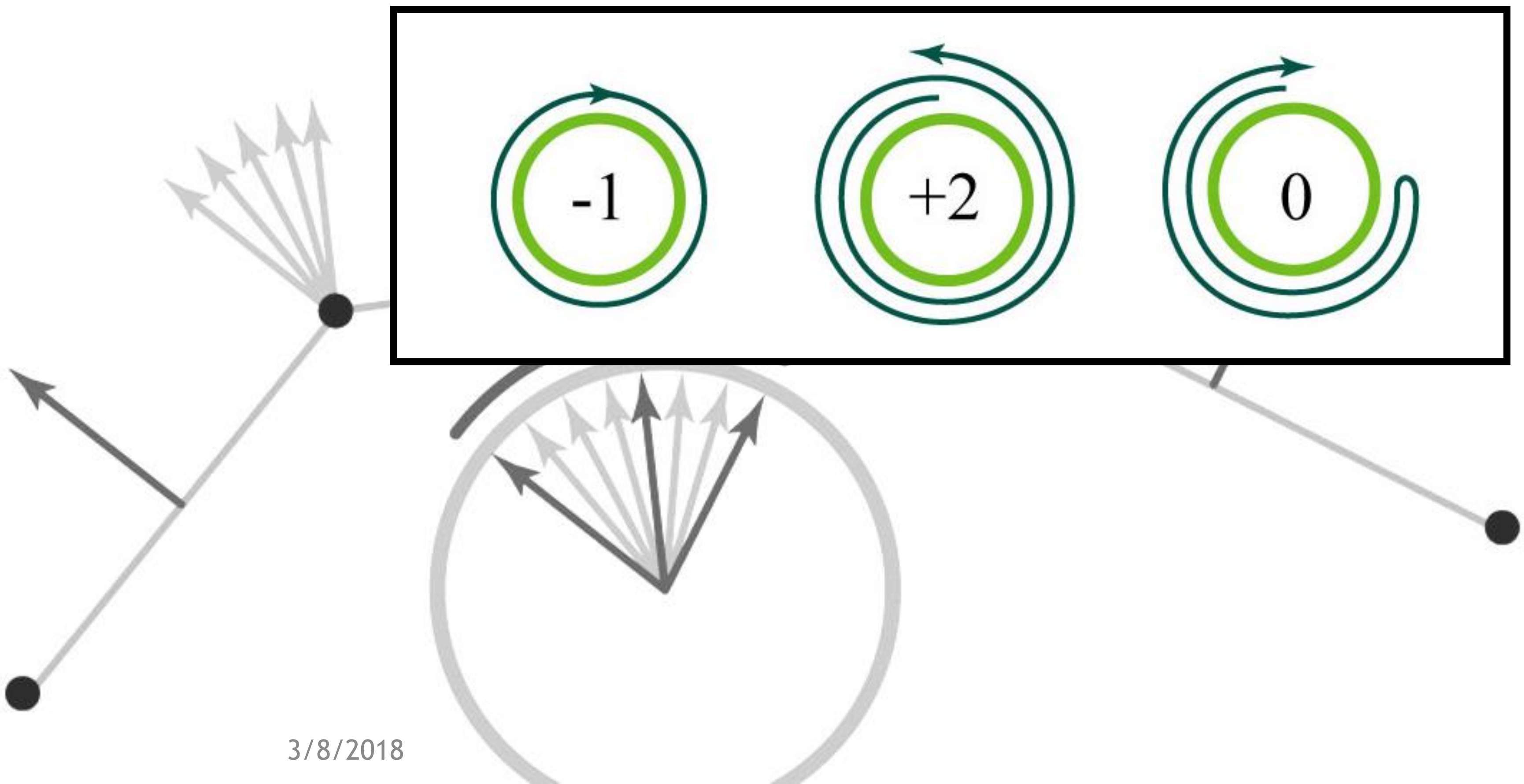
Discrete Gauss Map

Edges map to points, vertices map to arcs.



Discrete Gauss Map

Turning number well defined for discrete curves.

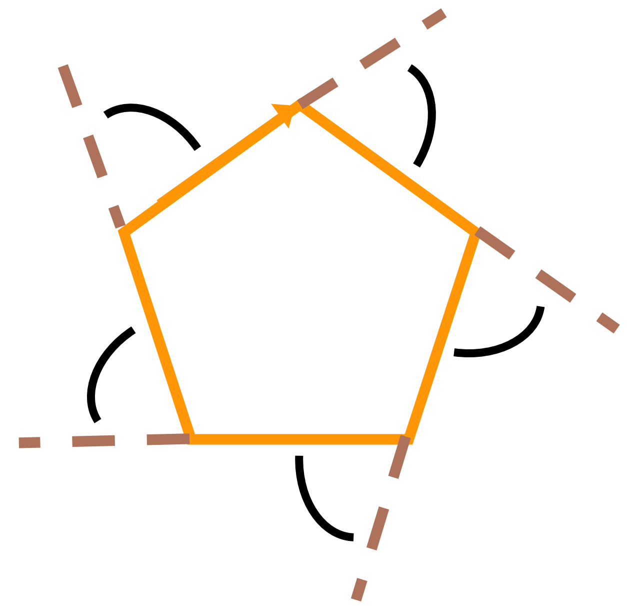


Discrete Turning Number Theorem

$$\mathrm{tsc}(p) = \sum_{i=1}^{n} \alpha_i = 2\pi k$$

For a closed curve,
the total signed curvature is
an integer multiple of 2π.

proof: sum of exterior angles



Turning Number Theorem

Continuous world

$$\int_\gamma \kappa \, dt = 2\pi k$$

Discrete world

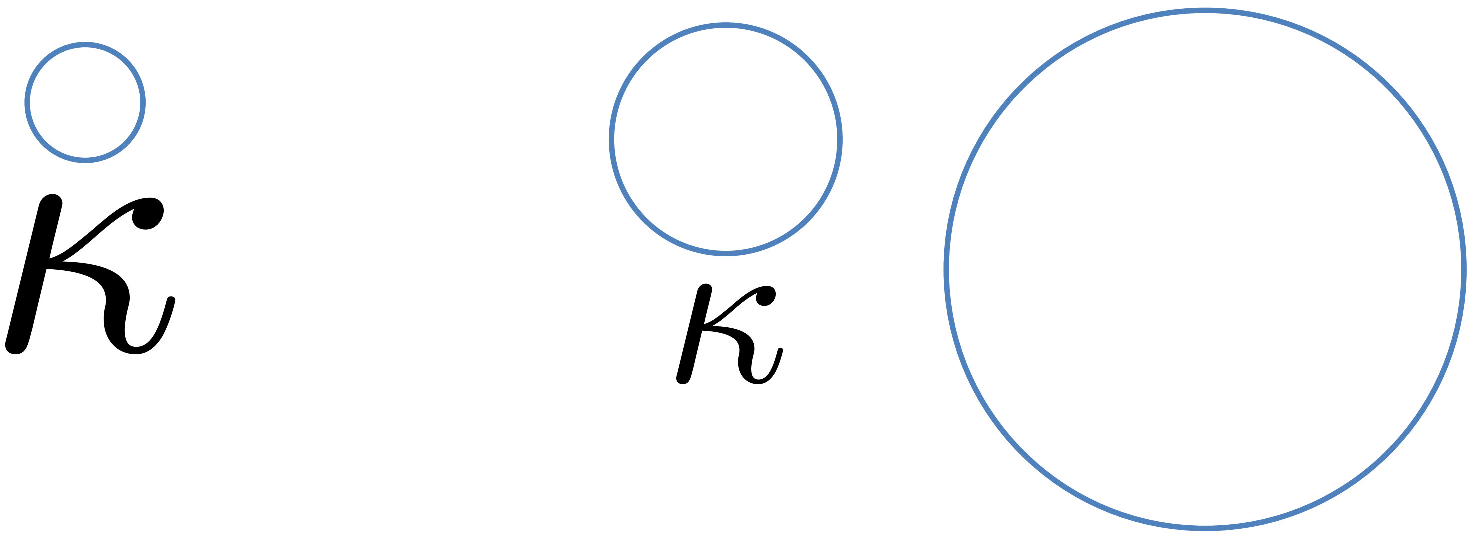
$$\sum_{i=1}^{n} \alpha_i = 2\pi k$$

k:

$$\kappa = \alpha_i \quad ??$$

Curvature is scale dependent

$$\kappa = \frac{1}{r}$$



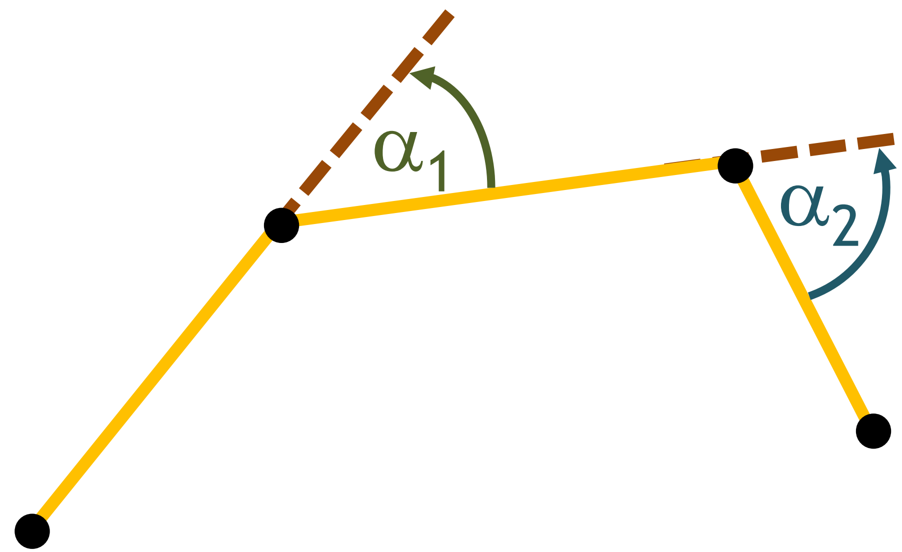
κ

α_i is scale-independent

Discrete Curvature – Integrated Quantity!

Cannot view α_i as pointwise curvature

It is *integrated curvature* over a local area associated with vertex i



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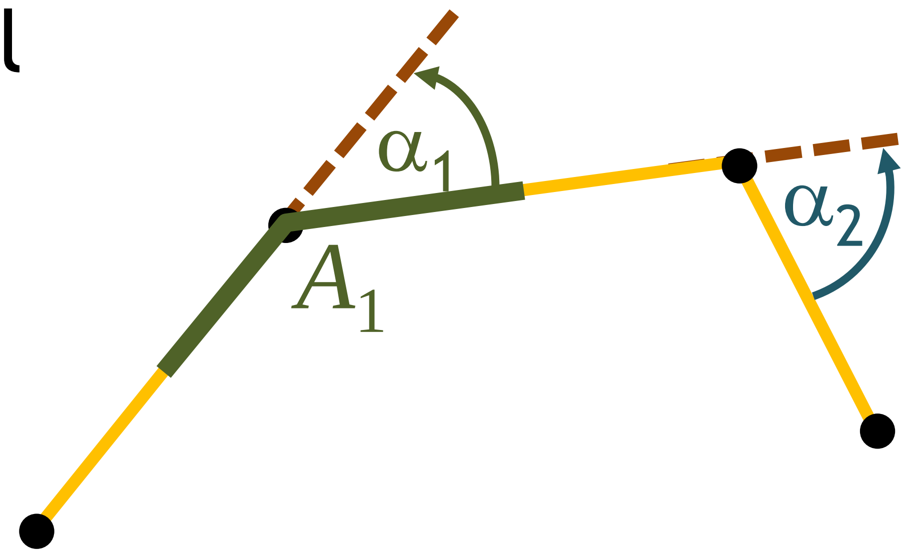
Discrete Curvature – Integrated Quantity!

Integrated over a local area associated with vertex *i*

$$\alpha_1 = A_1 \cdot \kappa_1$$



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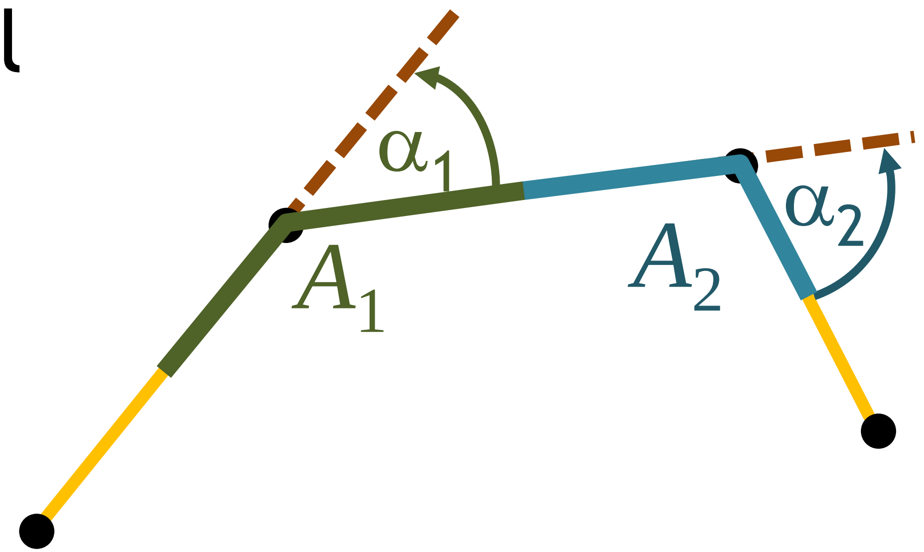
Discrete Curvature – Integrated Quantity!

Integrated over a local area associated with vertex *i*

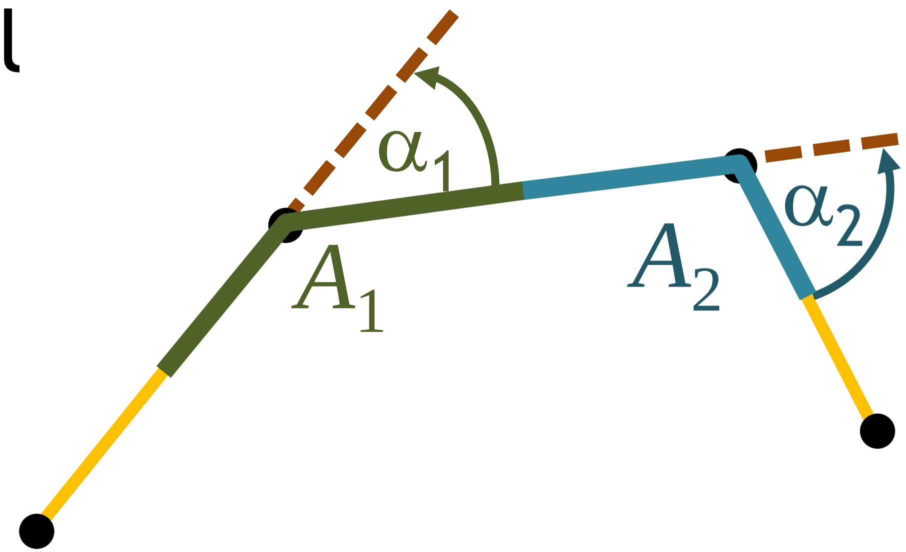


$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$

Discrete Curvature – Integrated Quantity!

Integrated over a local area associated with vertex *i*



$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$

$$\sum A_i = \text{len}(p)$$

The vertex areas A_i form a covering of the curve.
They are pairwise disjoint (except endpoints).

Discrete analogues

- Arbitrary discrete curve
  - total signed curvature obeys discrete turning number theorem
    - even coarse mesh (curve)
  - which continuous theorems to preserve?
    - that depends on the application…

Convergence

- length of sampled polygon approaches length of smooth curve
- in general, discrete measures approaches continuous analogues
- How to refine?
  - depends on discrete operator
  - pathological sequences may exist
  - in what sense does the operator converge?
    (pointwise, L_2; linear, quadratic)

Differential Geometry of Surfaces

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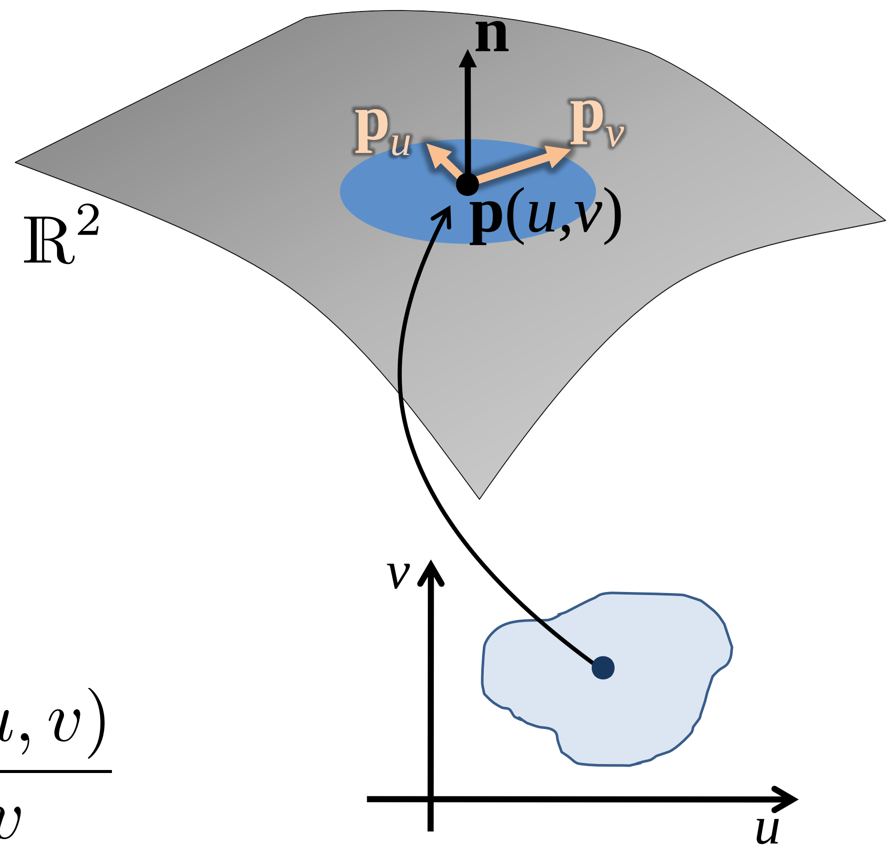
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Surfaces, Parametric Form

Continuous surface

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$



Tangent plane at point $\mathbf{p}(u,v)$ is spanned by

$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \quad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

These vectors don't have to be orthogonal

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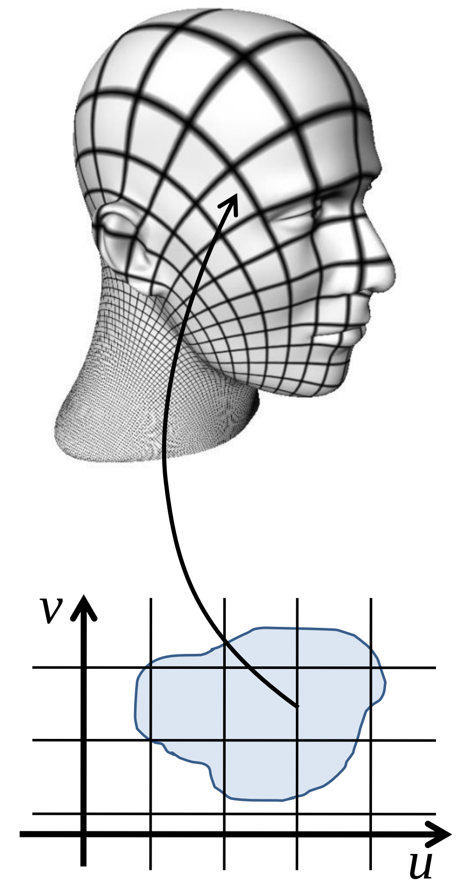
Isoparametric Lines

Lines on the surface when keeping one parameter fixed

$$\gamma_{u_0}(v) = \mathbf{p}(u_0, v)$$

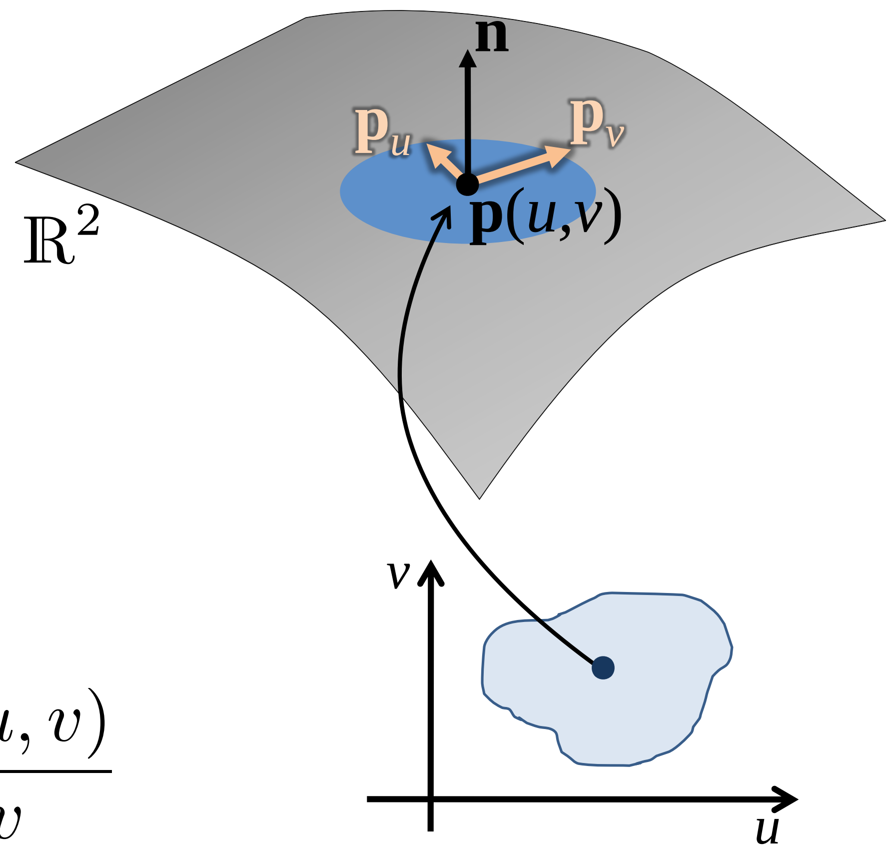
$$\gamma_{u_0}(u) = \mathbf{p}(u, v_0)$$



Surfaces, Parametric Form

Continuous surface

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \;\; (u,v) \in \mathbb{R}^2$$



Tangent plane at point $\mathbf{p}(u,v)$ is spanned by

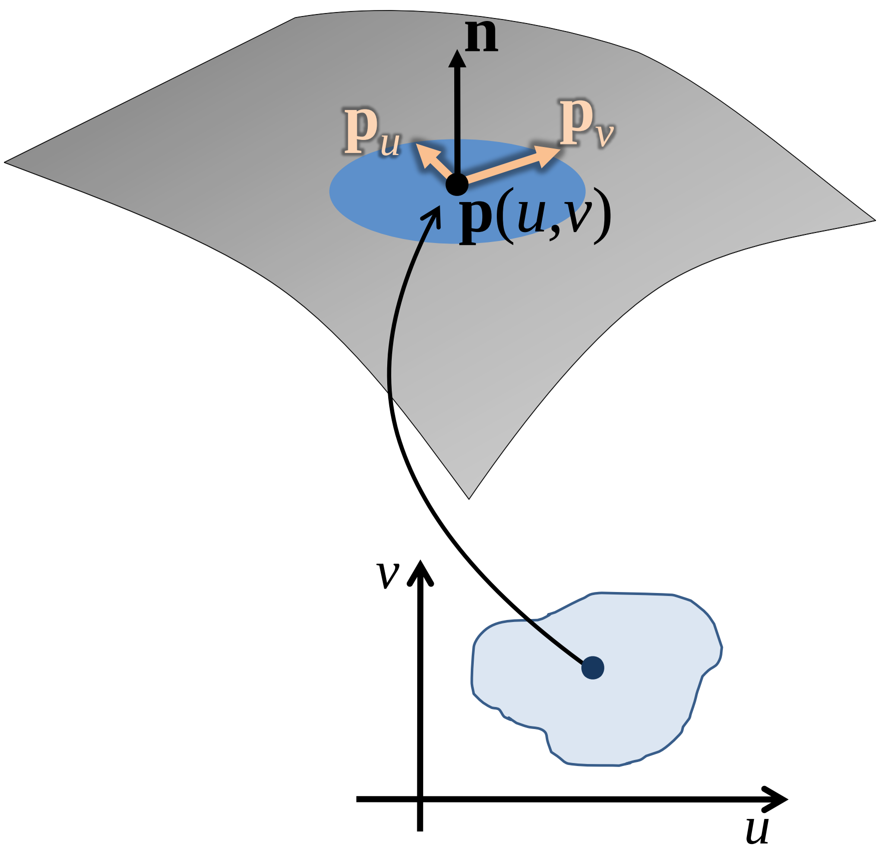
$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \quad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

These vectors don't have to be orthogonal

Surface Normals

Surface normal:

$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$



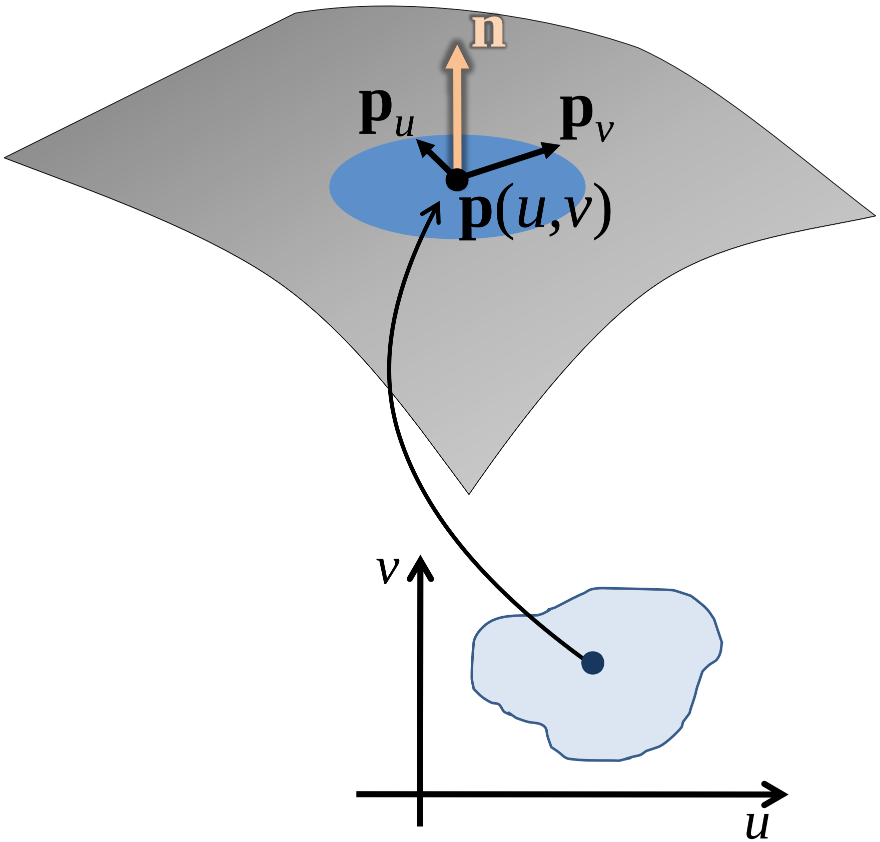
Surface Normals

Surface normal:

$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

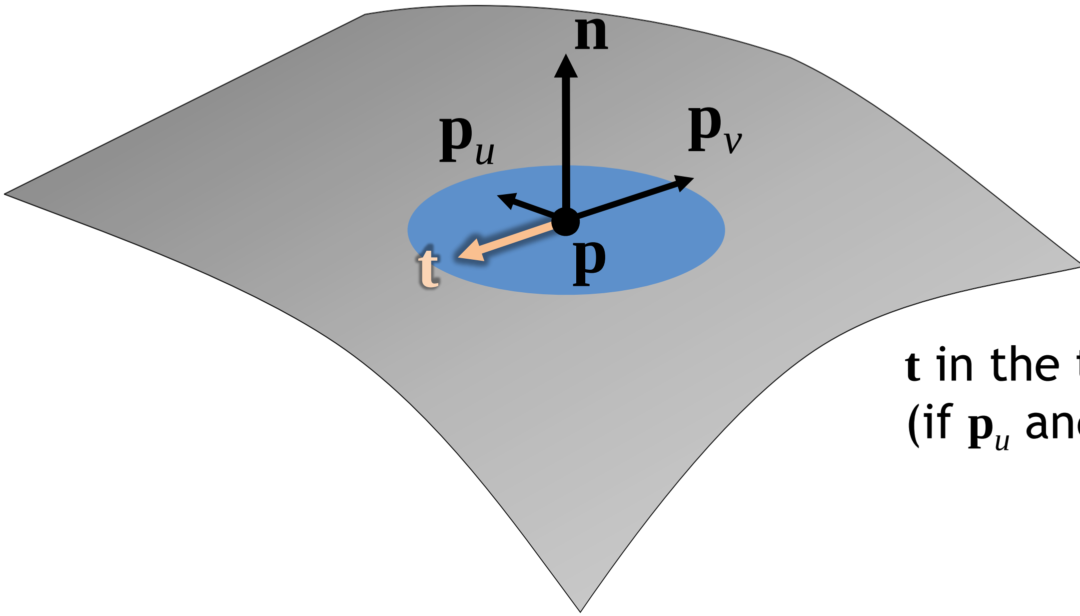
$$\mathbf{p}_u \times \mathbf{p}_v \neq 0$$

Regular parameterization



Normal Curvature

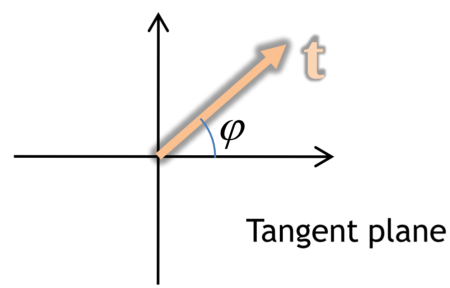
Normal Curvature



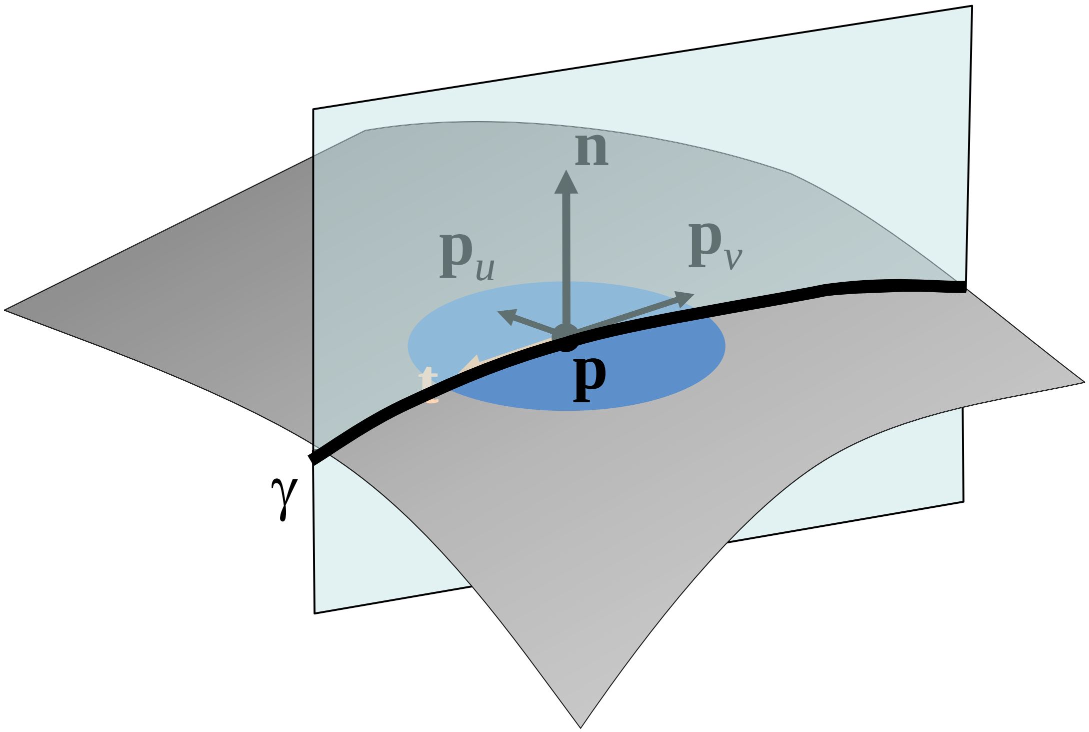
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

$\mathbf{t}$ in the tangent plane
(if $\mathbf{p}_u$ and $\mathbf{p}_v$ are orthogonal):

$$\mathbf{t} = \cos\varphi \frac{\mathbf{p}_u}{\|\mathbf{p}_u\|} + \sin\varphi \frac{\mathbf{p}_v}{\|\mathbf{p}_v\|}$$



Normal Curvature



The curve γ is the intersection of the surface with the plane through $\mathbf{n}$ and $\mathbf{t}$.

Normal curvature:

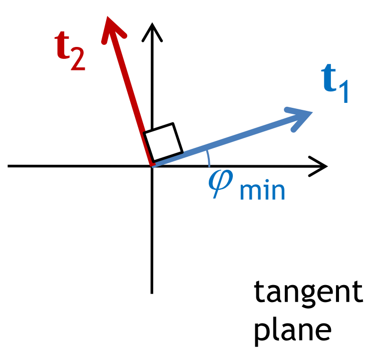
$$\kappa_n(\varphi) = \kappa(\gamma(\mathbf{p}))$$

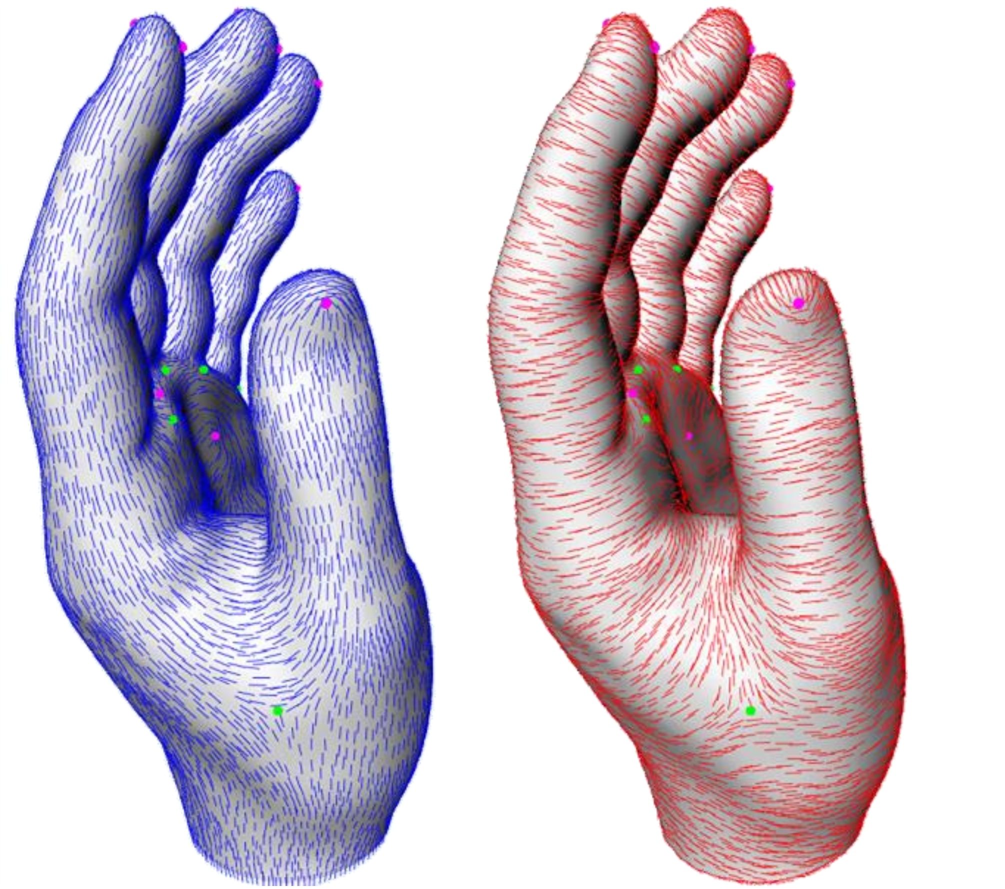
Surface Curvatures

- Principal curvatures
  - Minimal curvature $\kappa_1 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$
  - Maximal curvature $\kappa_2 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$
- Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\varphi) d\varphi$
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

Principal Directions

Principal directions: tangent vectors corresponding to φ_{max} and φ_{min}



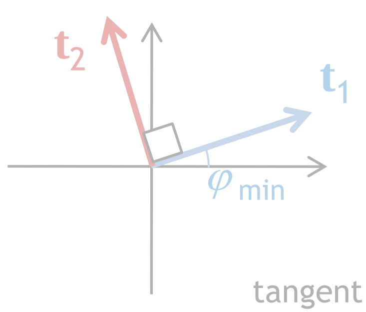


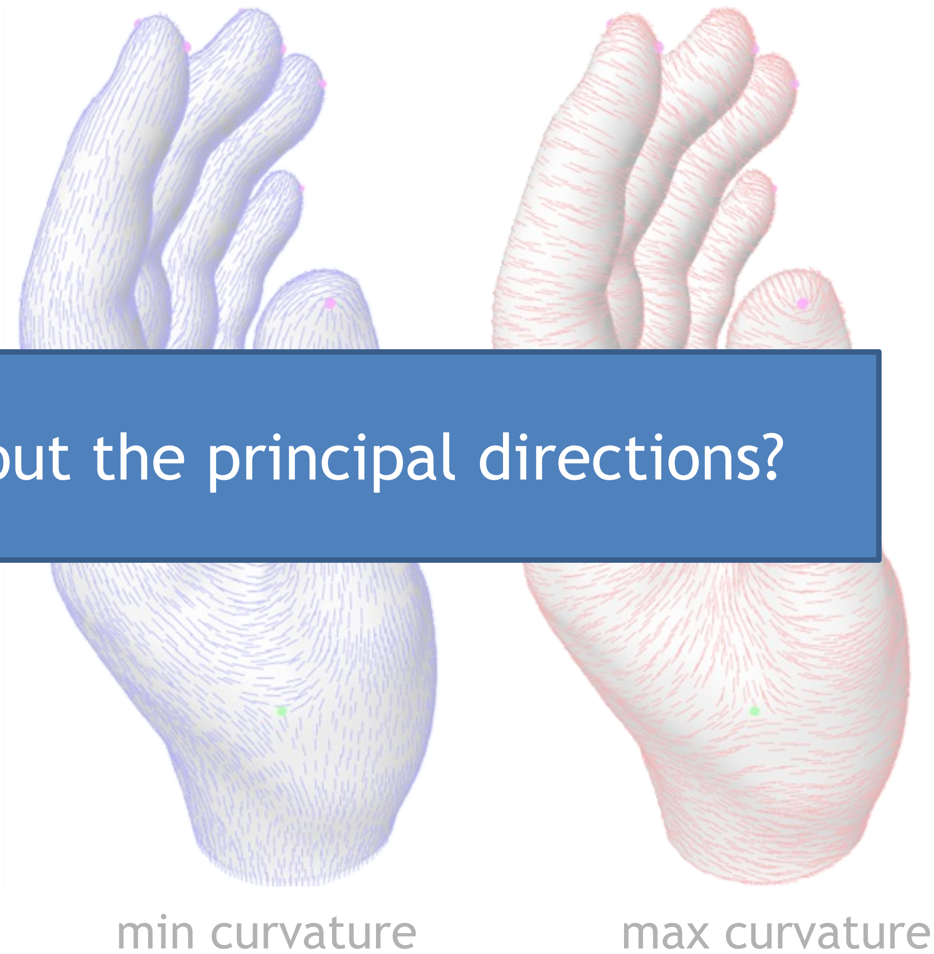
min curvature

max curvature

Principal Directions

Principal directions: tangent vectors corresponding to φ_{max} and φ_{min}





What can we say about the principal directions?

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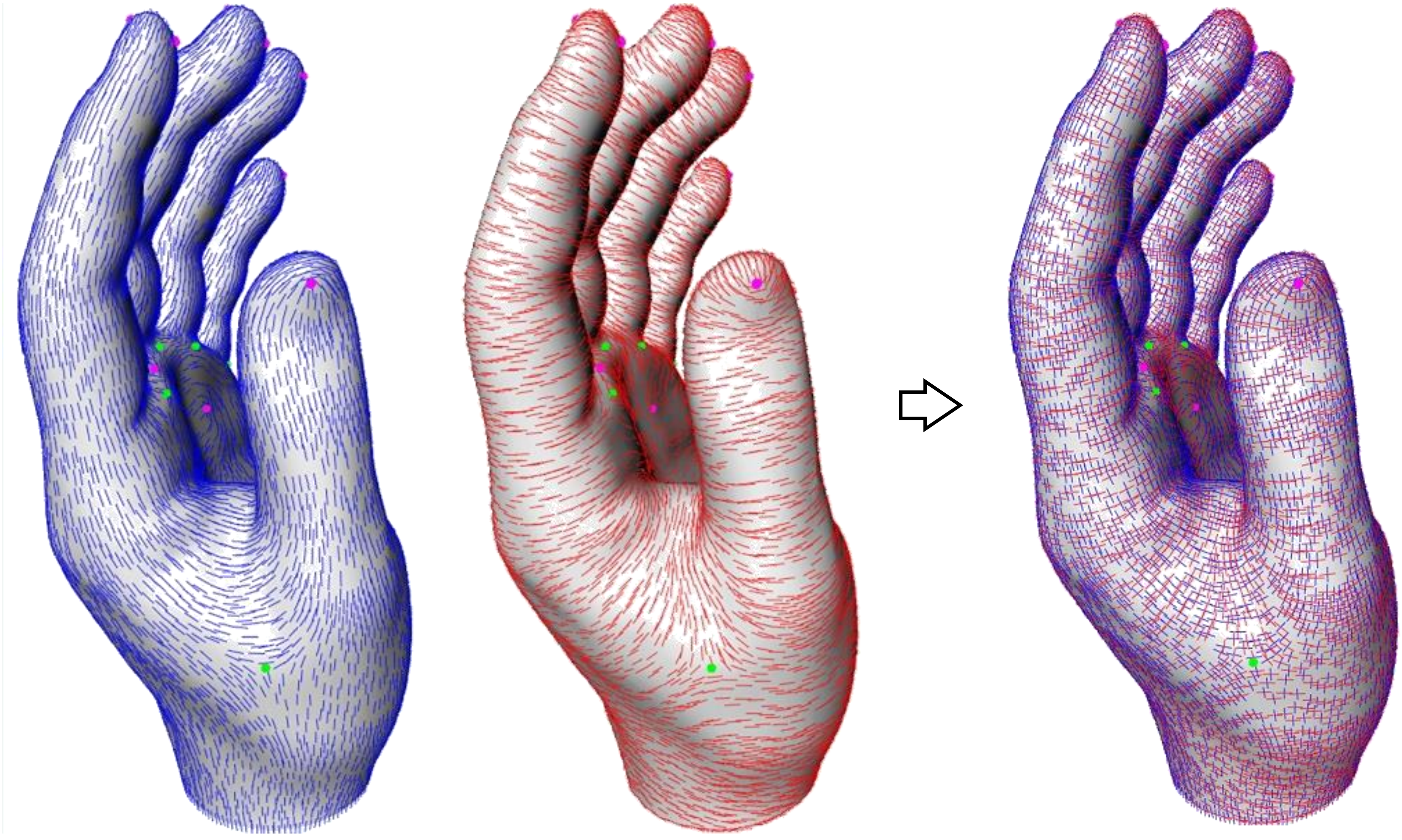
Principal Directions

Euler's Theorem: Principal directions are **orthogonal.**

$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi$$

Principal Directions



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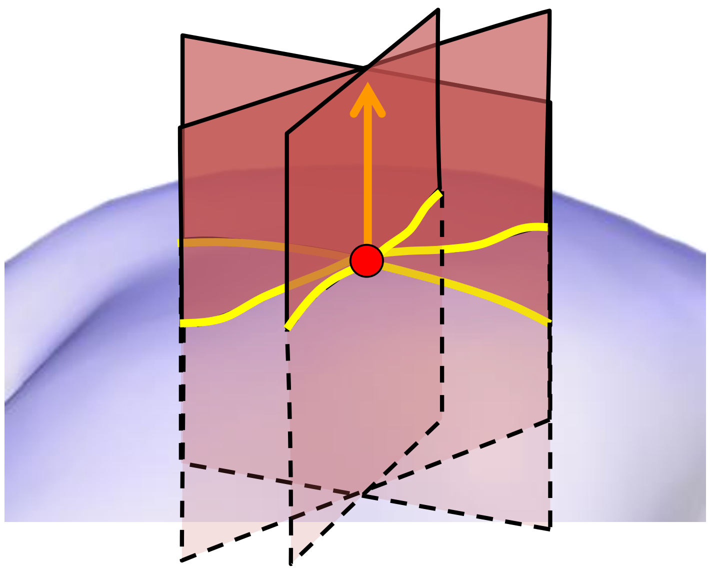
Mean Curvature

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\varphi)d\varphi$$

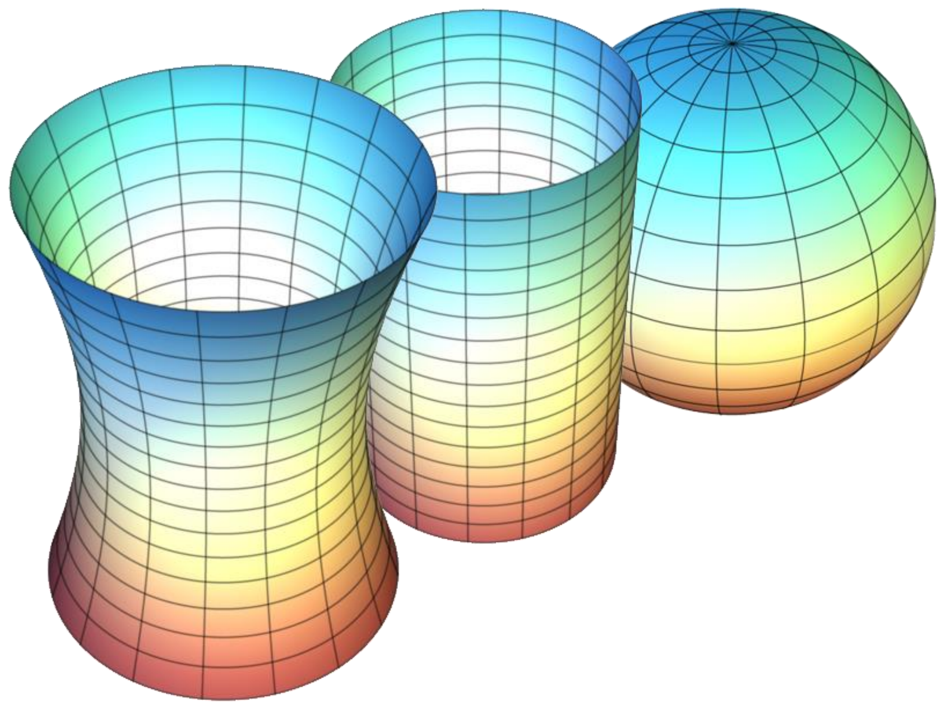


Gaussian Curvature

Classification $$K = \kappa_1 \cdot \kappa_2$$

A point $\mathbf{p}$ on the surface is called

- Elliptic, if $K > 0$
- Parabolic, if $K = 0$
- Hyperbolic, if $K < 0$

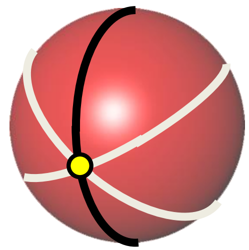


Developable surface
iff $K = 0$

Local Surface Shape By Curvatures

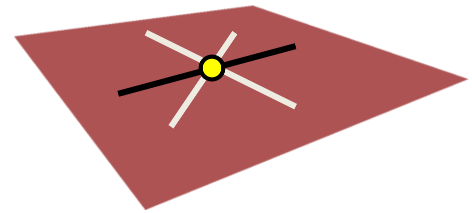
Isotropic:
all directions are principal directions

$K > 0, \kappa_1 = \kappa_2$



spherical (umbilical)

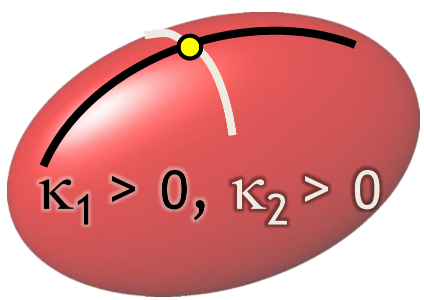
$K = 0$



planar

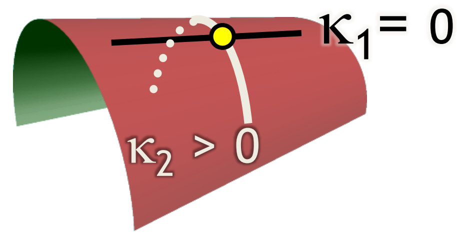
Anisotropic:
2 distinct principal directions

$K > 0$



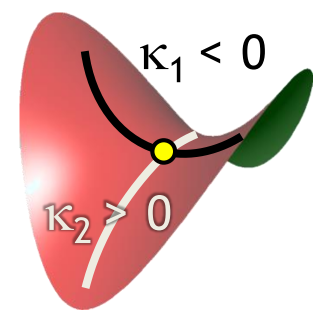
elliptic

$K = 0$



parabolic

$K < 0$



hyperbolic

Theorema Egregium

"Remarkable theorem"

$$K = \lim_{r \to 0^+} 3 \frac{2\pi r - C(r)}{\pi r^3}$$

Reminder: Euler-Poincaré Formula

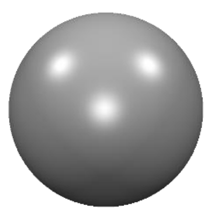
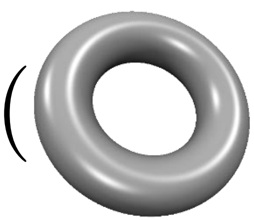
For orientable meshes:

$$v - e + f = 2(c - g) - b = \chi(M)$$

c = number of connected components

g = genus

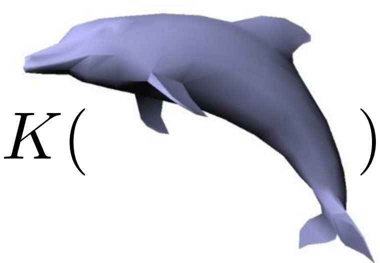
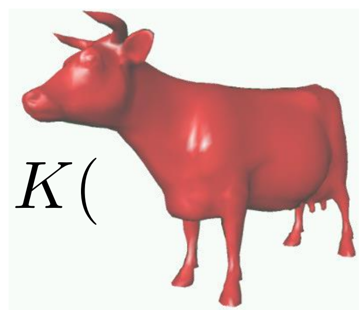
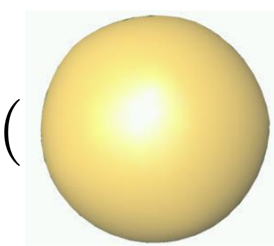
b = number of boundary loops

$\chi($  $) = 2$ $\chi($  $) = 0$

Gauss-Bonnet Theorem

For a closed surface M:

$$\int_{\mathcal{M}} K \, dA = 2\pi \, \chi(\mathcal{M})$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

Gauss-Bonnet Theorem

For a closed surface M:

$$\int_{\mathcal{M}} K\, dA = 2\pi\, \chi(\mathcal{M})$$

Compare with planar curves:

$$\int_{\gamma} \kappa\, ds = 2\pi\, k$$

Fundamental Forms

First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{p}_u^T \mathbf{p}_u & \mathbf{p}_u^T \mathbf{p}_v \\ \mathbf{p}_u^T \mathbf{p}_v & \mathbf{p}_v^T \mathbf{p}_v \end{pmatrix}$$

Fundamental Forms

I is a generalization of the dot product

allows to measure

- length, angles, area, curvature
- arc element

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2$$

- area element

$$dA = \sqrt{EG - F^2}\,dudv$$

Intrinsic Geometry

Properties of the surface that only depend on the first fundamental form

- length
- angles
- Gaussian curvature (Theorema Egregium)

Fundamental Forms

First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{p}_u^T \mathbf{p}_u & \mathbf{p}_u^T \mathbf{p}_v \\ \mathbf{p}_u^T \mathbf{p}_v & \mathbf{p}_v^T \mathbf{p}_v \end{pmatrix}$$

Second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{uu}^T \mathbf{n} & \mathbf{p}_{uv}^T \mathbf{n} \\ \mathbf{p}_{uv}^T \mathbf{n} & \mathbf{p}_{vv}^T \mathbf{n} \end{pmatrix}$$

Together, they define a surface (if some compatibility conditions hold)

Laplace Operator

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad \Delta f : \mathbb{R}^3 \to \mathbb{R}$$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f$$

function in Euclidean space

divergence operator

$$\mathrm{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \qquad \mathrm{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Laplace-Beltrami Operator

Extension of Laplace to functions on manifolds

$f : \mathcal{M} \to \mathbb{R} \qquad \Delta f : \mathcal{M} \to \mathbb{R}$

$$\Delta_{\mathcal{M}} f = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} f$$

Laplace-Beltrami

gradient operator

function on surface M

divergence operator

Laplace-Beltrami Operator

For coordinate functions: $\mathbf{p}(x, y, z) = (x, y, z)$

$$\Delta_{\mathcal{M}}\mathbf{p} = \mathrm{div}_{\mathcal{M}}\nabla_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n} \;\in \mathbb{R}^3$$

- $\Delta_{\mathcal{M}}$: Laplace-Beltrami
- $\nabla_{\mathcal{M}}$: gradient operator
- H: mean curvature
- $\mathbf{p}$: function on surface M
- $\mathrm{div}_{\mathcal{M}}$: divergence operator
- $\mathbf{n}$: unit surface normal

Differential Geometry on Meshes

Assumption: meshes are piecewise linear approximations of smooth surfaces

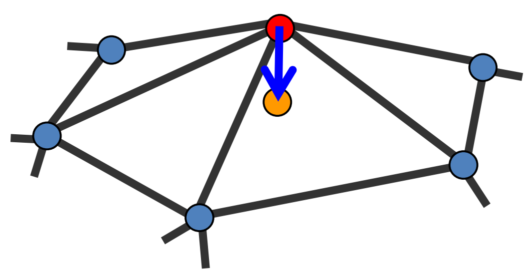
Can try fitting a smooth surface locally (say, a polynomial) and find differential quantities analytically

But: it is often too slow for interactive setting and error prone

Discrete Differential Operators

Approach: approximate differential properties at point $\mathbf{v}$ as spatial average over local mesh neighborhood $N(\mathbf{v})$ where typically

- $\mathbf{v}$ = mesh vertex
- $N_k(\mathbf{v})$ = k-ring neighborhood



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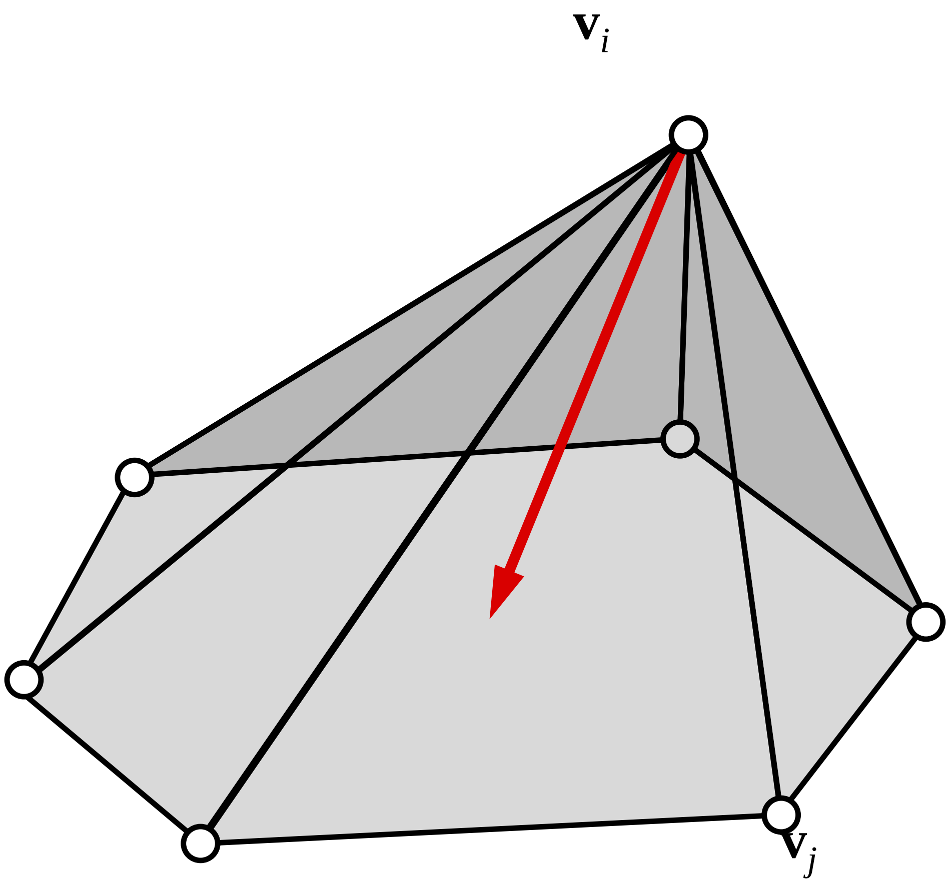
Discrete Laplace-Beltrami

$$\Delta_{\mathcal{M}} \mathbf{p} = -2H\mathbf{n}$$



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Discrete Laplace-Beltrami

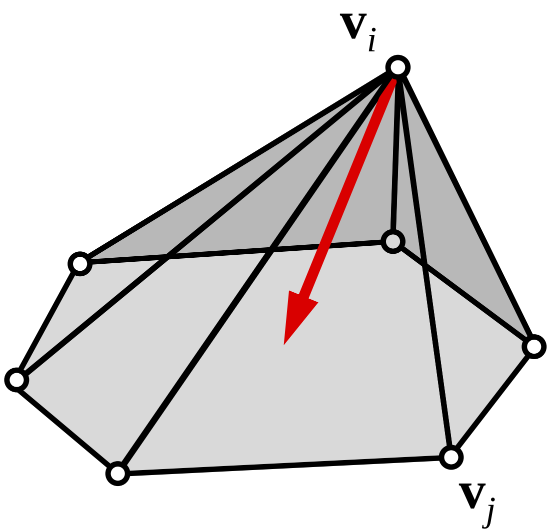
$$\boxed{\Delta_{\mathcal{M}} \mathbf{p} = -2H\mathbf{n}}$$

Uniform discretization: $L(\mathbf{v})$ or $\Delta\mathbf{v}$

$$L_u(\mathbf{v}_i) = \frac{1}{|\mathcal{N}(i)|} \sum_{j \in \mathcal{N}(i)} (\mathbf{v}_j - \mathbf{v}_i) = \left(\frac{1}{|\mathcal{N}(i)|} \sum_{j \in \mathcal{N}(i)} \mathbf{v}_j \right) - \mathbf{v}_i$$

Depends only on connectivity
= simple and efficient

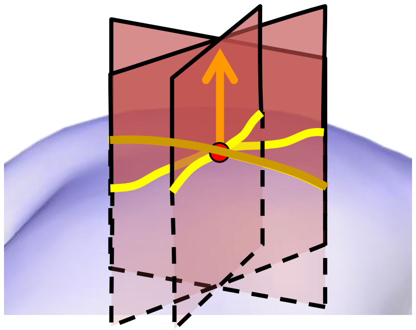
Bad approximation for
irregular triangulations



Discrete Laplace-Beltrami

$$\boxed{\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}}$$

Intuition for uniform discretization

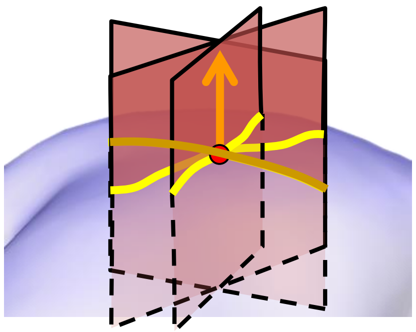


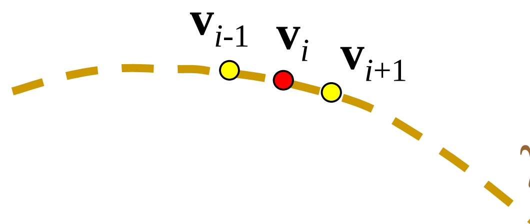
$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)d\varphi \qquad \boldsymbol{\kappa}\mathbf{n} = \gamma''$$

Discrete Laplace-Beltrami

$$\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}$$

Intuition for uniform discretization





$$H = \frac{1}{2\pi}\int_0^{2\pi}\kappa(\varphi)d\varphi \qquad \kappa\mathbf{n} = \gamma''$$

$$\gamma'' \approx \frac{1}{h}\left(\frac{\mathbf{v}_{i+1}-\mathbf{v}_i}{h} - \frac{\mathbf{v}_i-\mathbf{v}_{i-1}}{h}\right)$$

$$-2H\mathbf{n} = -2\left(\frac{1}{2\pi}\int_0^{2\pi}\kappa(\varphi)\,d\varphi\right)\mathbf{n} = -\frac{1}{\pi}\int_0^{2\pi}\kappa(\varphi)\mathbf{n}\,d\varphi = -\frac{1}{\pi}\int_0^{2\pi}\gamma''\,d\varphi$$

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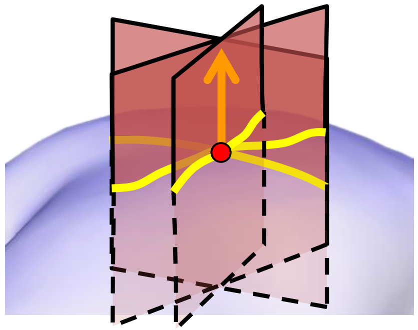
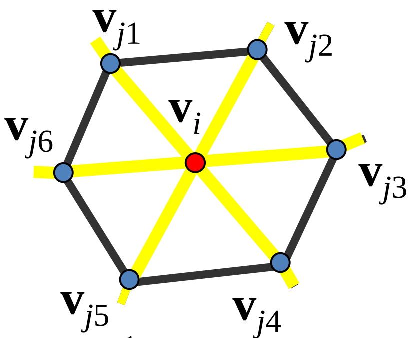
Discrete Laplace-Beltrami

$$\boxed{\Delta_{\mathcal{M}} \mathbf{p} = -2H\mathbf{n}}$$

Intuition for uniform discretization





$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)d\varphi$$

$$\frac{1}{2}(\mathbf{v}_{j_1} + \mathbf{v}_{j_4}) - \mathbf{v}_i \ +$$
$$\frac{1}{2}(\mathbf{v}_{j_2} + \mathbf{v}_{j_5}) - \mathbf{v}_i \ +$$
$$\frac{1}{2}(\mathbf{v}_{j_3} + \mathbf{v}_{j_6}) - \mathbf{v}_i =$$
$$= \frac{1}{2}\sum_{j \in \mathcal{N}(i)} \mathbf{v}_j - 3\mathbf{v}_i = 3\left(\frac{1}{6}\sum_{j \in \mathcal{N}(i)} \mathbf{v}_j - \mathbf{v}_i\right)$$

$\boxed{L_u(\mathbf{v}_i)}$

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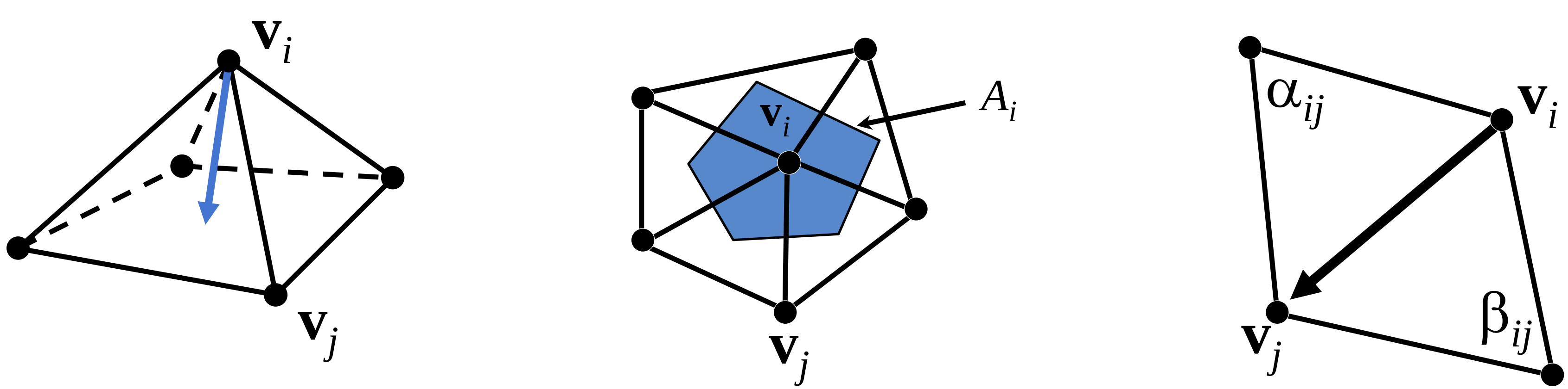
Discrete Laplace-Beltrami

Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{v}_j - \mathbf{v}_i)$$



Discrete Laplace-Beltrami

Cotangent formula

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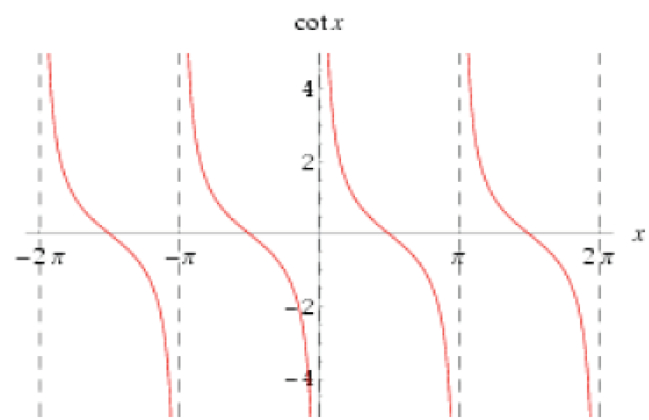
Accounts for mesh geometry

Potentially negative/ infinite weights



Discrete Laplace-Beltrami

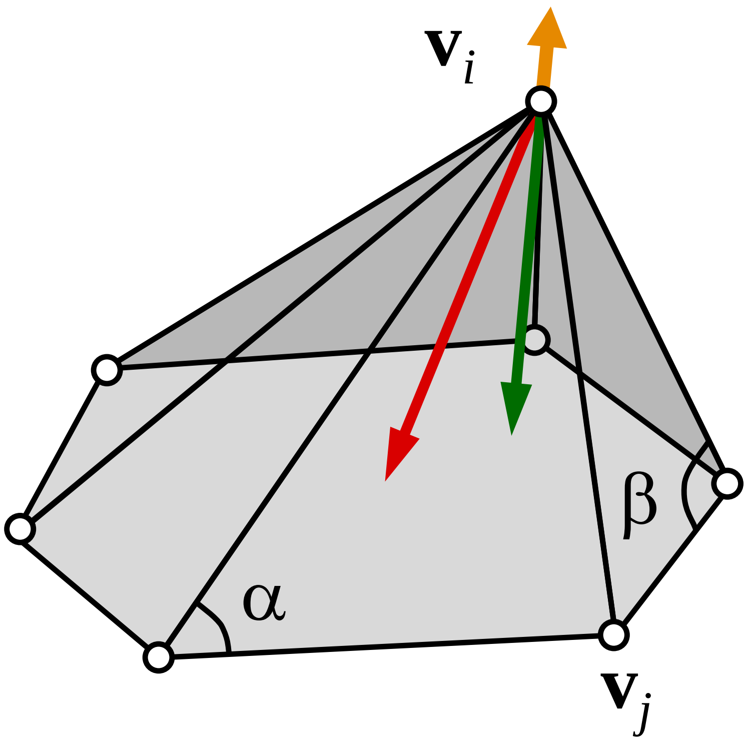
Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{v}_j - \mathbf{v}_i)$$

Can be derived using linear Finite Elements

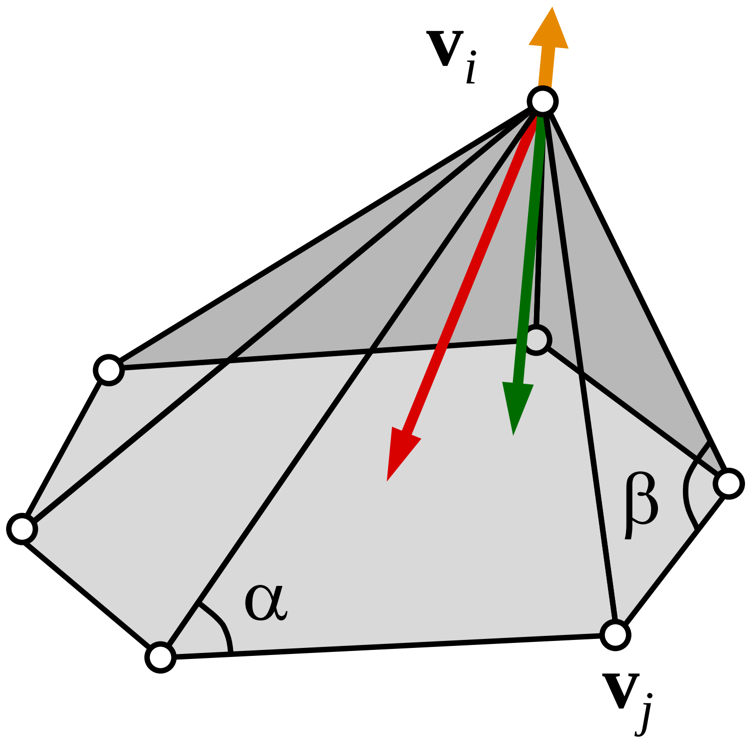
Nice property: gives zero for planar 1-rings!

Discrete Laplace-Beltrami



- Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
- Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
- Normal

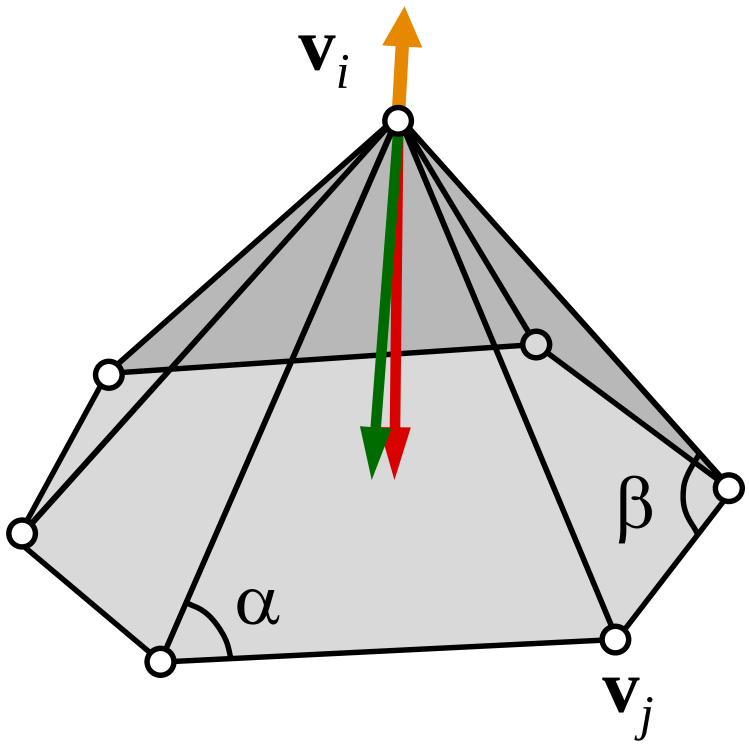
Discrete Laplace-Beltrami



- Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
- Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
- Normal

- For nearly equal edge lengths
  Uniform ≈ Cotangent

Discrete Laplace-Beltrami



- Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
- Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
- Normal

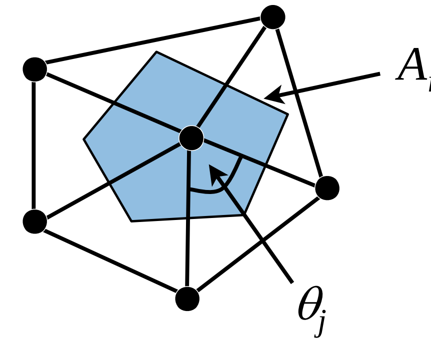
- **For nearly equal edge lengths**
  Uniform ≈ Cotangent

Cotan Laplacian allows computing discrete normal

Discrete Curvatures

Mean curvature

$$|H(\mathbf{v}_i)| = \|L_c(\mathbf{v}_i)\|/2$$



Gaussian curvature

$$K(\mathbf{v}_i) = \frac{1}{A_i}(2\pi - \sum_j \theta_j)$$

Principal curvatures

$$\kappa_1 = H - \sqrt{H^2 - K} \qquad \kappa_2 = H + \sqrt{H^2 - K}$$

Discrete Gauss-Bonnet Theorem

Total Gaussian curvature is fixed for a given topology

$$\int_{\mathcal{M}} K \, dA = 2\pi\chi(\mathcal{M})$$

Discrete Gauss-Bonnet Theorem

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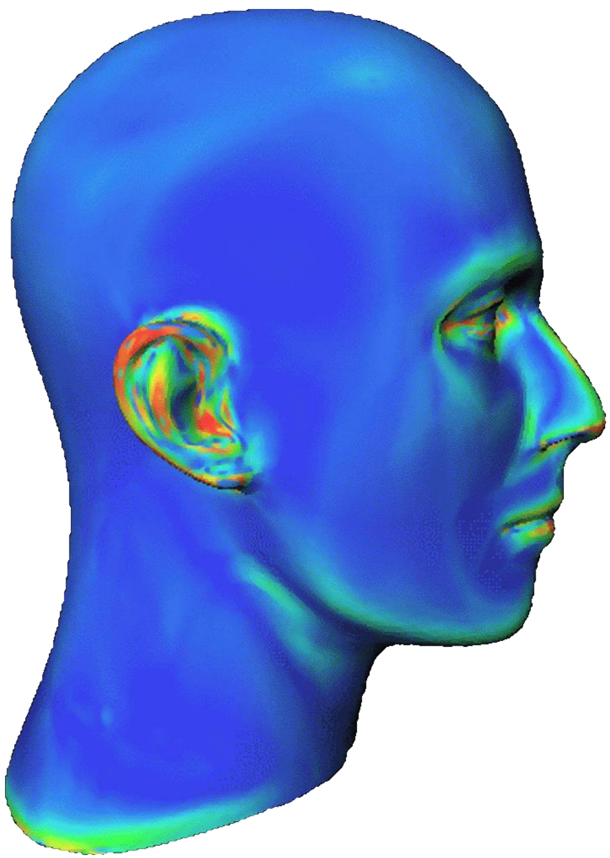
$$\int_{\mathcal{M}} K \, dA = \qquad 2\pi\chi(\mathcal{M})$$

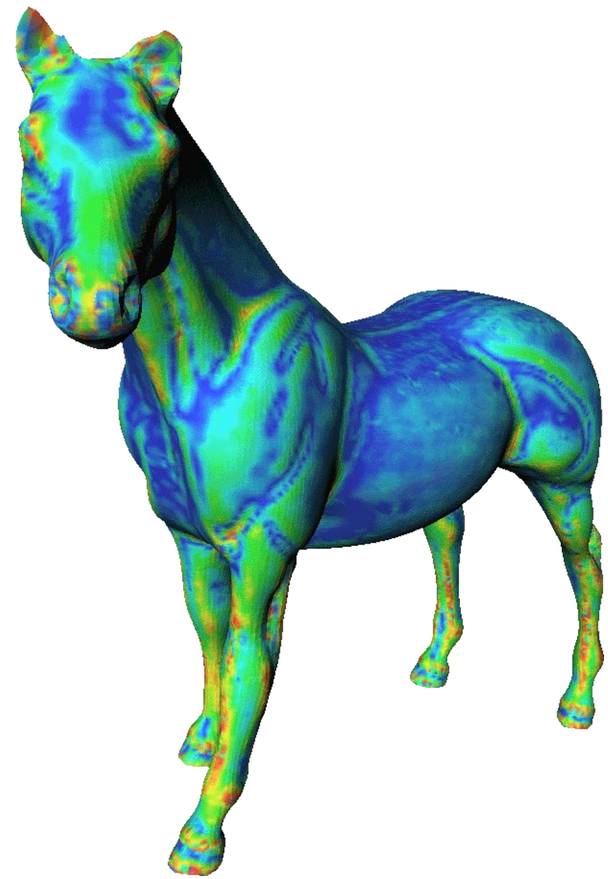
Discrete Gauss-Bonnet Theorem

Total Gaussian curvature is fixed for a given topology

$$\int_{\mathcal{M}} K \, dA = \sum_i A_i K(\mathbf{v}_i) = \sum_i \left[2\pi - \sum_{j \in \mathcal{N}(i)} \theta_j \right] == 2\pi\chi(\mathcal{M})$$

Example: Discrete Mean Curvature





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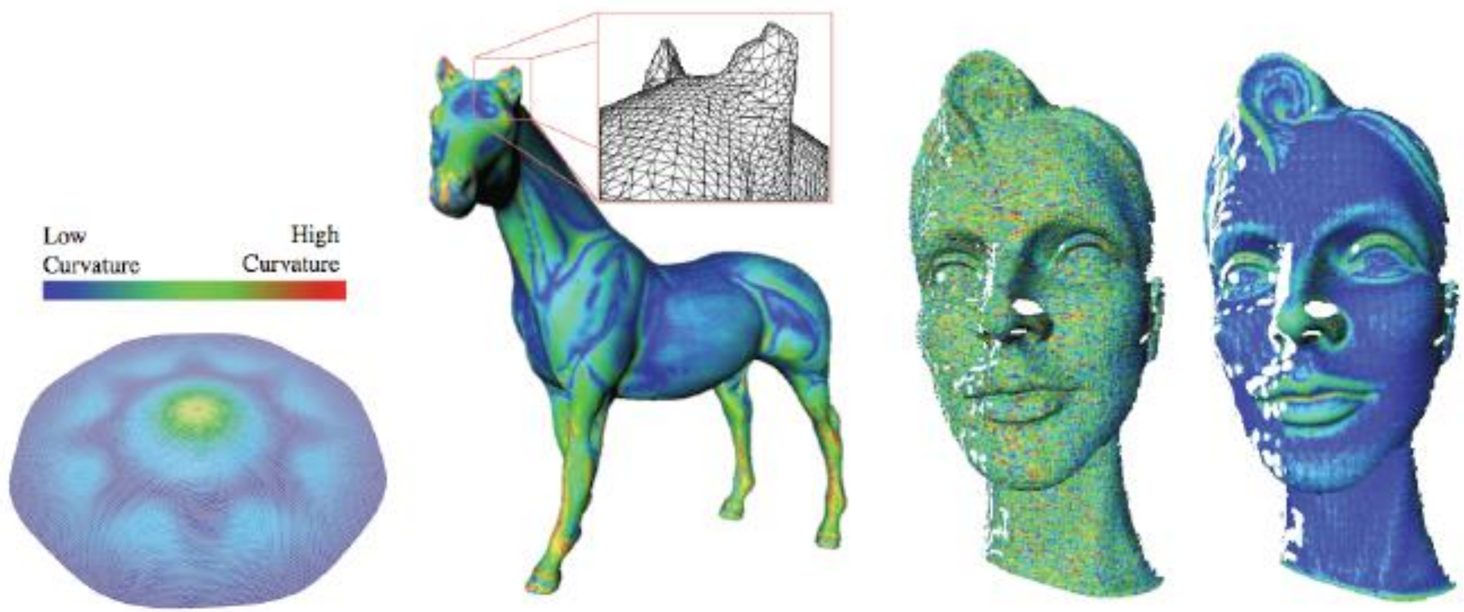
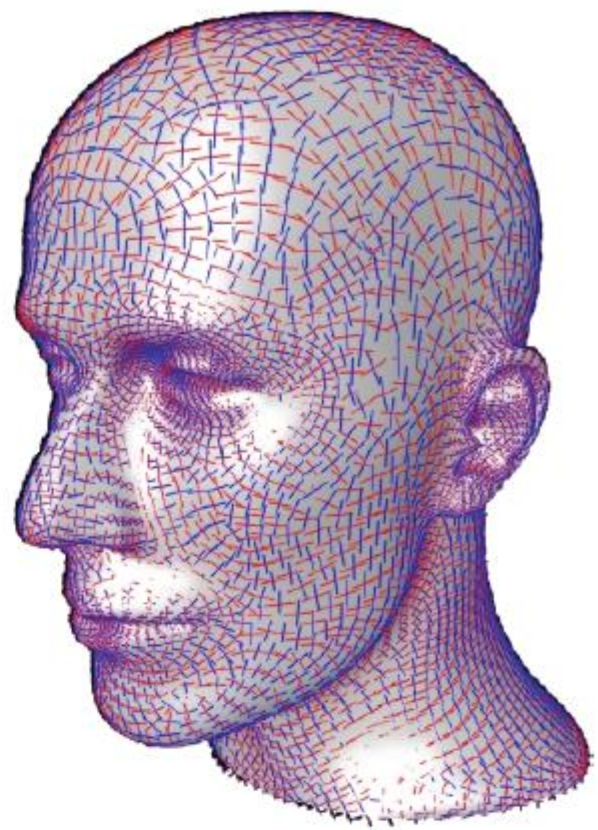
Links and Literature

- M. Meyer, M. Desbrun, P. Schroeder, A. Barr
  Discrete Differential-Geometry Operators for Triangulated 2-Manifolds, VisMath, 2002



Links and Literature

- libigl implements many discrete differential operators
- See the tutorial!
- http://libigl.github.io/libigl/tutorial/tutorial.html



principal directions

Thank You

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Swiss Federal Institute of Technology Zurich