252-0538-00L, Spring 2017

Shape Modeling and Geometry Processing

Space Deformations

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Space Deformation

Defined on the ambient space

$$\mathbf{d}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

Evaluate on points of shape embedded in space

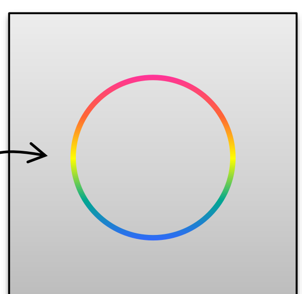
Displacement

$$\mathrm{x}' = x + d(\mathrm{x})$$

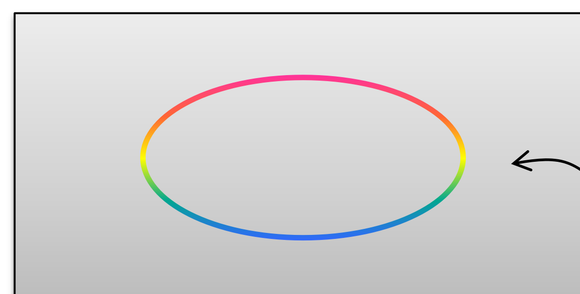
Mapping

$$\mathrm{x}' = d(\mathrm{x})$$

S

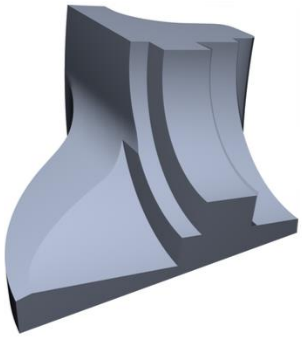


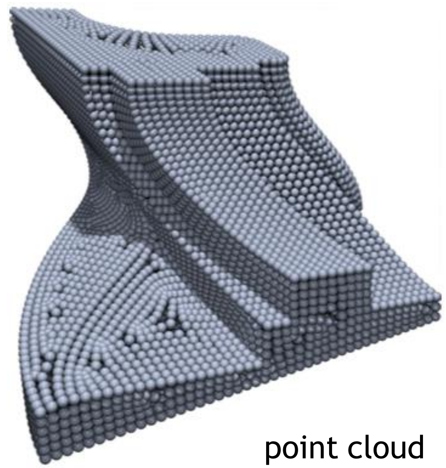
$$f\,(x, y) = (2x, y)$$



S'

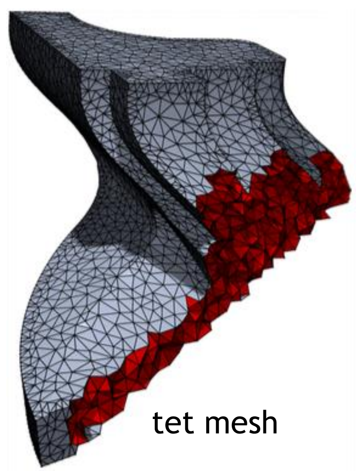
Works with many shape reps!



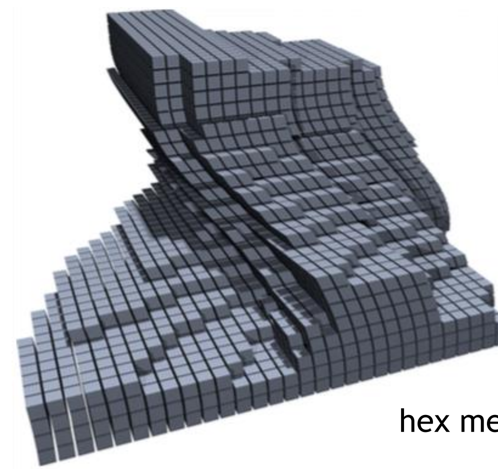


point cloud

triangle mesh



tet mesh



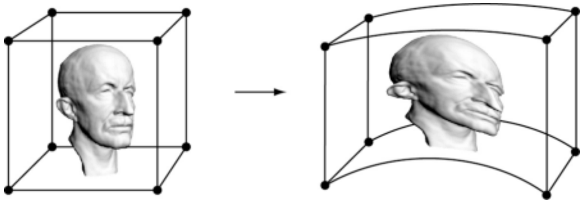
hex mesh

Daniel Sieger, PhD dissertation, 2016

ETH zürich

Freeform Deformation

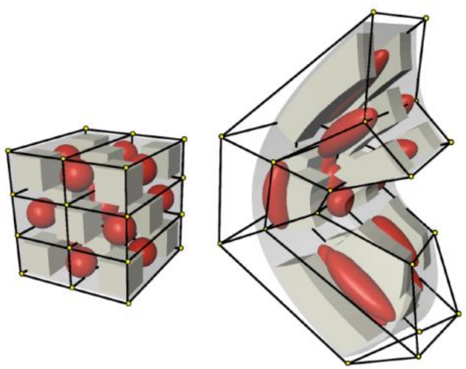
[Sederberg and Parry 86]

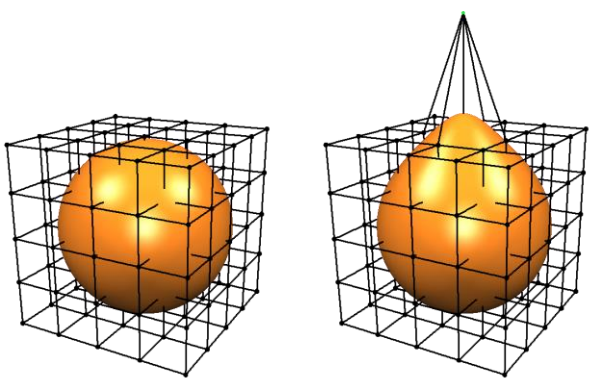


Control lattice

Basis functions $B_i(\mathbf{x})$ are trivariate tensor-product splines:

$$\mathbf{d}(x, y, z) = \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} \mathbf{d}_{ijk} N_i(x) N_j(y) N_k(z)$$





http://dl.acm.org/citation.cfm?id=15903

Volumetric Energy Minimization

[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]

Minimize similar energies to surface case

$$\int_{\mathbb{R}^3} \|\mathbf{d}_{xx}\|^2 + \|\mathbf{d}_{xy}\|^2 + \ldots + \|\mathbf{d}_{zz}\|^2 \; dx\,dy\,dz \to \min$$

How to minimize this in the space of all functions?

Radial Basis Functions

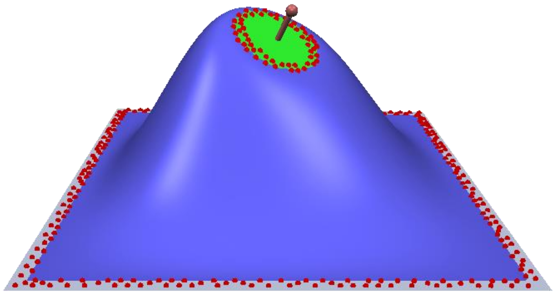
[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]

Represent deformation by RBFs

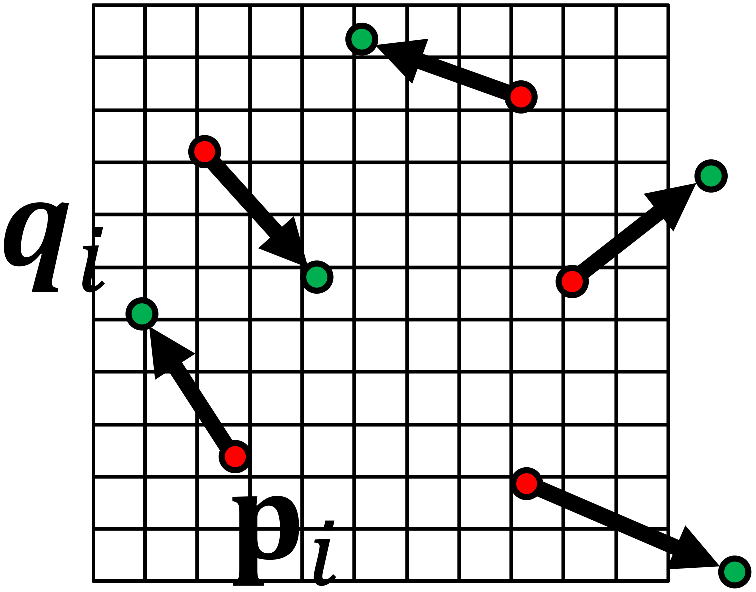
$$\mathbf{d}(\mathbf{x}) = \sum_j \mathbf{w}_j \, \varphi(\|\mathbf{c}_j - \mathbf{x}\|) + \mathbf{p}(\mathbf{x})$$

RBF fitting

- Interpolate constraints
- Solve linear system for $\mathbf{w}_j$ and $\mathbf{p}$



Deformation as an interpolation problem



$$\mathbf{f}(\mathbf{p}_i) = \sum \mathbf{w}_i f_i(\mathbf{x})$$

$$\mathbf{w}_i = ?$$

$$\mathbf{f}(\mathbf{p}_i) = \mathbf{q}_i, \forall i$$

$$\sum \mathbf{w}_i f_i(\mathbf{p}_i) = \mathbf{q}_i, \forall i$$

ETH zürich

Example: Thin Plate Spline

Solve the problem

$$\min E_{\mathrm{TPS}}(\mathbf{f}) = \iint \left[\left(\frac{\partial^2 \mathbf{f}}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 \mathbf{f}}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 \mathbf{f}}{\partial y^2} \right)^2 \right]$$

Bending energy

$$\text{s.t. } \mathbf{f}(\mathbf{p}_i) = \mathbf{q}_i, \forall i$$

General solution

$$\mathbf{f}(\mathbf{p}_i) = \mathbf{c}_0 + \mathbf{c}_x \boldsymbol{x} + \mathbf{c}_y \boldsymbol{y} + \sum \mathbf{c}_i \phi(\|\mathbf{x} - \mathbf{p}_i\|)$$

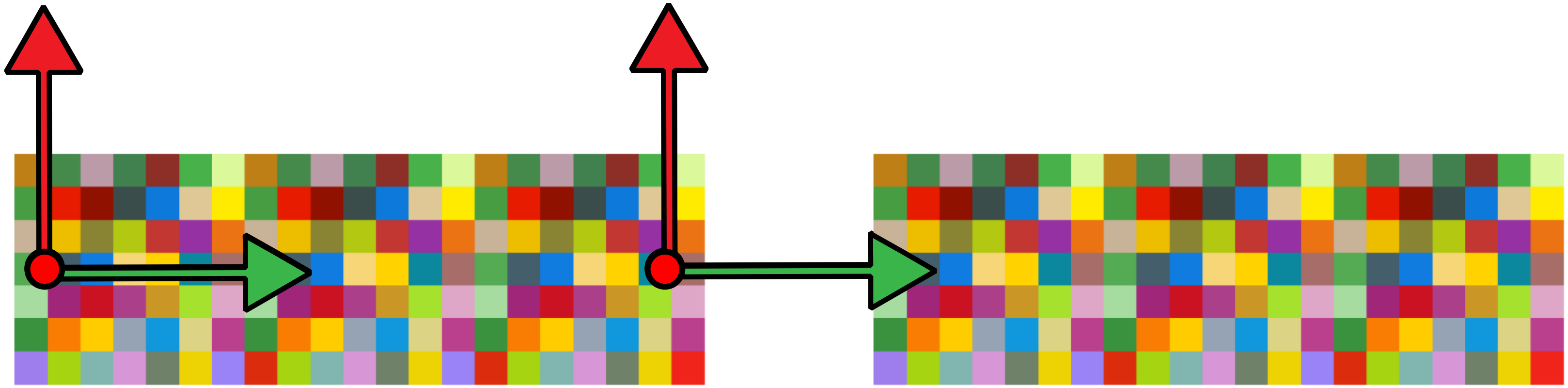
$$\phi(r) = r^2 \log r$$

Hermite interpolation

Interpolate derivatives

Hermite interpolation

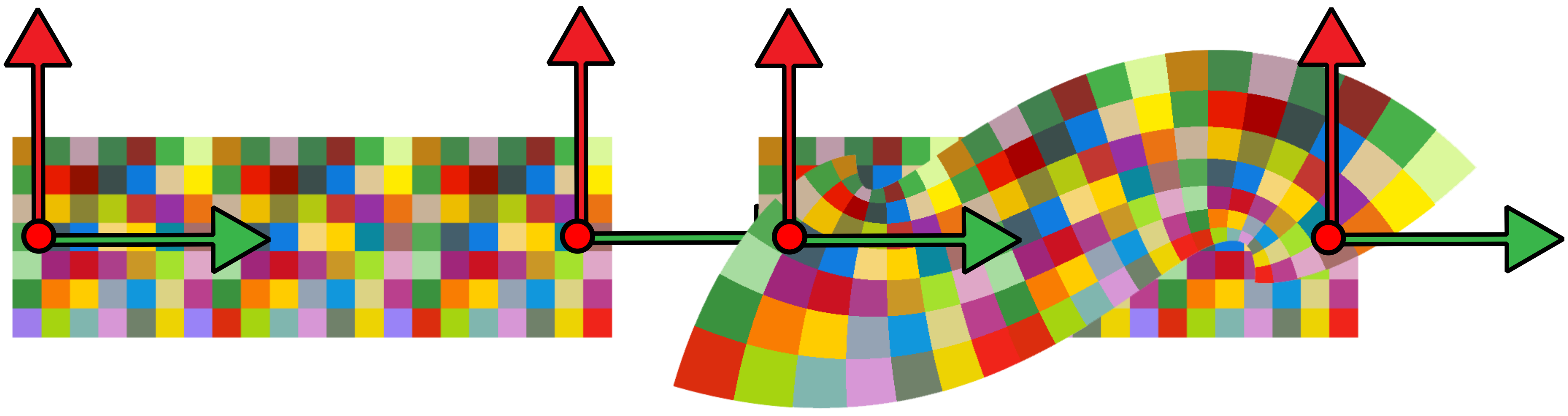
Interpolate derivatives



ETH zürich

Hermite interpolation

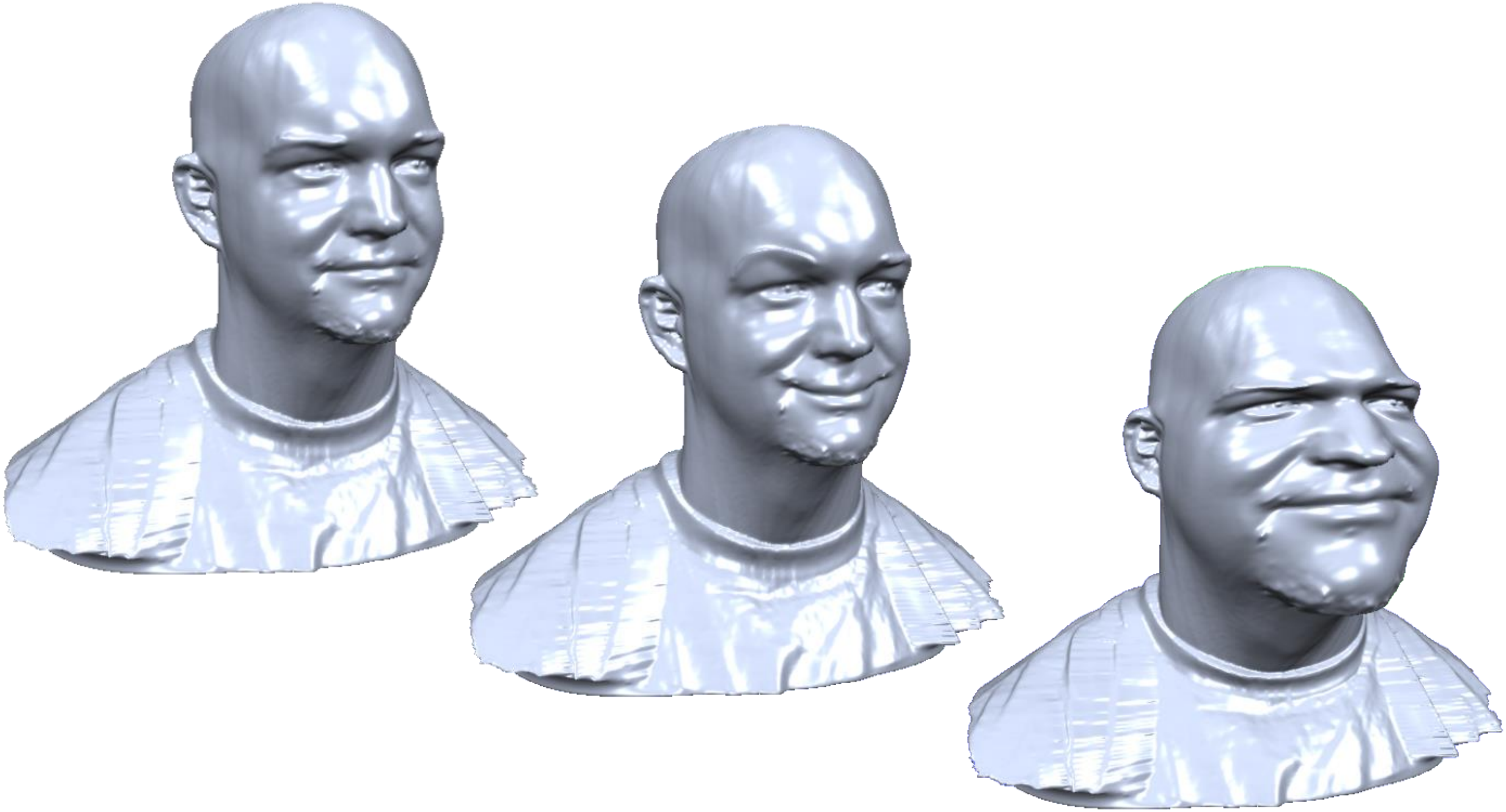
Interpolate derivatives



$$\sum \mathbf{c}_i f_i(\mathbf{p}_i) = \mathbf{q}_i \qquad \sum \mathbf{c}_i \nabla f_i(\mathbf{p}_i) = \mathbf{D}_i$$

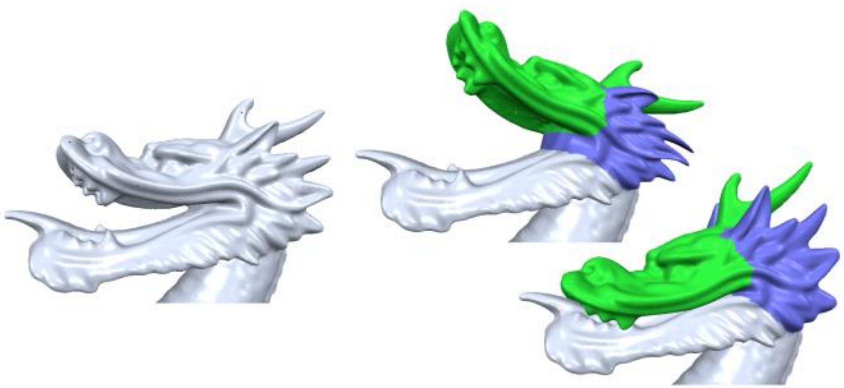
Local & Global Deformations

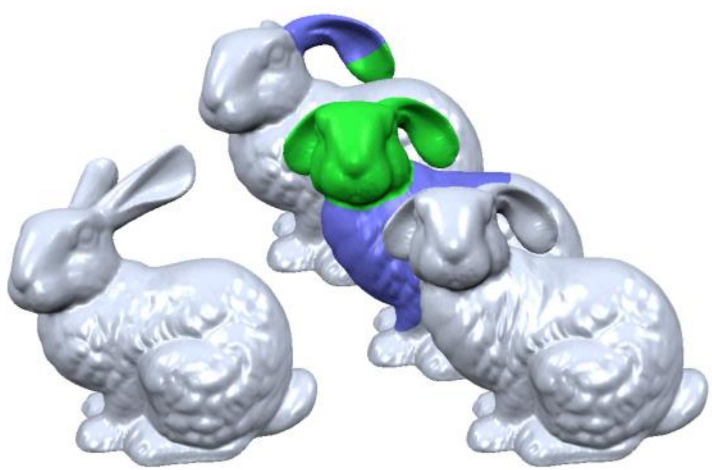
[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]

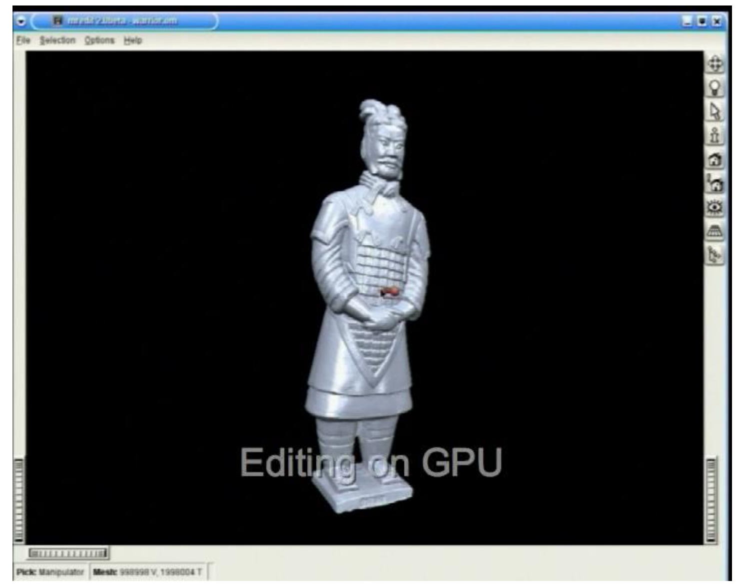


Local & Global Deformations

[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]



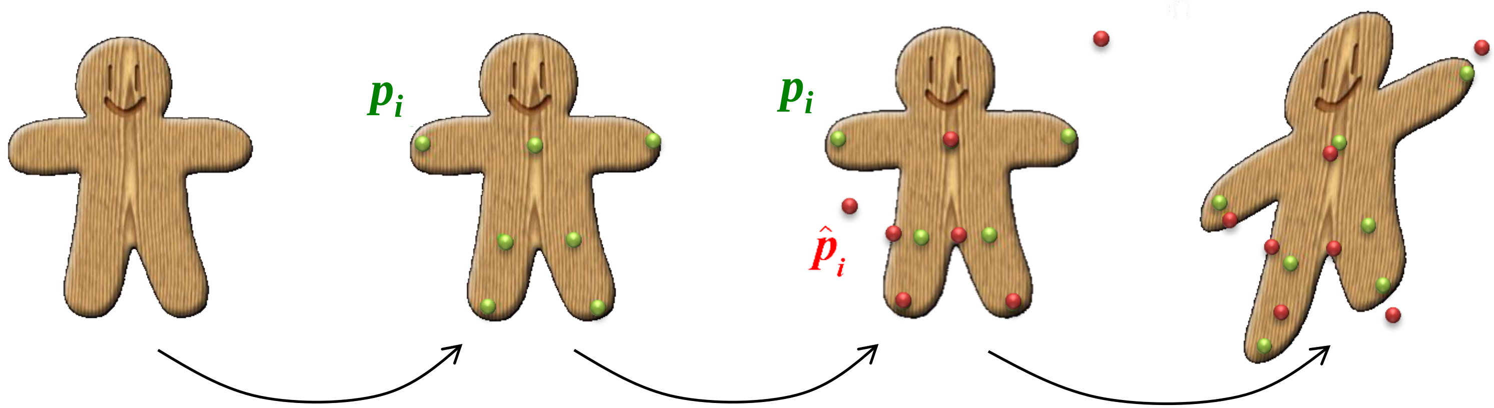




1M vertices
movie

MLS Deformation

[Schaeffer et al. '06]



1. Handles $\boldsymbol{p}_i$ 2. Target locations $\hat{p}_i$

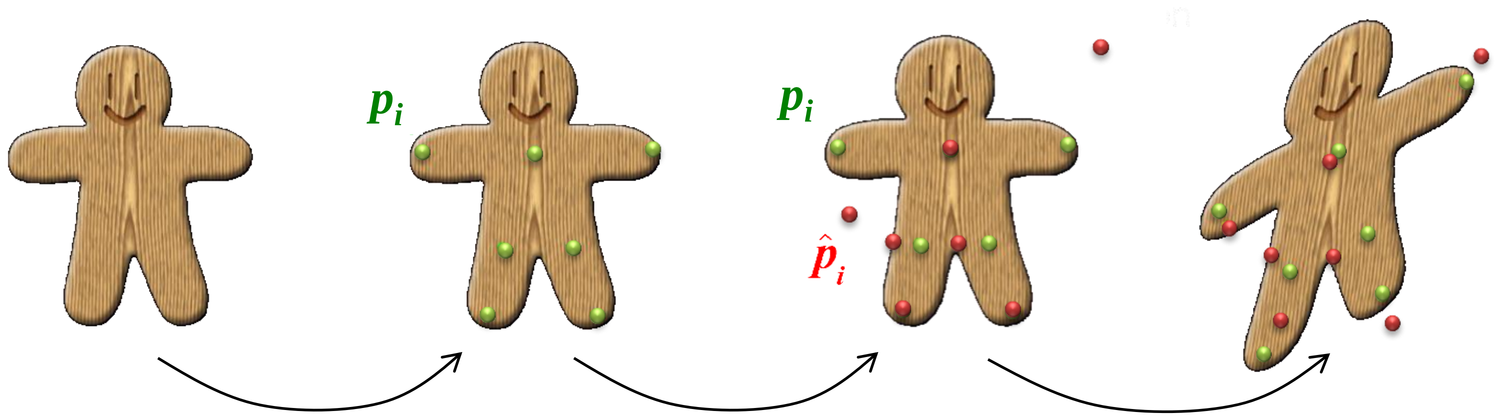
3. Find best affine transformation that maps $\boldsymbol{p}_i$ *to* $\hat{p}_i$

$$\min_{M,T} \sum_i \|M\boldsymbol{p}_i + T - \hat{p}_i\|^2$$

4. Deform $\quad f(v) = M\boldsymbol{v} + T$

ETH zürich

MLS Deformation
[Schaeffer et al. '06]



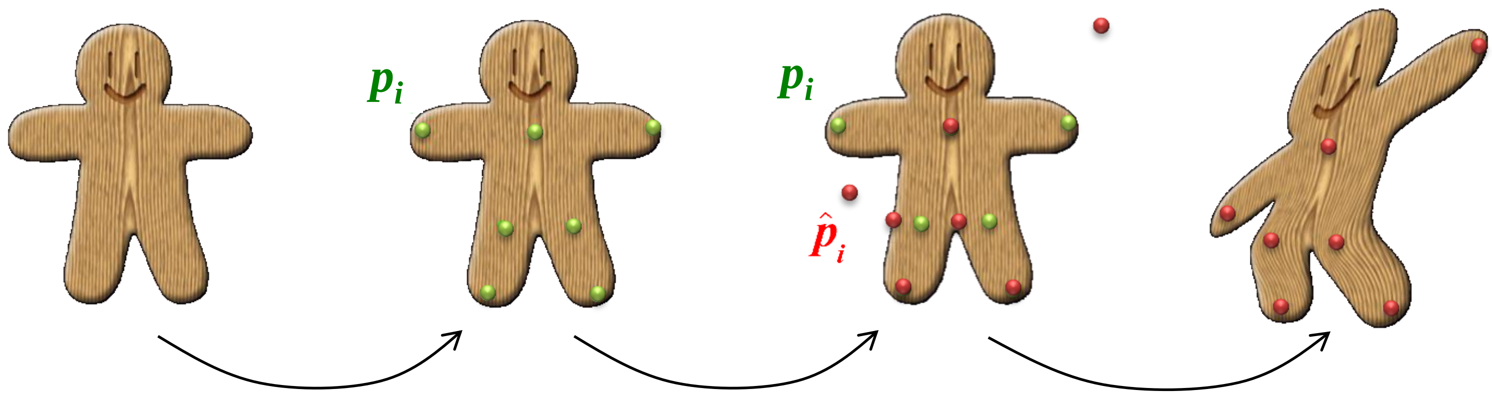
1. Handles $\boldsymbol{p}_i$
2. Target locations $\hat{p}_i$
3. Find best affine transformation that maps $\boldsymbol{p}_i$ *to* $\hat{p}_i$

$$\min_{M,T} \sum_i \left\| \frac{1}{\|\boldsymbol{p}_i - v\|} (M\boldsymbol{p}_i + T) - \hat{p}_i \right\|^2$$

4. Deform $\qquad f(v) = M\boldsymbol{v} + T$

Closed form solution

MLS Deformation
[Schaeffer et al. '06]



1. Handles p_i 2. Target locations $\hat{p}_i$

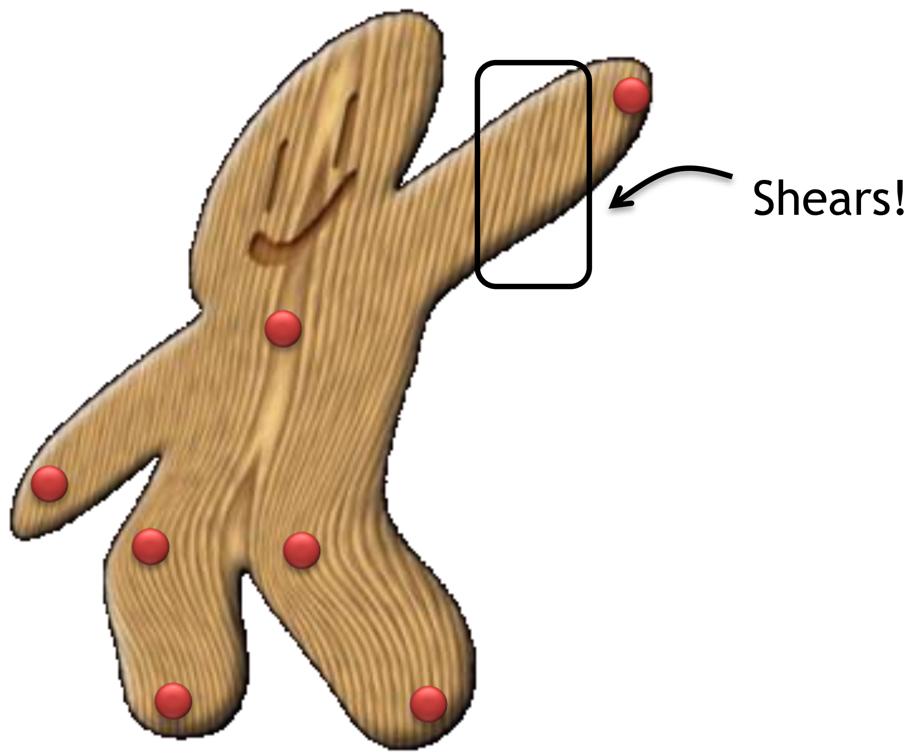
3. Find best affine transformation that maps p_i *to* $\hat{p}_i$

$$\min_{M,T} \sum_i \left\| \frac{1}{\|p_i - v\|} (M p_i + T) - \hat{p}_i \right\|^2$$

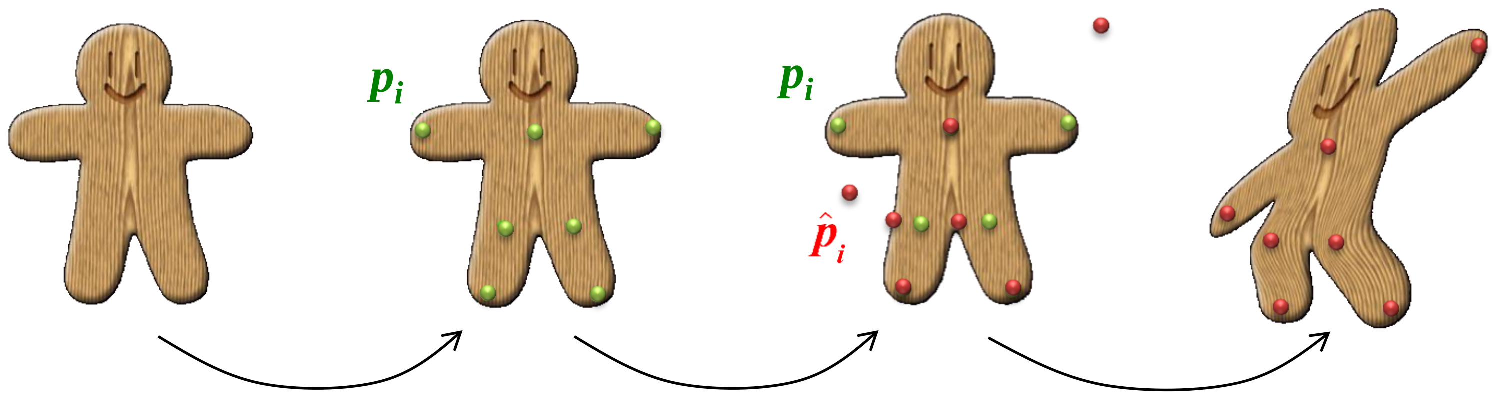
4. Deform $f(v) = Mv + T$

Closed form solution

~~Similarity~~ Affine Transformations?



Similarity Transformations



1. Handles $\boldsymbol{p}_i$
2. Target locations $\hat{p}_i$
3. Find best similarity transformation that maps $\boldsymbol{p}_i$ *to* $\hat{p}_i$

$$\min_{M,T} \sum_i \left\| \frac{1}{\|\boldsymbol{p}_i - v\|} (M\boldsymbol{p}_i + T) - \hat{p}_i \right\|^2$$

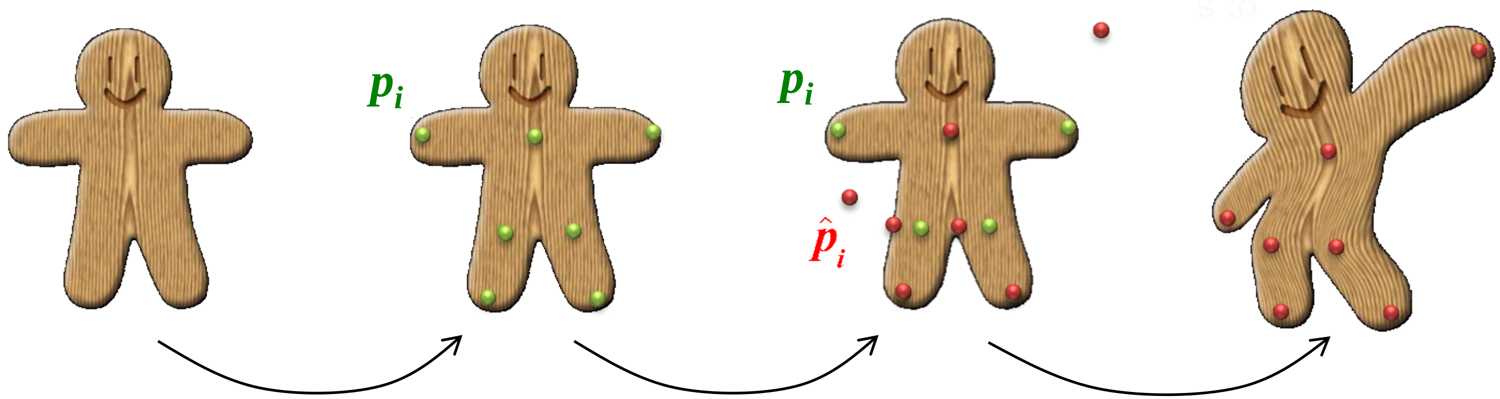
$$M = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

Closed form solution

4. Deform $\quad f(v) = M\boldsymbol{v} + T$

ETH zürich

Similarity Transformations



1. Handles $\boldsymbol{p}_i$ 2. Target locations $\hat{p}_i$

3. Find best similarity transformation that maps $\boldsymbol{p}_i$ *to* $\hat{p}_i$

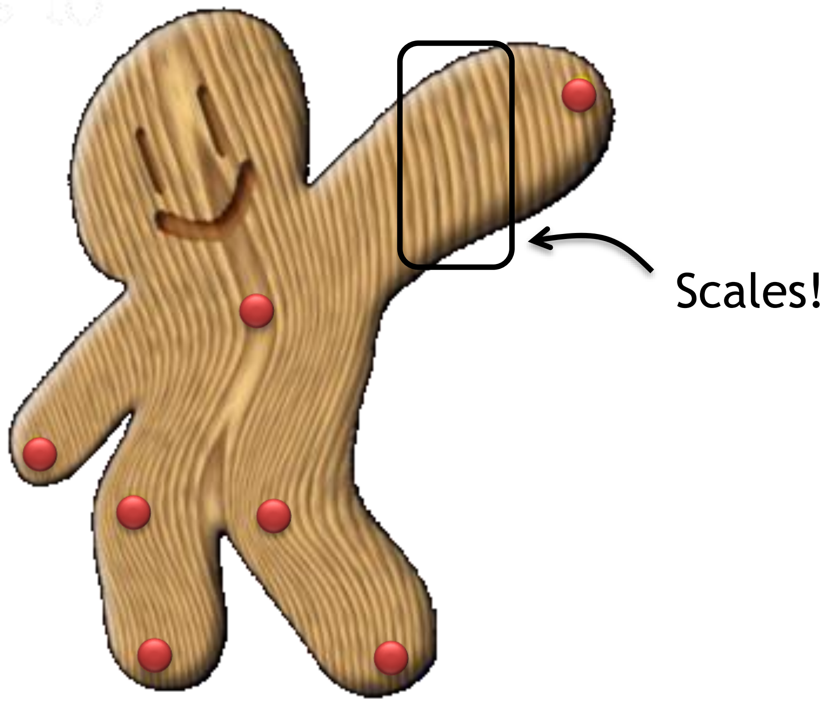
$$\min_{M,T} \sum_i \left\| \frac{1}{\|\boldsymbol{p}_i - v\|} (M\boldsymbol{p}_i + T) - \hat{p}_i \right\|^2$$

$$M = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

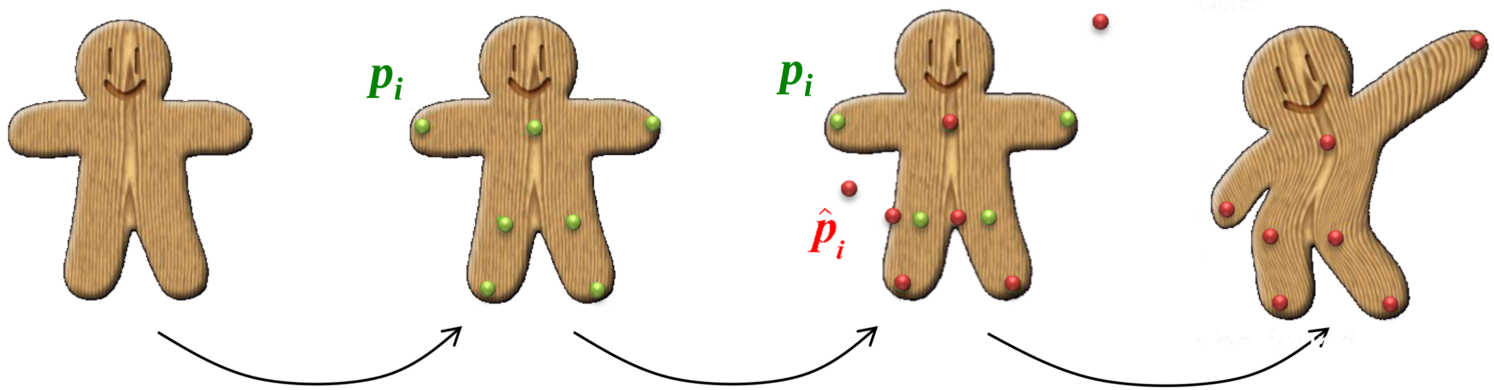
Closed form solution

4. Deform $f(v) = M\boldsymbol{v} + T$

Rigid ~~Similarity~~ Transformations?



Rigid Transformations



1. Handles $\boldsymbol{p}_i$
2. Target locations $\hat{p}_i$
3. Find best rigid transformation that maps $\boldsymbol{p}_i$ *to* $\hat{p}_i$

$$\min_{M,T} \sum_i \left\| \frac{1}{\|\boldsymbol{p}_i - v\|} (M\boldsymbol{p}_i + T) - \hat{p}_i \right\|^2$$

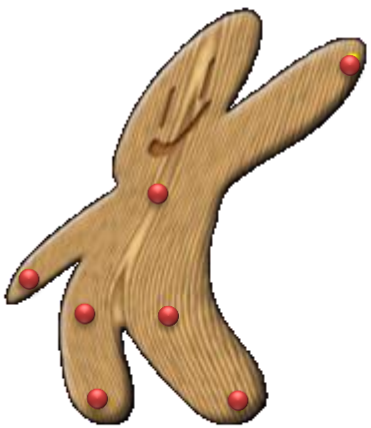
$$M = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$c^2 + s^2 = 1$$

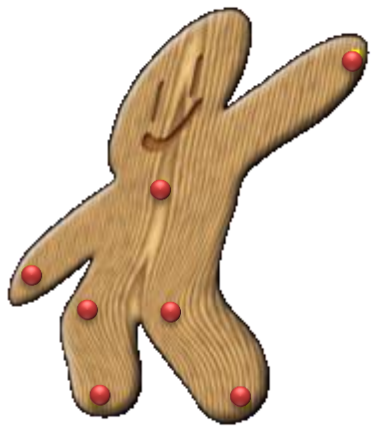
4. Deform $f(v) = M\boldsymbol{v} + T$

ETH zürich

Comparison



Thin-Plate
[Bookstein ’89]

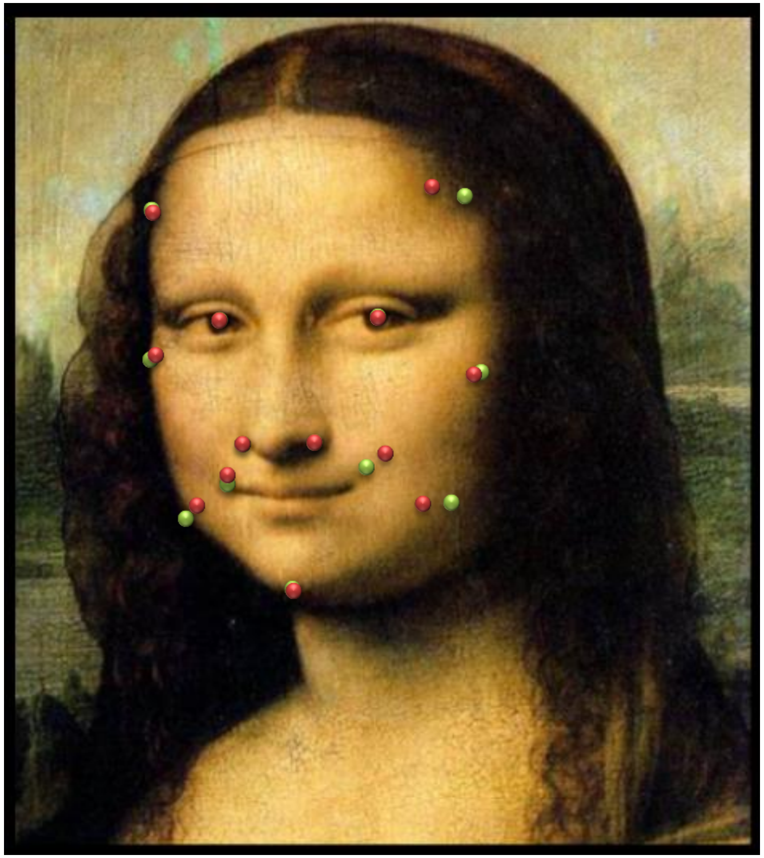


Affine MLS

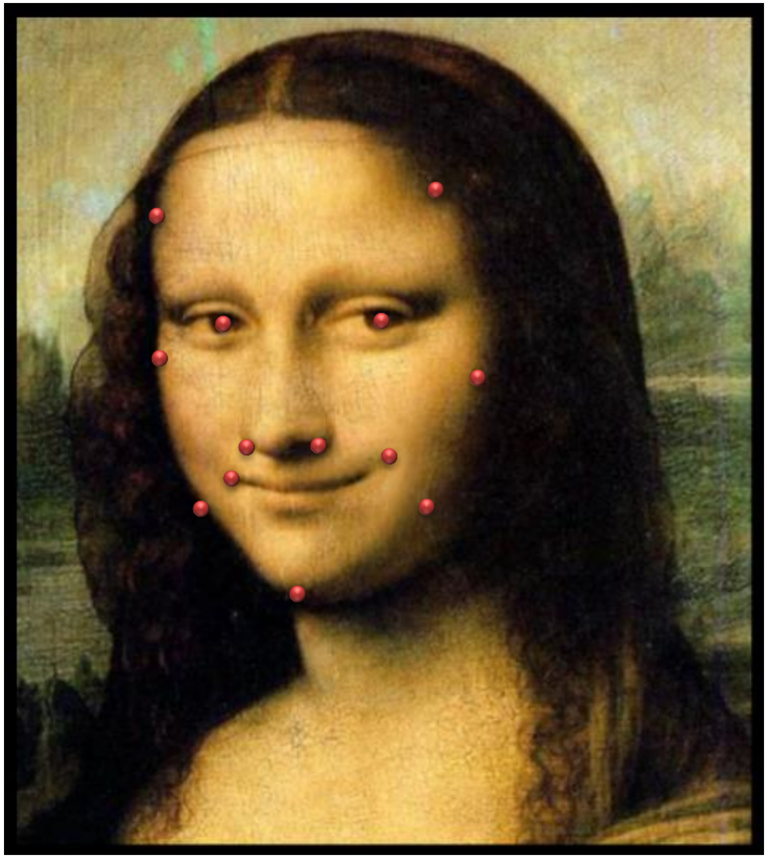
Similarity
MLS

Rigid MLS

Examples

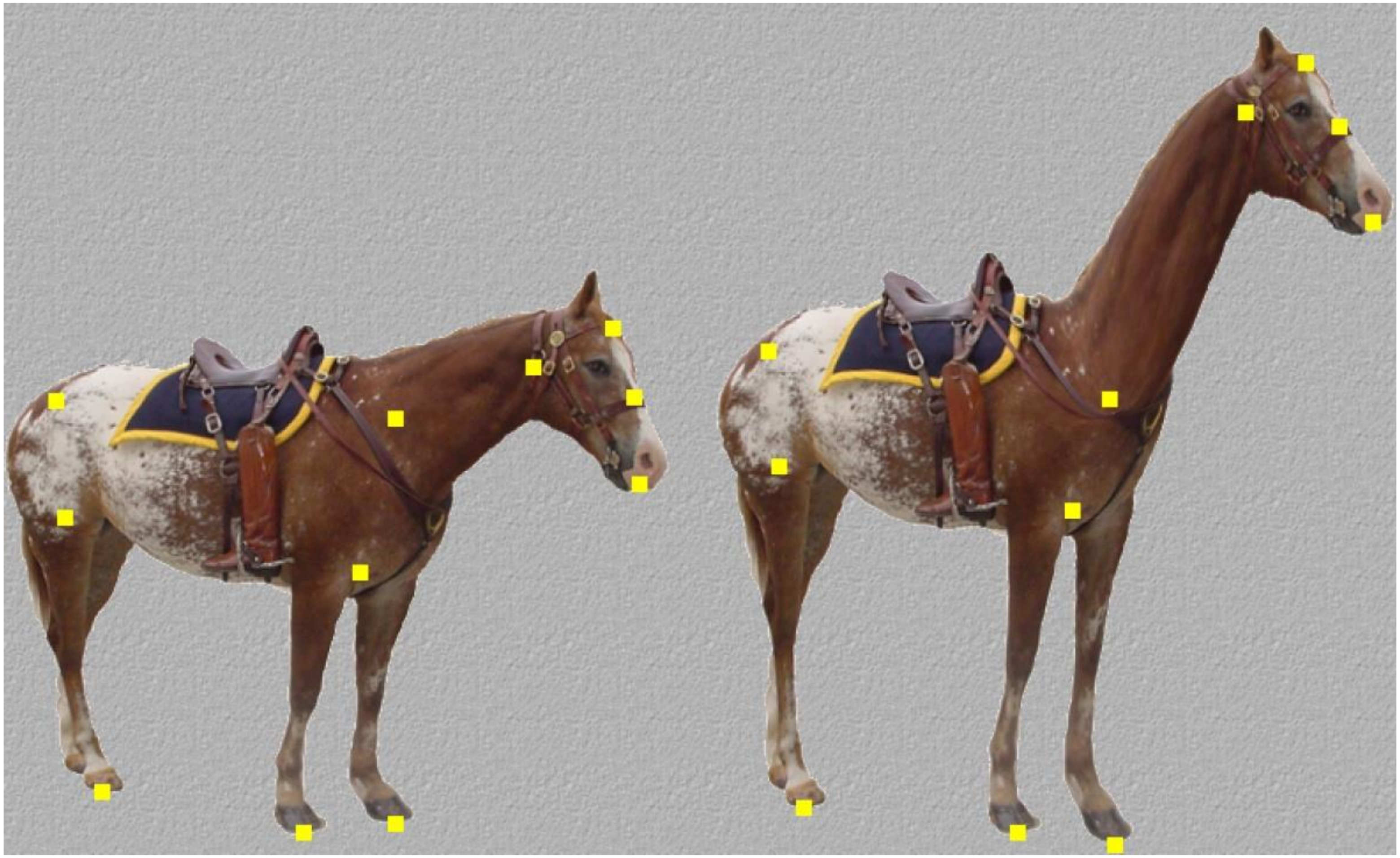


Before



After

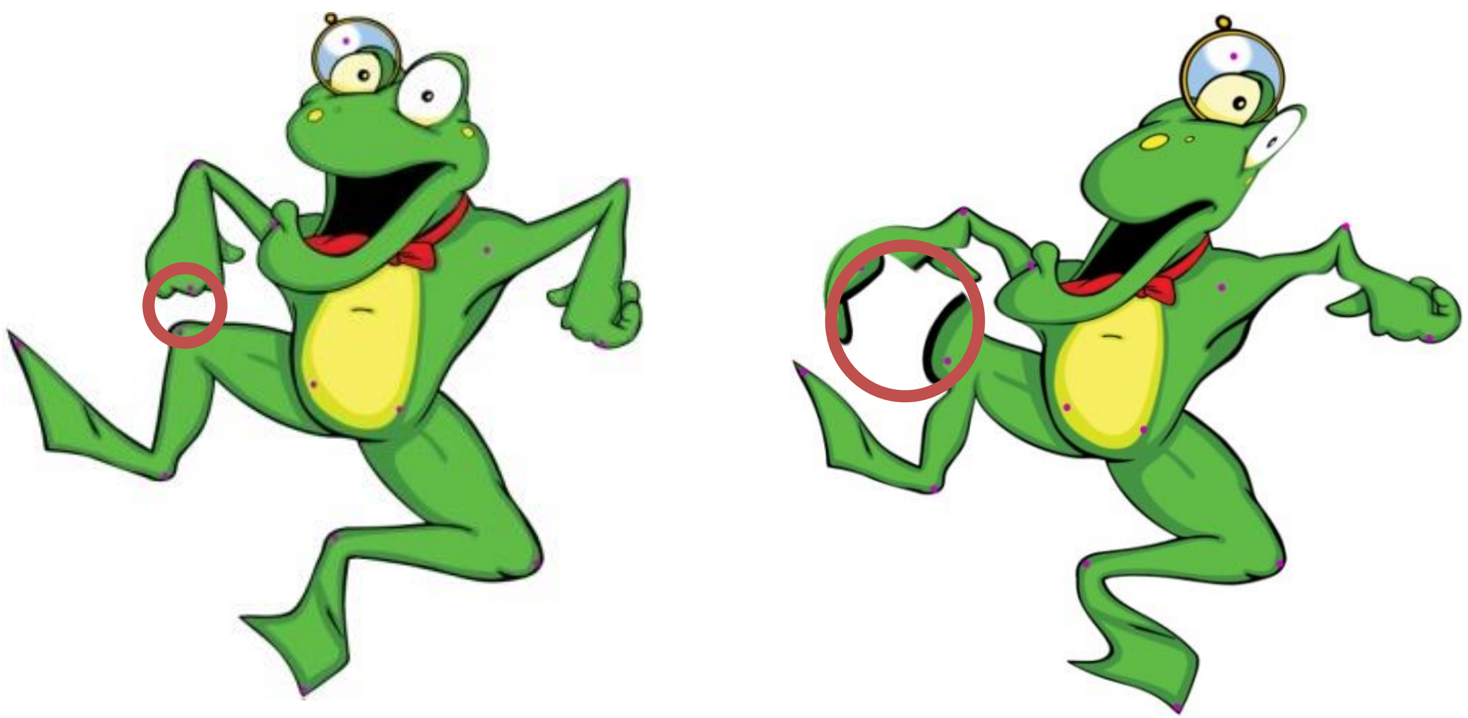
Examples



Horse Giraffe

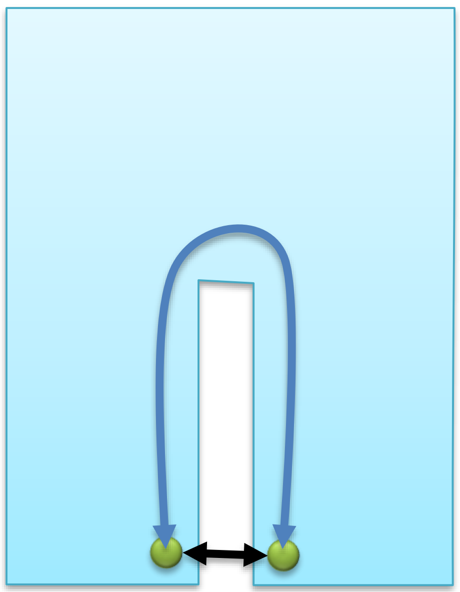
Limitations

Deforms all space - is not “shape aware”

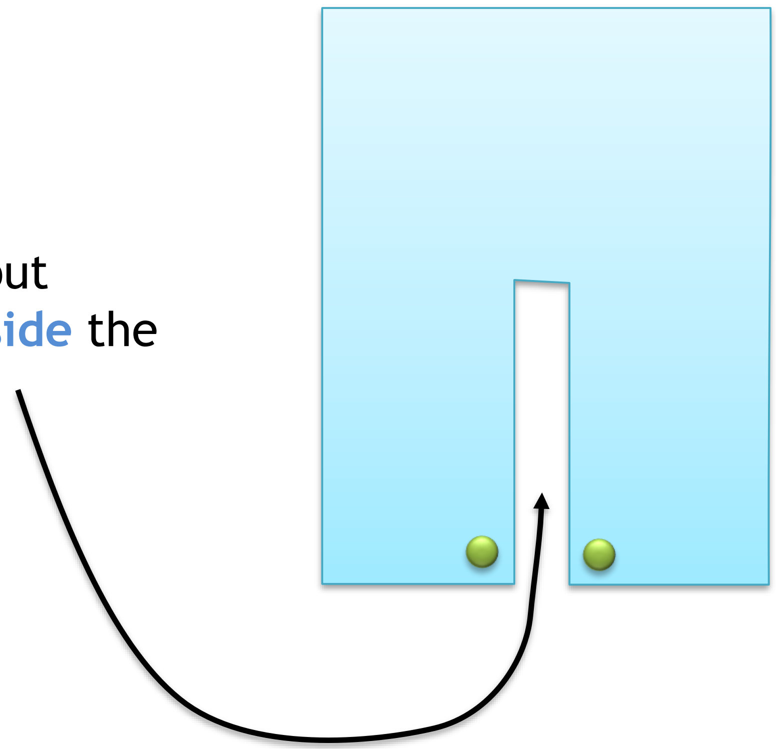


The "Pants" Problem

Small Euclidean distance
Large geodesic distance



The “Pants” Problem



Don’t care about distortion **outside** the shape

Solution: Cages

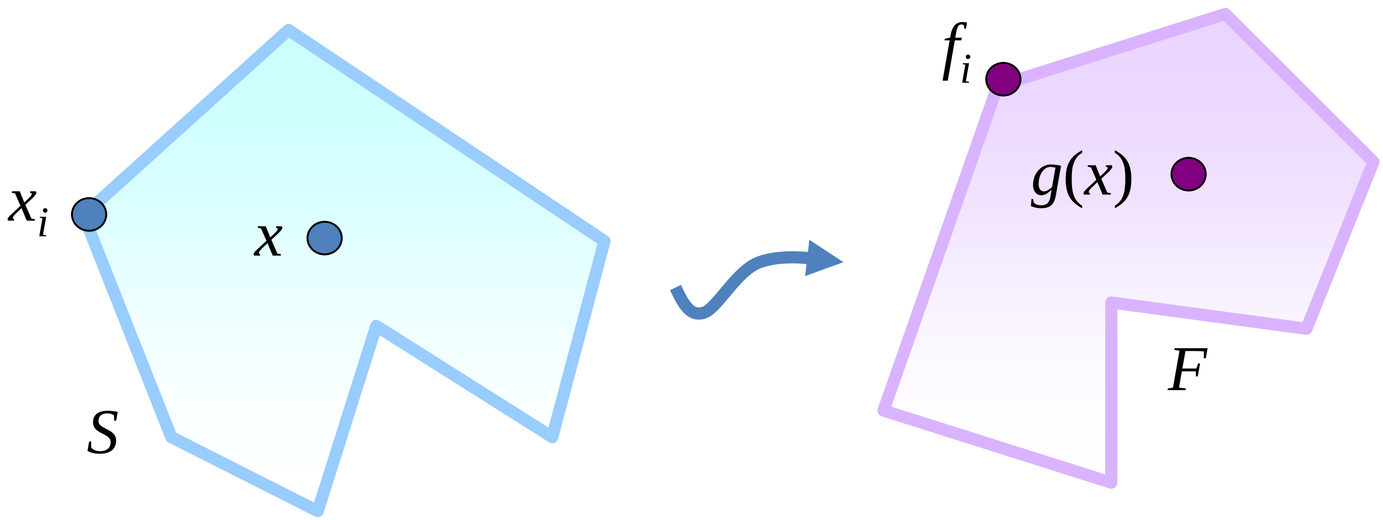
Enclose the shape in a “cage” $\Omega \subset \mathbb{R}^n$

Deformation function defined only on cage

$$f : \Omega \rightarrow \mathbb{R}^n$$

New problem: how to build the cage?

Deformation with a Cage



$S = \{x_1, x_2, \ldots, x_n\}$

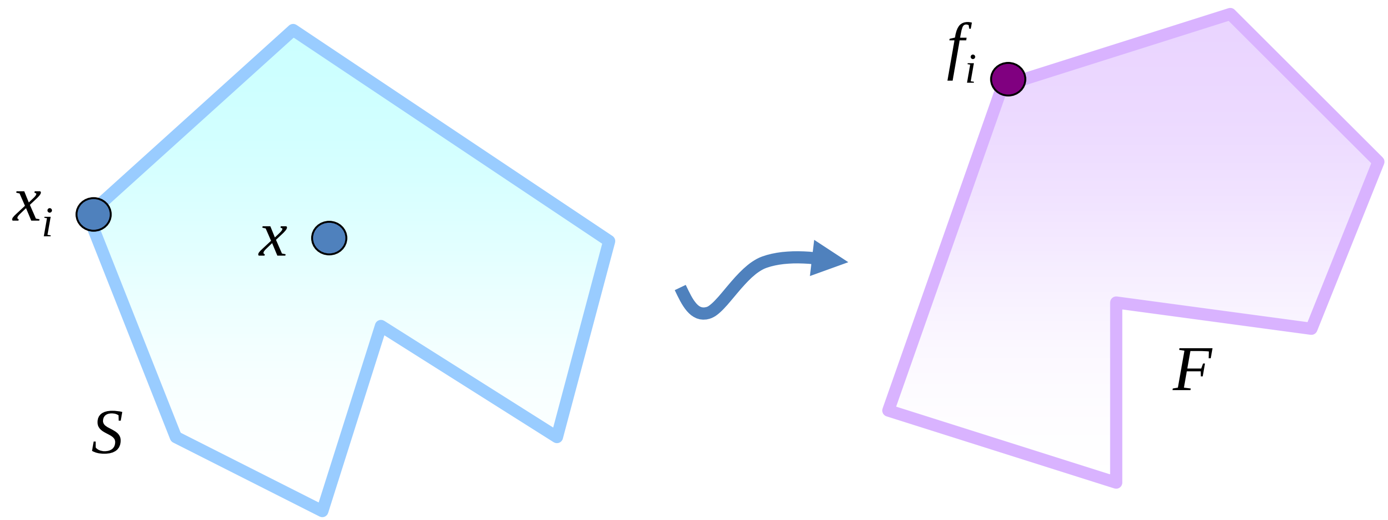
Source polygon

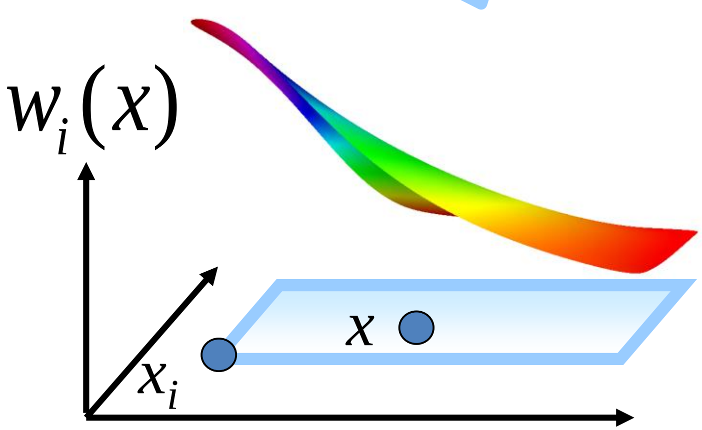
$x_i \rightarrow f_i$

Target polygon

$g(x) = ?$ Interior?

Generalized barycenric coordinates

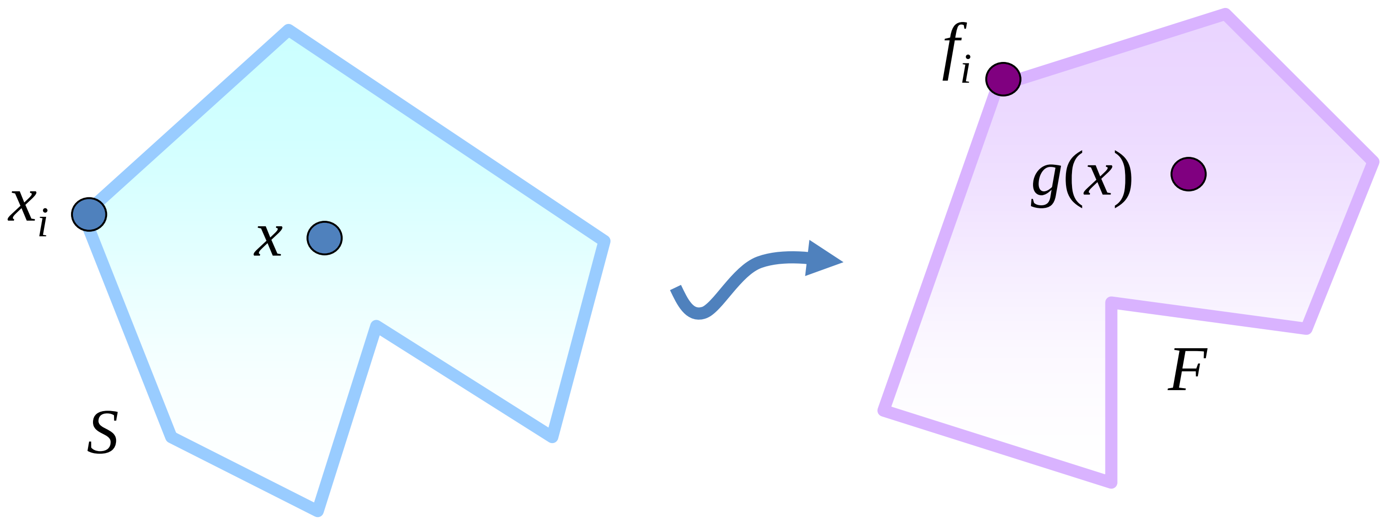


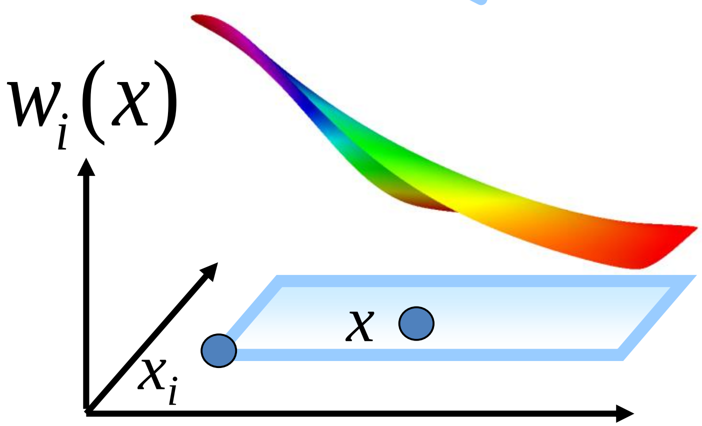


$$w_i(x) : \Omega \to \mathbb{R}^n$$

Barycentric Coords Function

Generalized barycenric coordinates



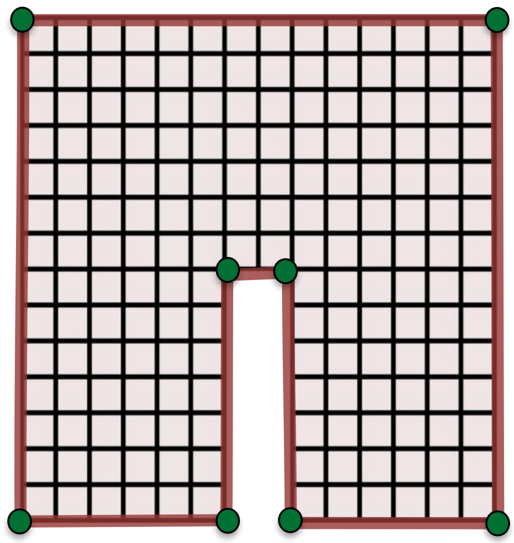


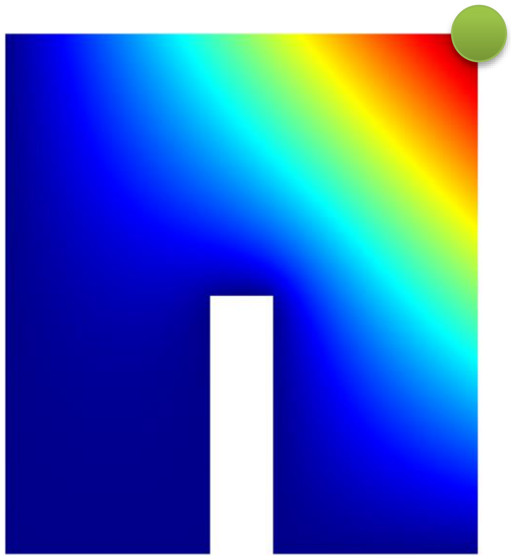
$$g_F(x) = \sum_{i=1}^{n} w_i(x) f_i$$

Generalized barycenric coordinates

Stages:

- Source shape
- Polygonal cage
- Coordinates



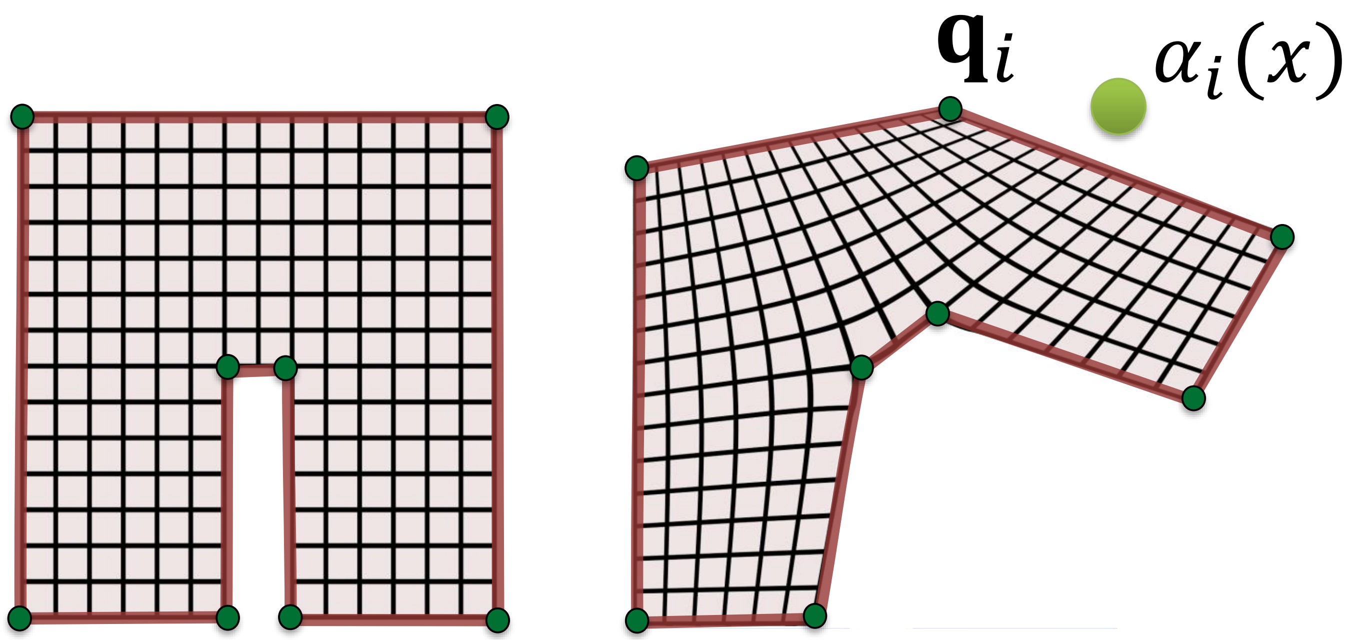


$$f(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i(\mathbf{x})\mathbf{q}_i$$

Generalized barycenric coordinates

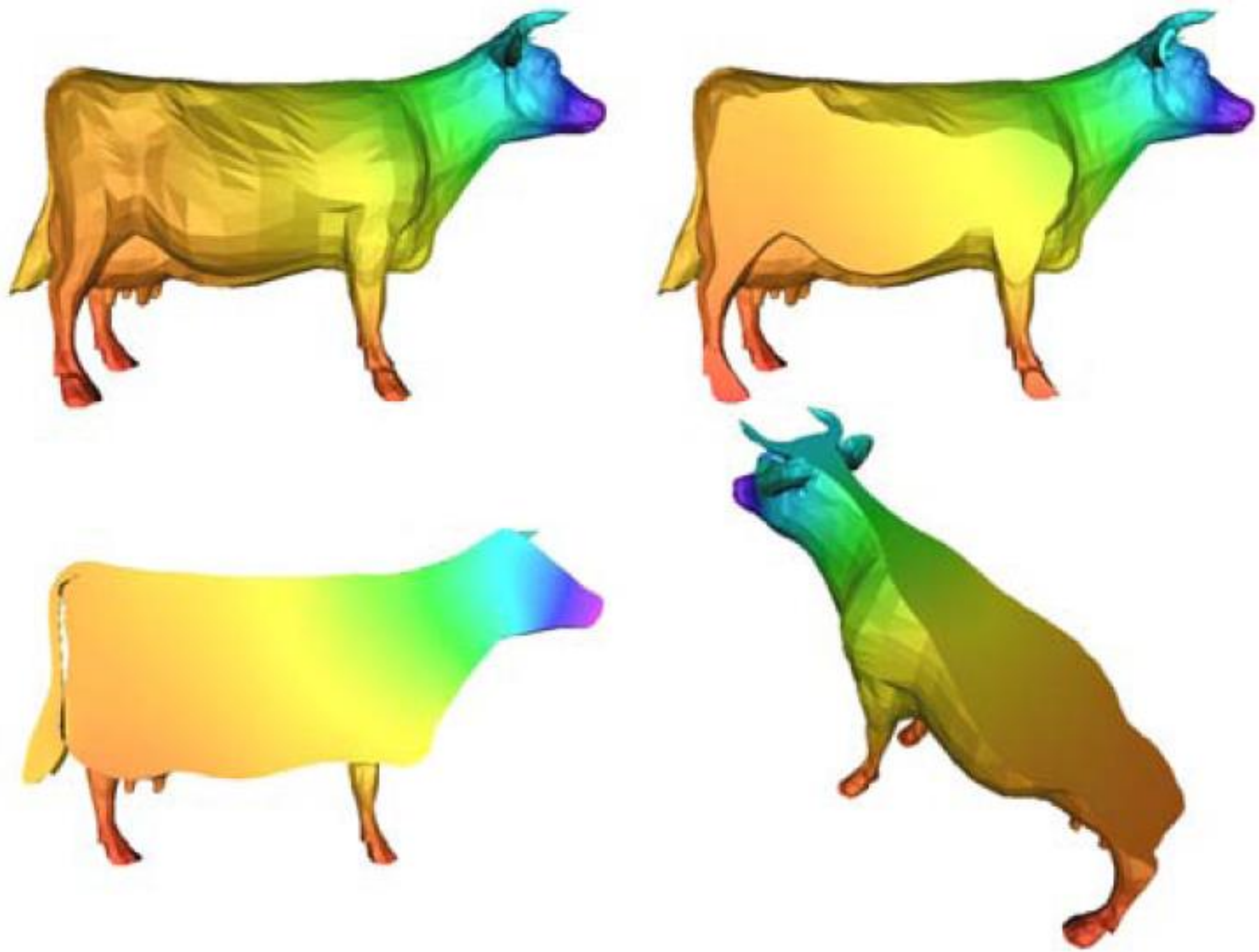
Stages:

- Source shape
- Polygonal cage
- Coordinates
- Manipulate cage
- Apply deformation



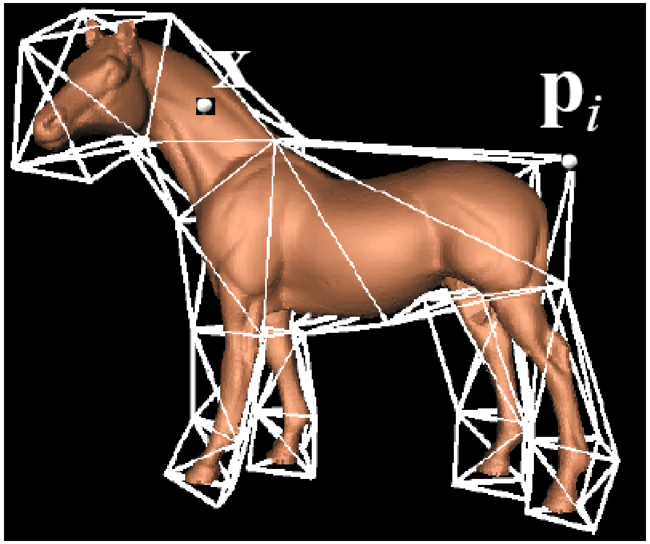
$$f(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i(\mathbf{x})\mathbf{q}_i$$

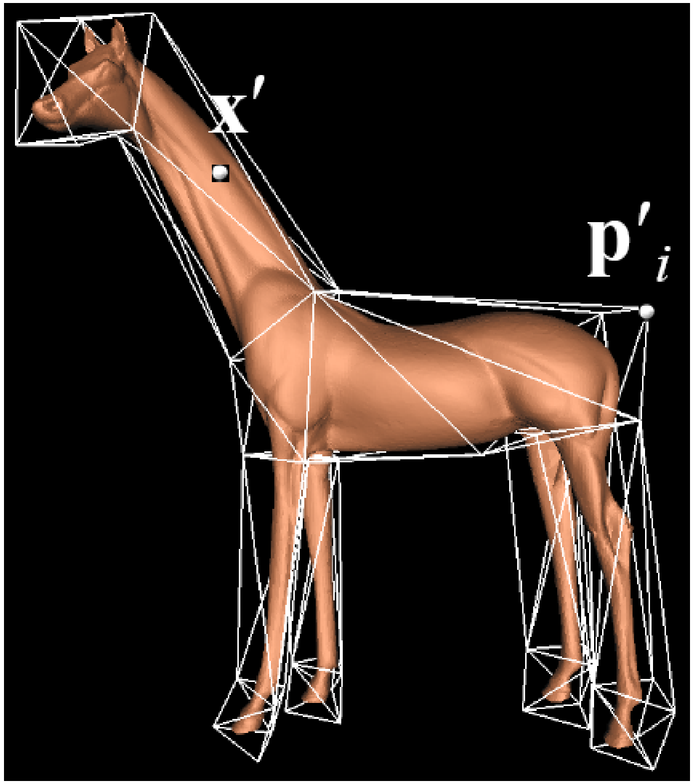
3D Example



3D Example

$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\mathbf{p}'_i$$

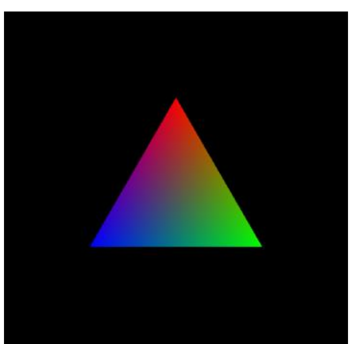


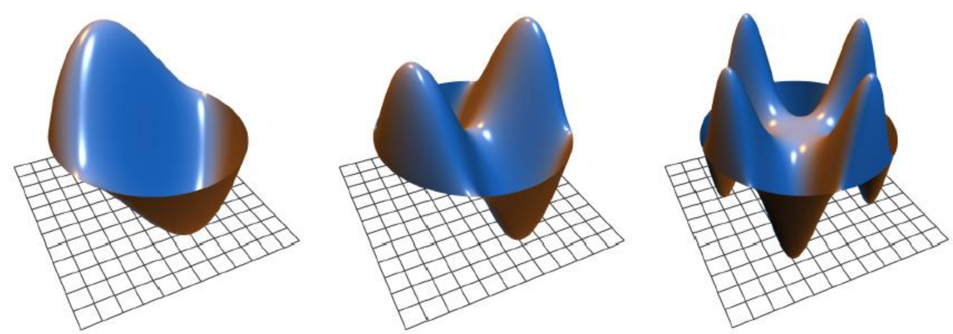


Generalized barycenric coordinates

Data interpolation from the vertices of a boundary polygon to its interior

- Boundary value problems
- Shading
- Space deformations
- Parametrization





ETH zürich

Generalized barycenric coordinates

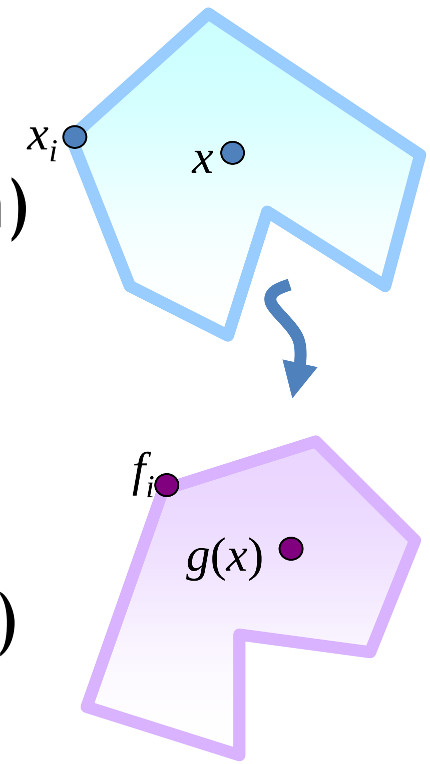
Required properties

Translation invariance (constant precision)

$$\sum_{i=1}^{n} w_i(x) = 1$$

Reproduction of identity (linear precision)

$$\sum_{i=1}^{n} w_i(x) x_i = x$$



$$g(x) = \sum_{i=1}^{n} w_i(x) f_i$$

Generalized barycenric coordinates

Constant + linear precision = affine invariance

$$g_{Ax+T}(x) = \sum_{i=1}^{n} w_i(x)(Ax_i + T) =$$

ETH zürich

Generalized barycenric coordinates

Constant + linear precision = affine invariance

$$g_{Ax+T}(x) = \sum_{i=1}^{n} w_i(x)(Ax_i + T) =$$

$$= A \underbrace{\sum_{i=1}^{n} w_i(x) x_i}_{x} + T \underbrace{\sum_{i=1}^{n} w_i(x)}_{1} =$$

Generalized barycenric coordinates

Constant + linear precision = affine invariance

$$g_{Ax+T}(x) = \sum_{i=1}^{n} w_i(x)(Ax_i + T) =$$

$$= A\underbrace{\sum_{i=1}^{n} w_i(x)x_i}_{x} + T\underbrace{\sum_{i=1}^{n} w_i(x)x_i}_{1} =$$

$$= Ax + T$$

Generalized barycenric coordinates

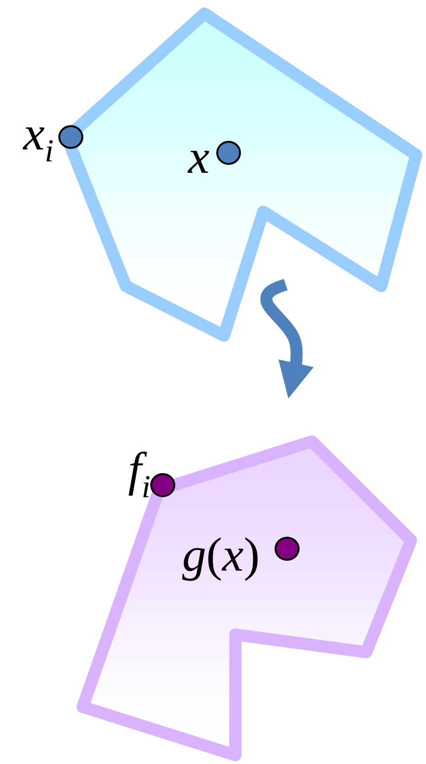
Required properties

Smoothness – at least C1

Interpolation (Lagrange property)

$$f(x_j) = f_j$$

$$w_i(x_j) = \delta_{ij}$$



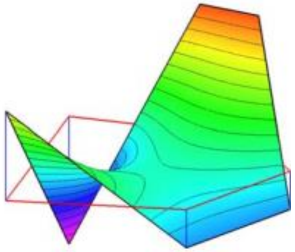
$$g(x) = \sum_{i=1}^{n} w_i(x) f_i$$

Generalized barycenric coordinates

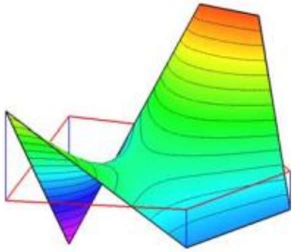
Not unique. Many recipies proposed.

We will show three main coordinates:

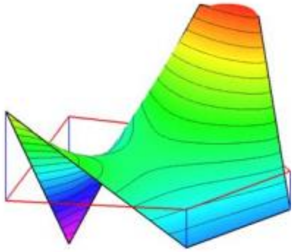
Wachspress



Harmonic



Mean Value

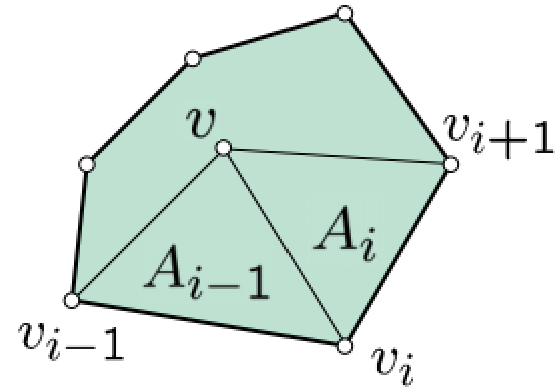


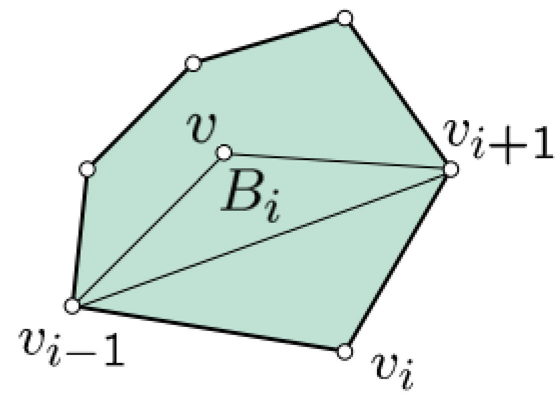
Wachspress Coordinates

Only apply to convex polygons

Three points construction:

$$w_i = \frac{A_{i+1} + A_i - B_i}{A_{i+1} A_i}$$





The areas are signed!

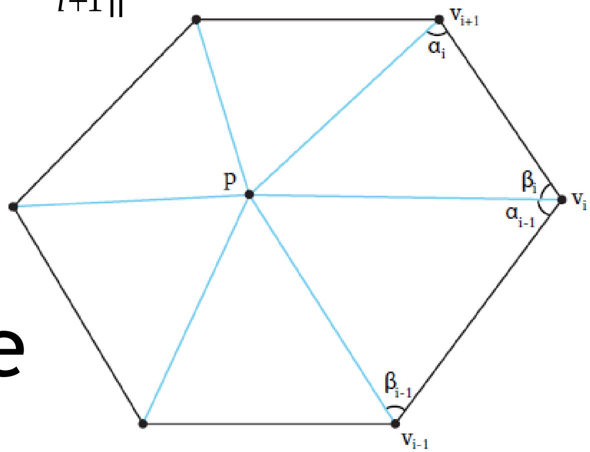
ETH zürich

Wachspress Coordinates

Rephrasing the expression:

$$\frac{A_i + A_{i+1} - B_i}{A_i A_{i+1}} = \frac{\sin(\alpha_{i-1} + \beta_i)\|v_i - v_{i-1}\| \cdot \|v_i - v_{i+1}\|}{\sin(\alpha_{i-1})\|v_i - v_{i-1}\|\|v_i - p\|^2 \sin(\beta_i)\|v_i - v_{i+1}\|} =$$

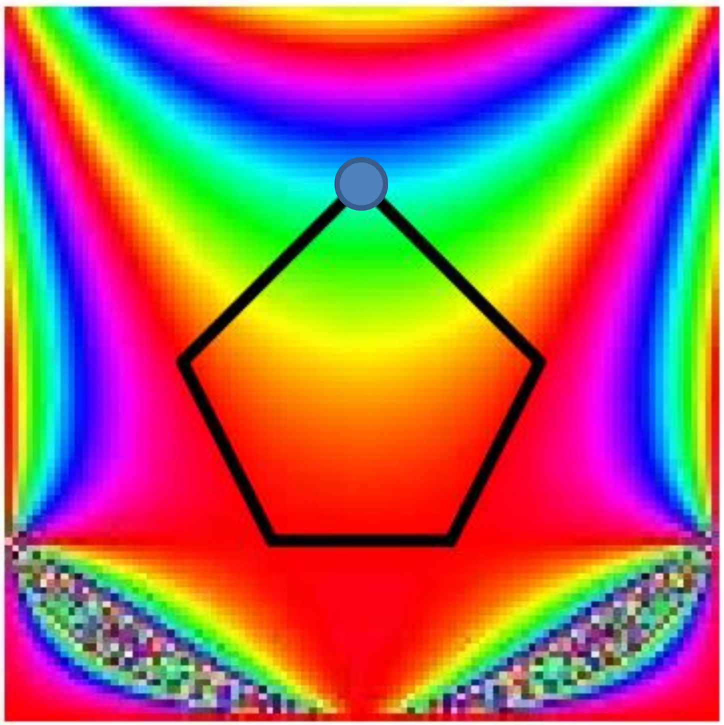
$$= \frac{\sin(\alpha_{i-1} + \beta_i)}{\sin(\alpha_{i-1})\sin(\beta_i)\|v_i - p\|^2} = \frac{\cot(\alpha_{i-1}) + \cot(\beta_i)}{\|v_i - p\|^2}$$



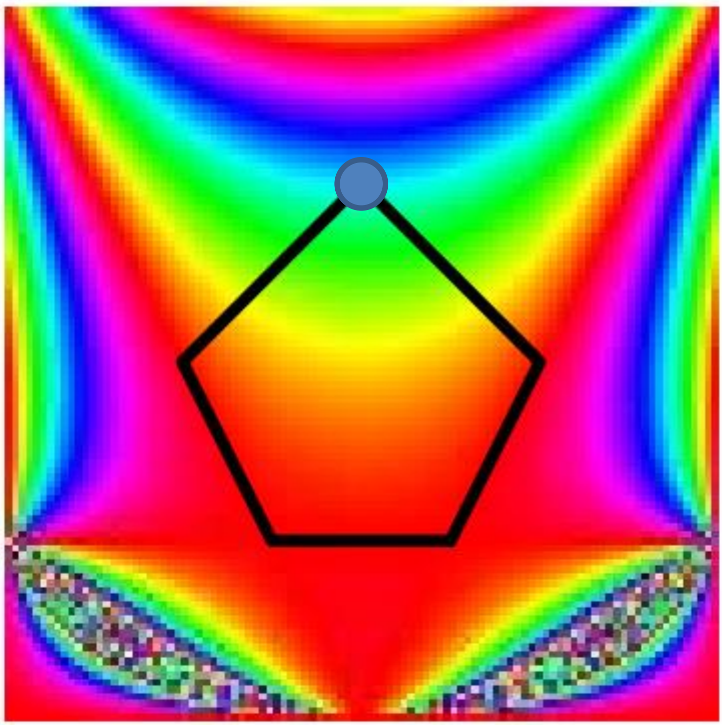
Every coordinate depends on the vertex its neighbors

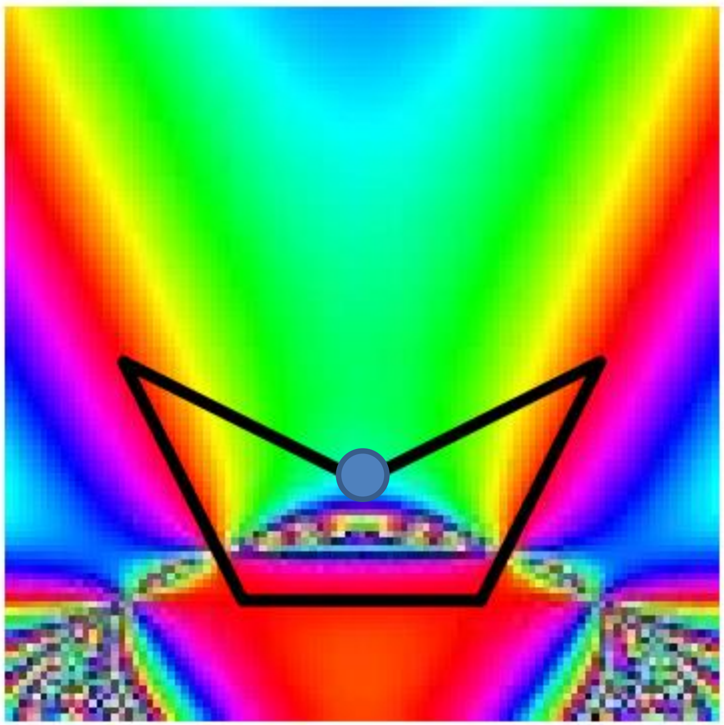
Such coordinates are called **three-point coordinates.**

Wachspress Interpolation



Wachspress Interpolation



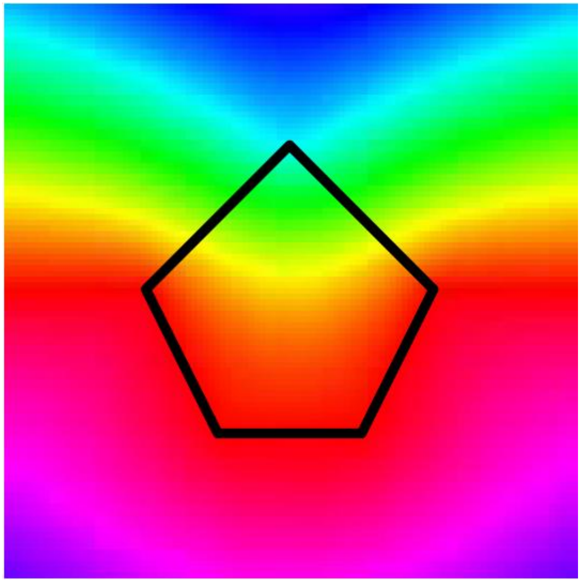


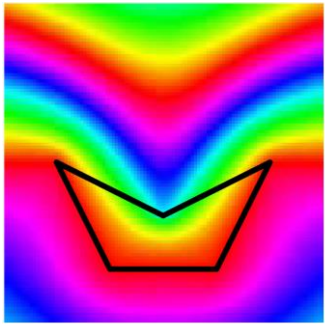
Coordinates blow-up for non convex polygons

ETH zürich

Mean Value Coordinates

$$w_i(x) = \frac{\tan\left(\frac{\alpha_{i-1}}{2}\right) + \tan\left(\frac{\alpha_i}{2}\right)}{\left\|v_i - x\right\|}$$



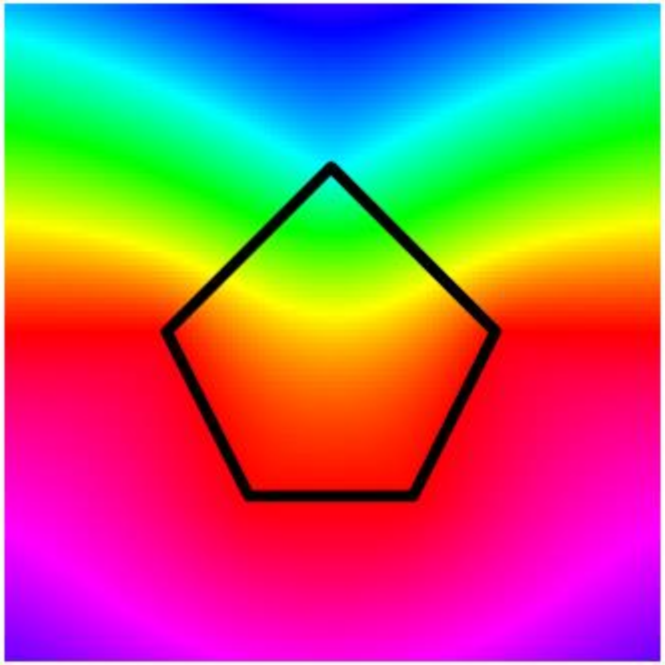


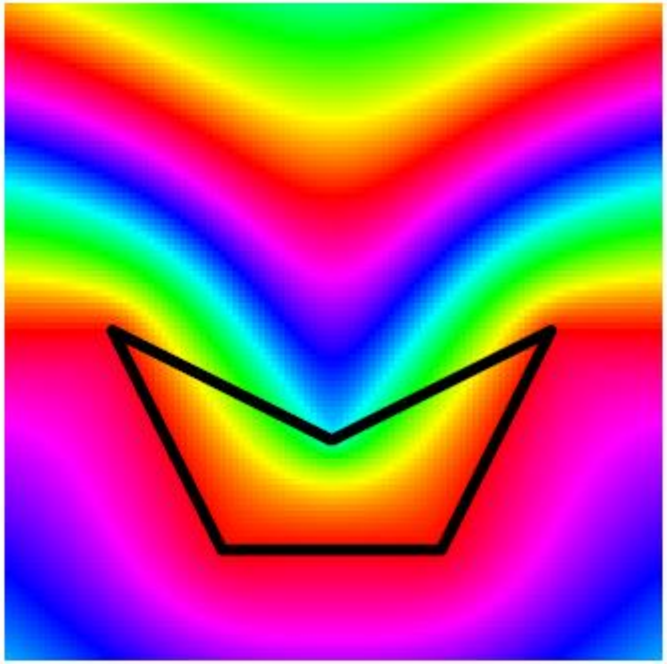
Closed form!

ETH zürich

Mean Value Coordinates

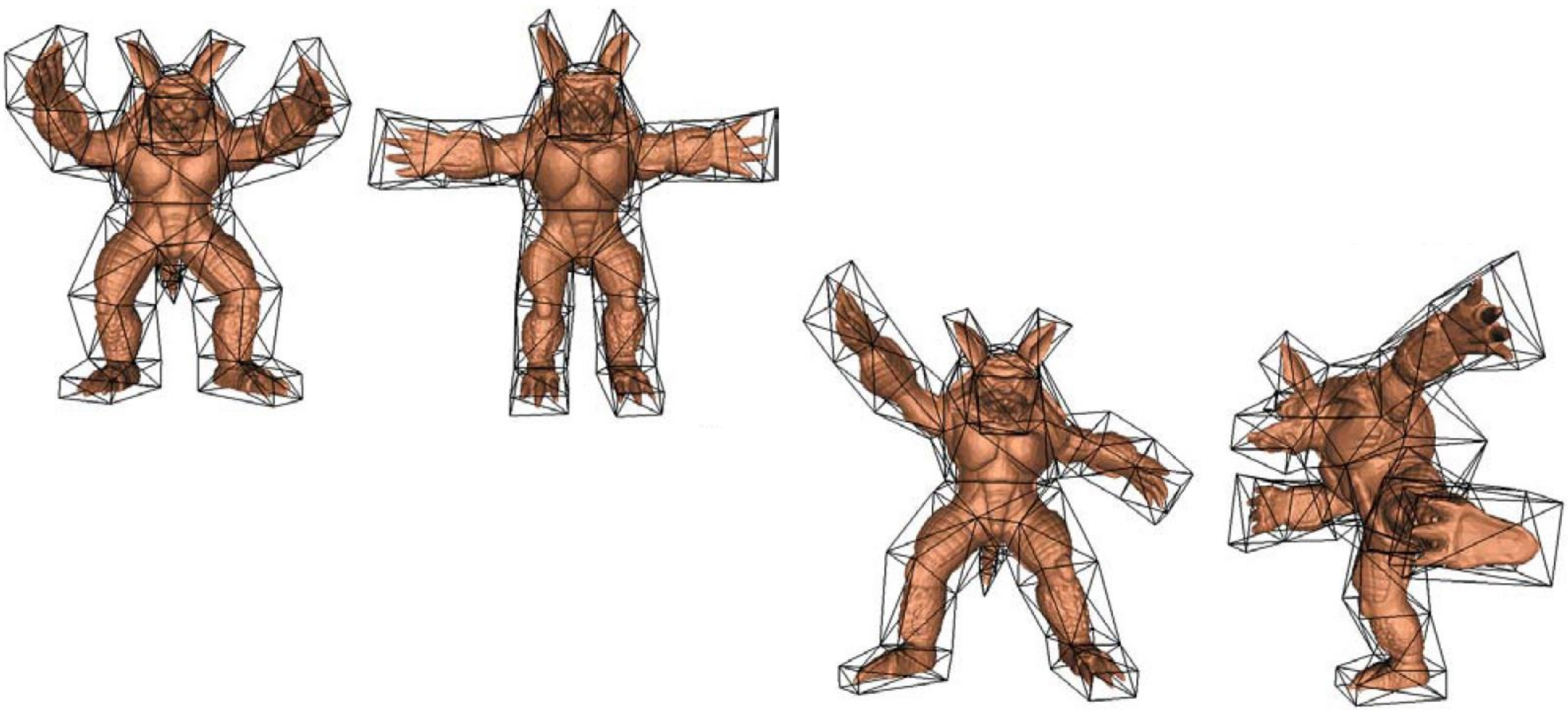
Defined anywhere in the plane





ETH zürich

Example: 3D Mean Value Coords

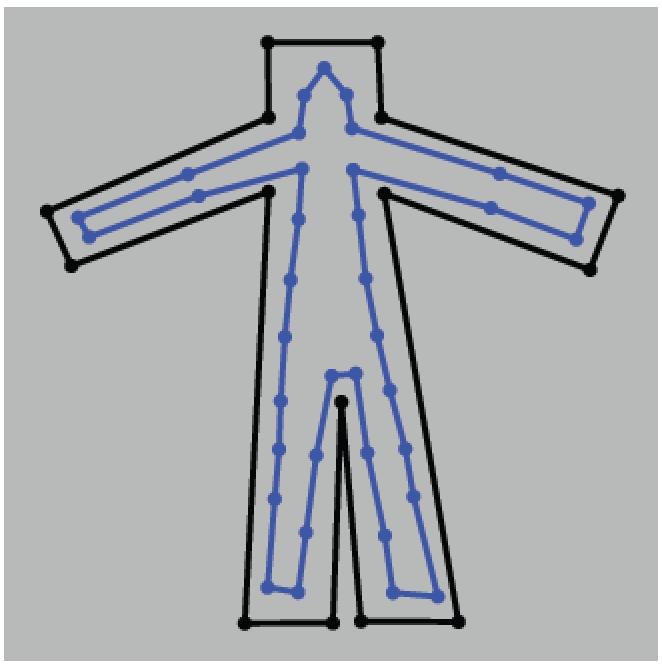


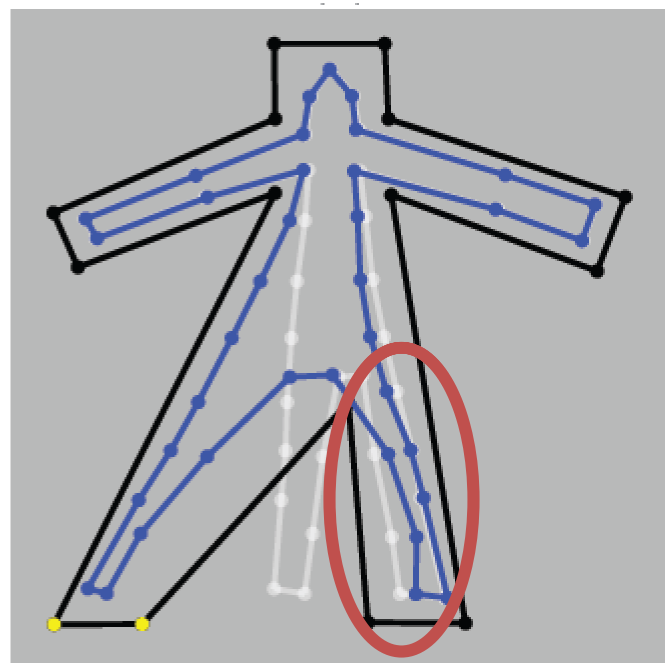
ETH zürich

MV - Limitations

Back to the pants problem

- MV **negative** on concave polygons

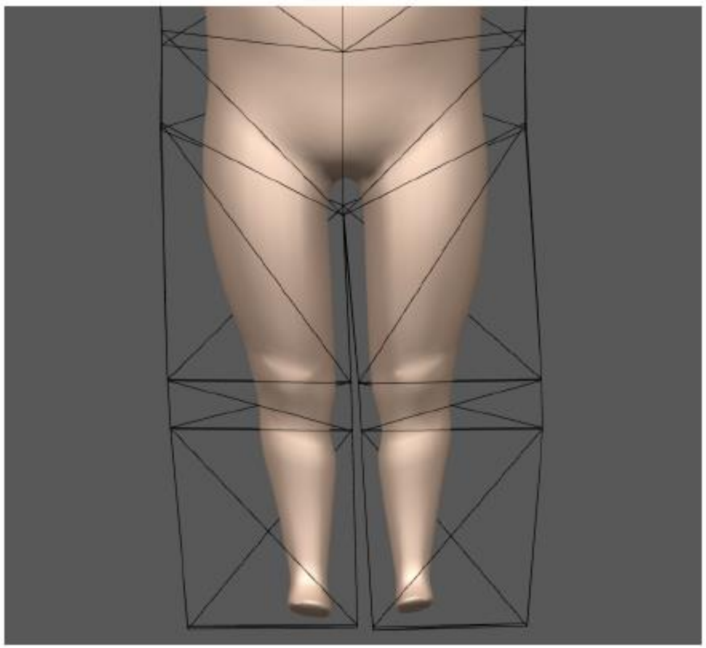
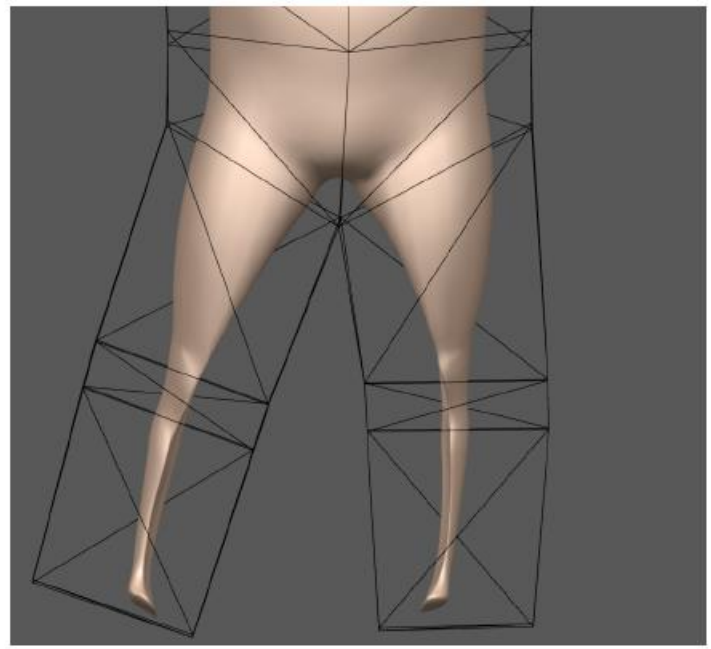




ETH zürich

MV - Limitations

Other leg moves in opposite (!) direction

ETH zürich

Positivity

Additional property required:

$$w_i(x) > 0$$

Mean value coords only positive
on convex polygons

Harmonic Coordinates

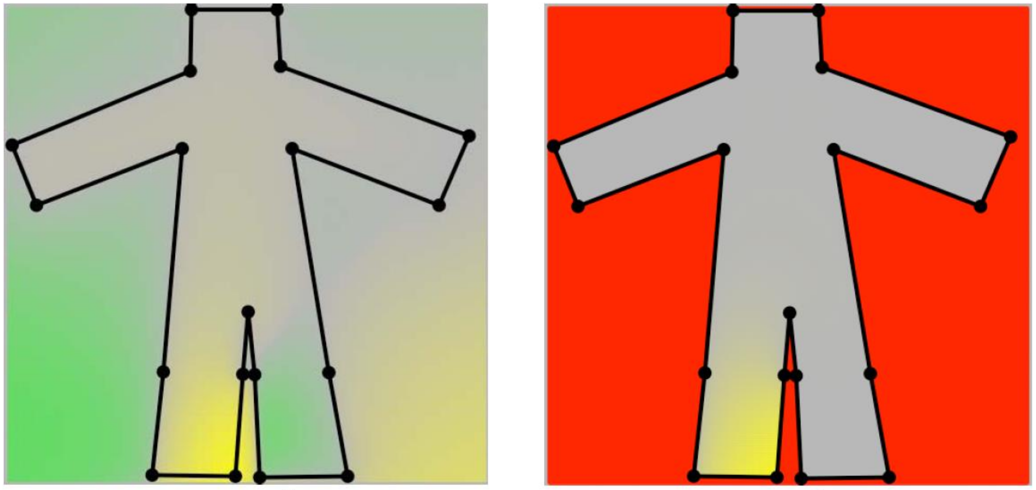
[Joshi et al '07]

Solve for $w_i(x)$:

$$\nabla^2 w_i(x) = 0$$

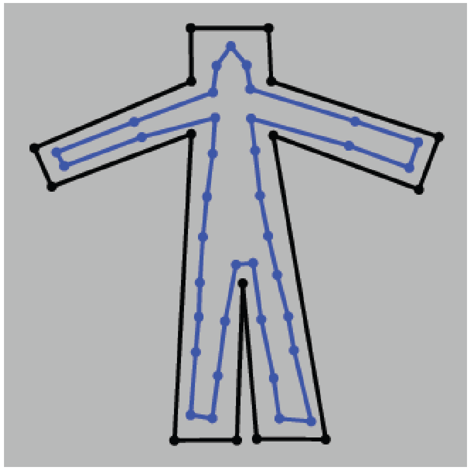
subject to: w_i linear on the boundary and

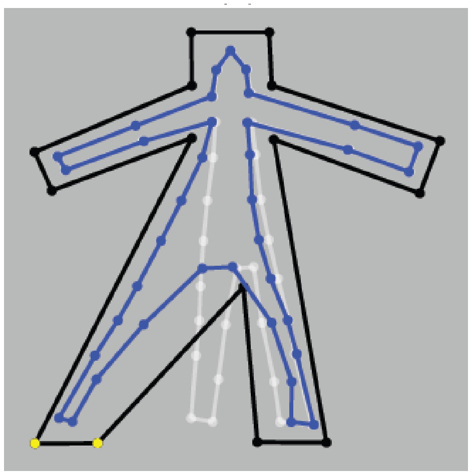
$$w_i(x_j) = \delta_{ij}$$



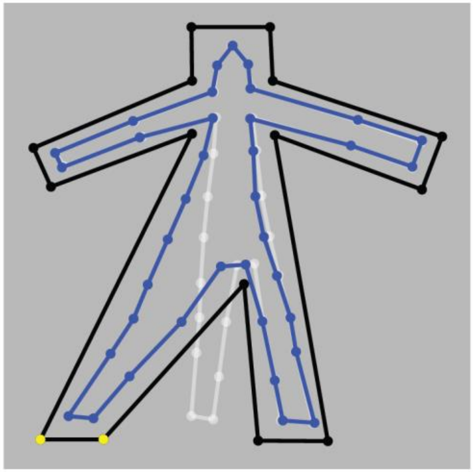
MVC HC

Harmonic Coordinates





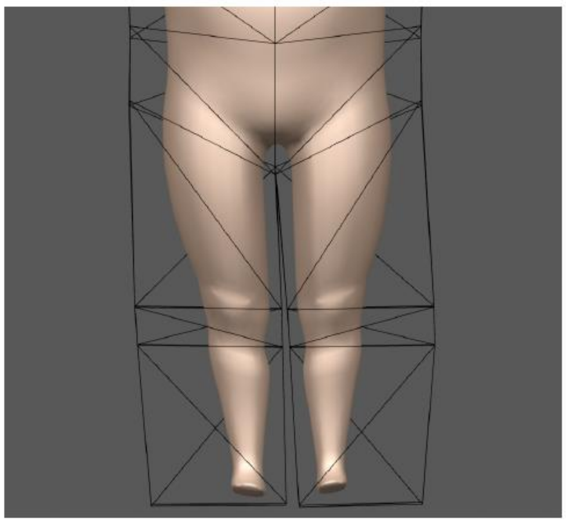
MVC

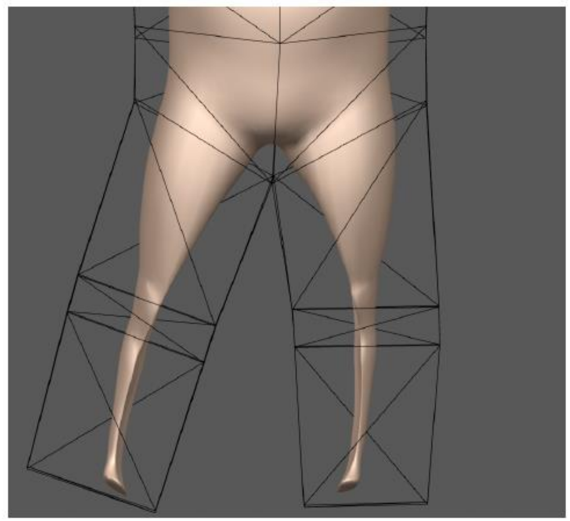


HC

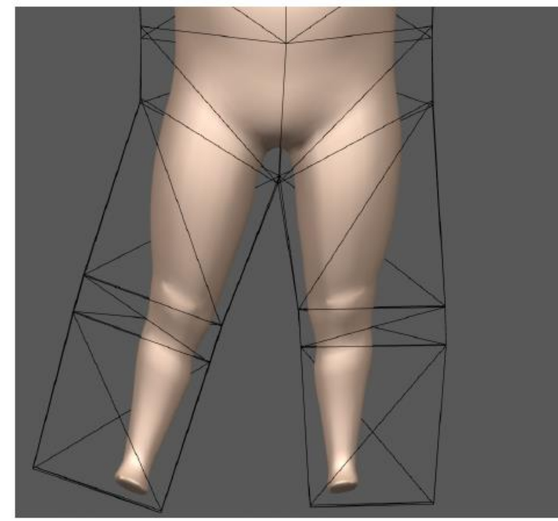
ETH zürich

Harmonic Coordinates





MVC



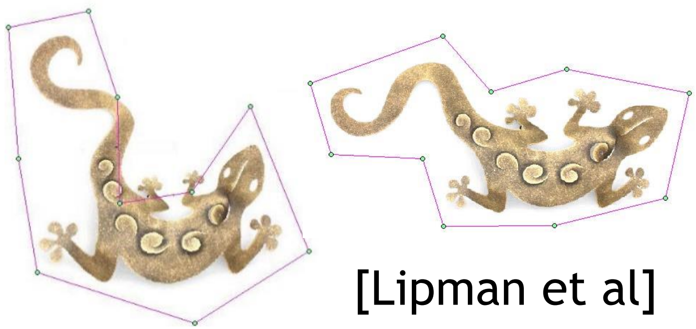
HC

Harmonic Coordinates

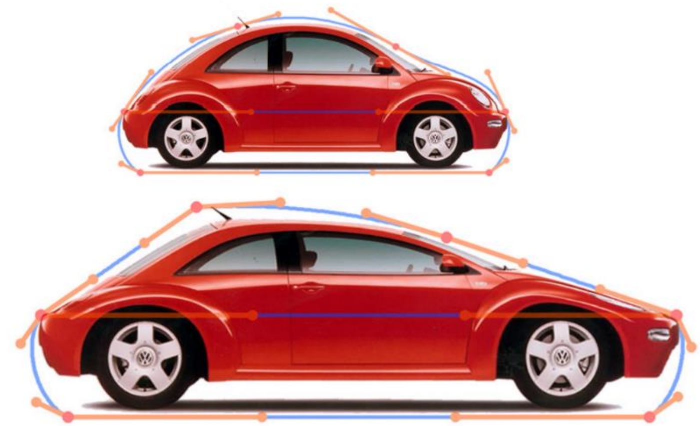
Properties:

- All required properties
  - Smooth, translation + rotation invariant
- Positive everywhere
- No closed form, need to solve a PDE

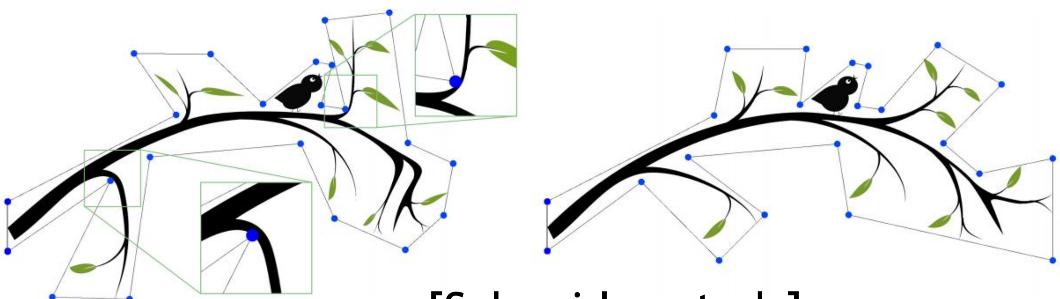
More Examples



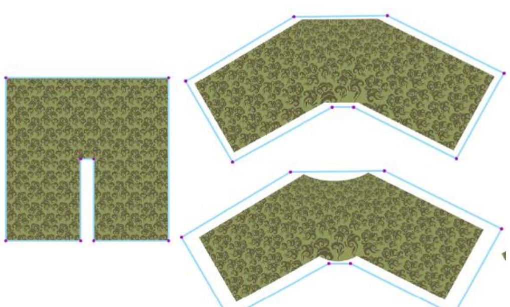
[Lipman et al]



[Li et al.]

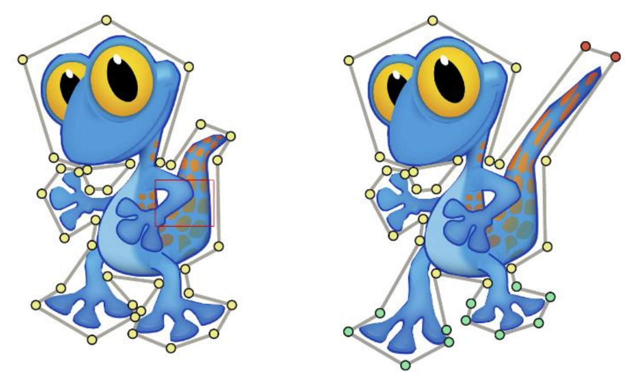


[Schneider et al.]



[Weber et al.]

[Weber et. al]



[Li et al.]

ETH zürich

References

"On Linear Variational Surface Deformation Methods" [Botsch & Sorkine '08]

Tutorial: "Interactive Shape Modeling and Deformation" [Sorkine & Botsch '09]

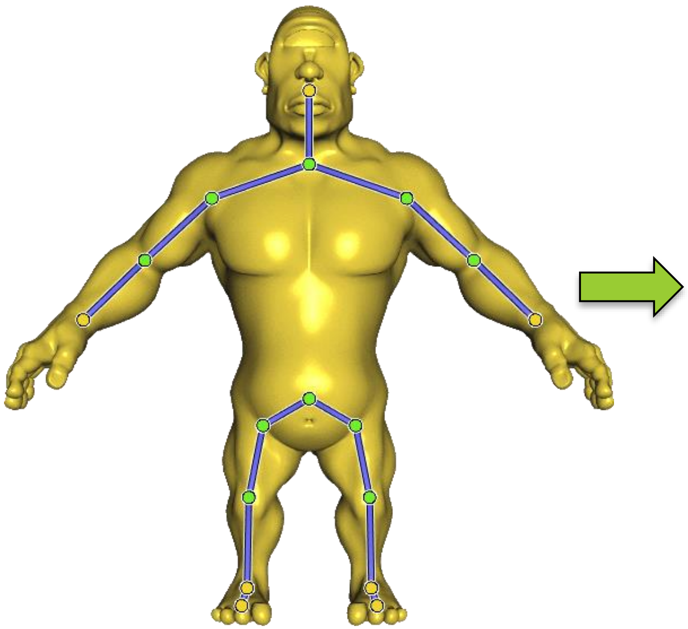
"Image deformation using moving least squares" [Schaefer et al '06]

"Mean Value Coordinates for Closed Triangular Meshes" [Ju et al '05]

"Harmonic coordinates for character articulation" [Joshi et al '07]

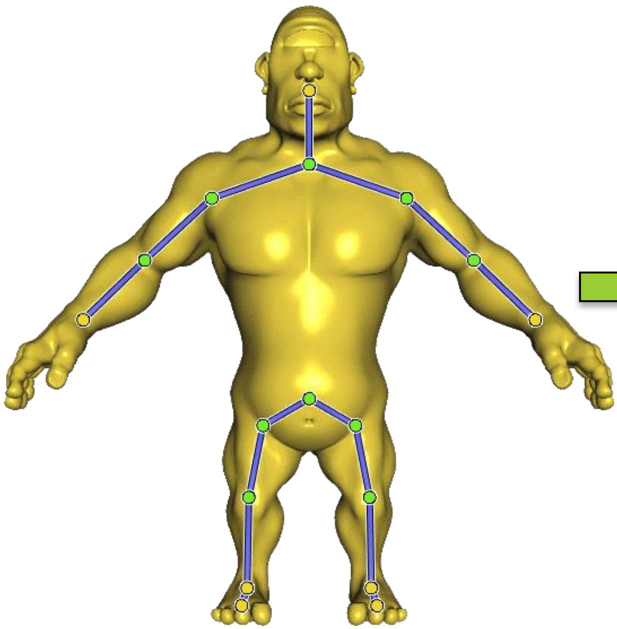
Excellent webpage on barycentric coordinates:
http://www.inf.usi.ch/hormann/barycentric/

Linear Blend Skinning

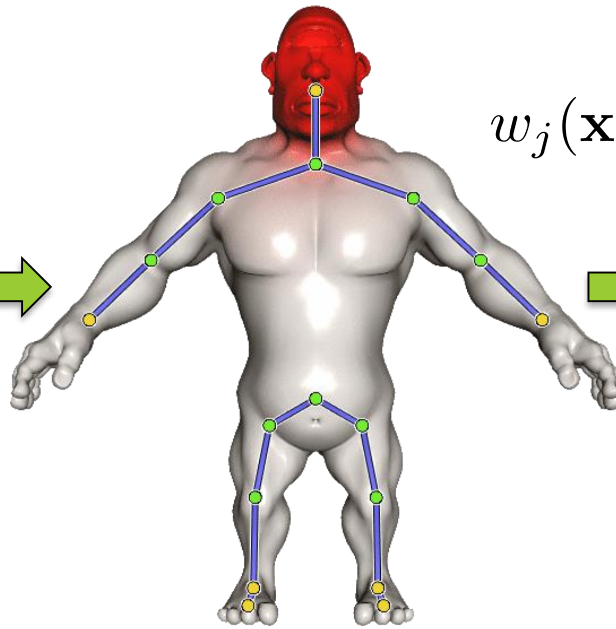


place handles in shape

Linear Blend Skinning



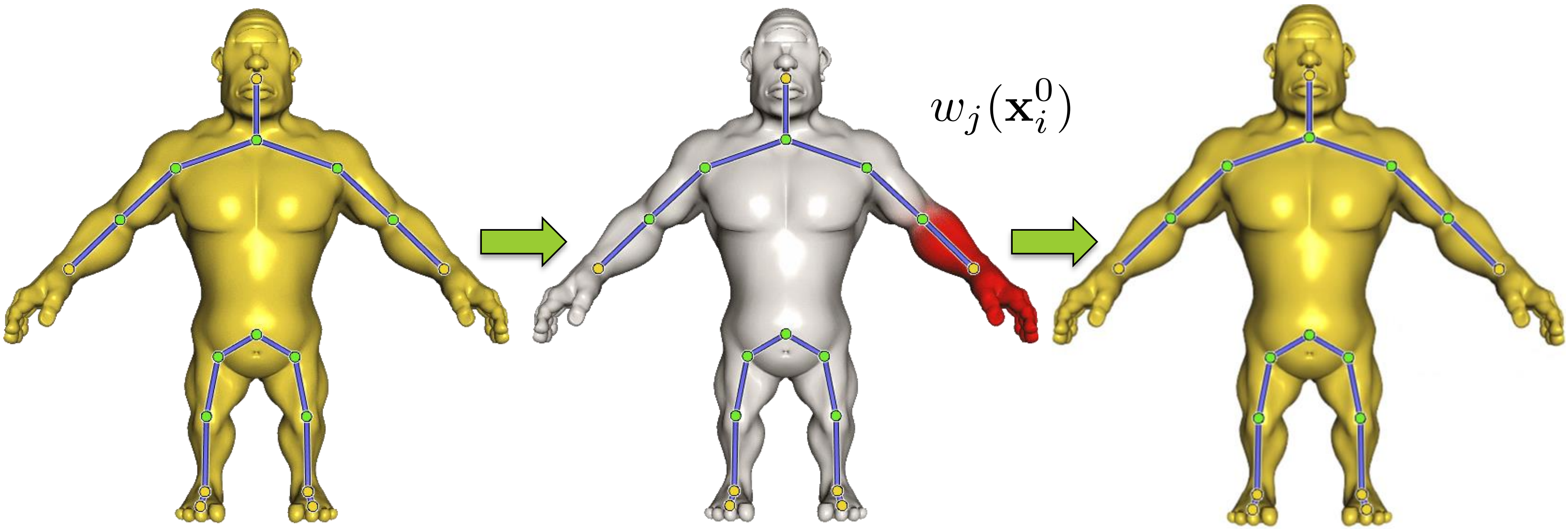
place handles in shape



$$w_j(\mathbf{x}_i^0)$$

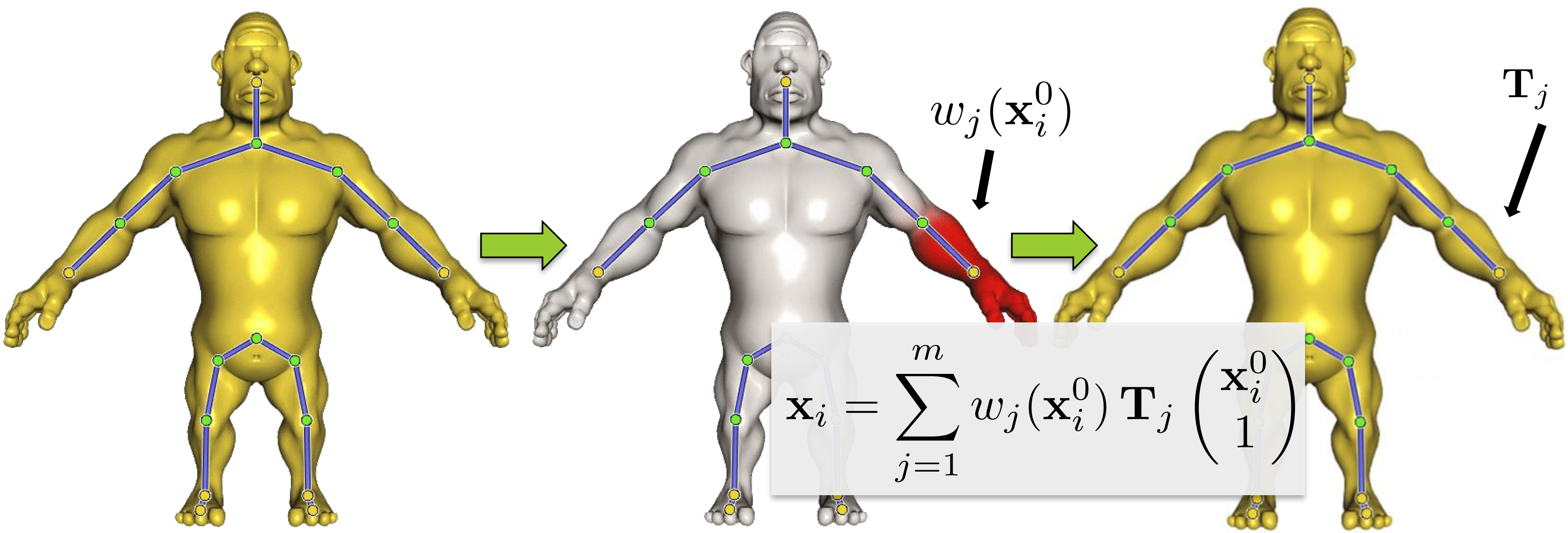
paint weights

Linear Blend Skinning



place handles in shape | paint weights | deform handles

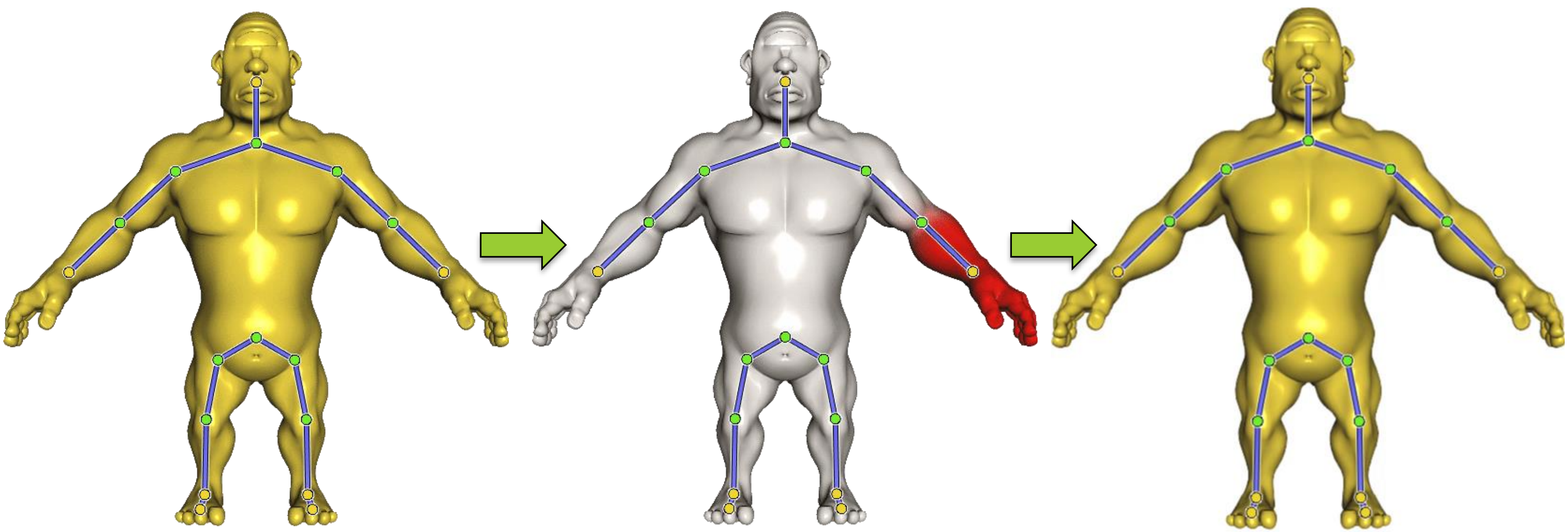
Linear Blend Skinning



$$\mathbf{x}_i = \sum_{j=1}^{m} w_j(\mathbf{x}_i^0)\,\mathbf{T}_j \begin{pmatrix} \mathbf{x}_i^0 \\ 1 \end{pmatrix}$$

place handles in shape | paint weights | deform handles

Linear Blend Skinning



place handles in shape — paint weights — deform handles

Challenges with LBS

$$\mathbf{x}_i = \sum_{j=1}^{m} \underline{w_j(\mathbf{x}_i^0)}\, \underline{\mathbf{T}_j} \begin{pmatrix} \mathbf{x}_i^0 \\ 1 \end{pmatrix}$$

Weight functions w_j

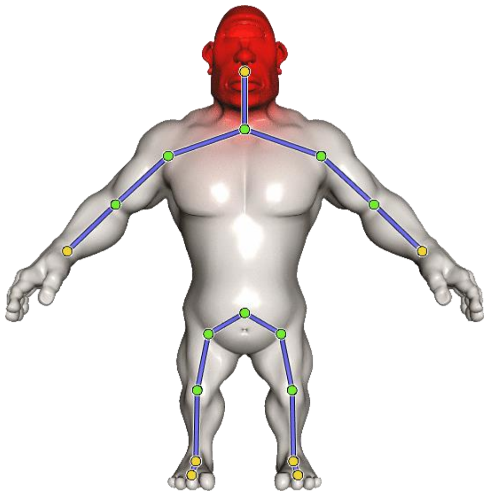
- Need intuitive, general and automatic weights

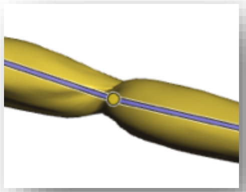
Degrees of freedom $\mathbf{T}_j$

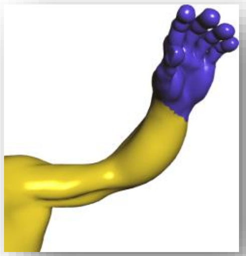
- Decide via optimization?

Richness of achievable deformations

- Want to avoid common LBS pitfalls – candy wrapper, collapses

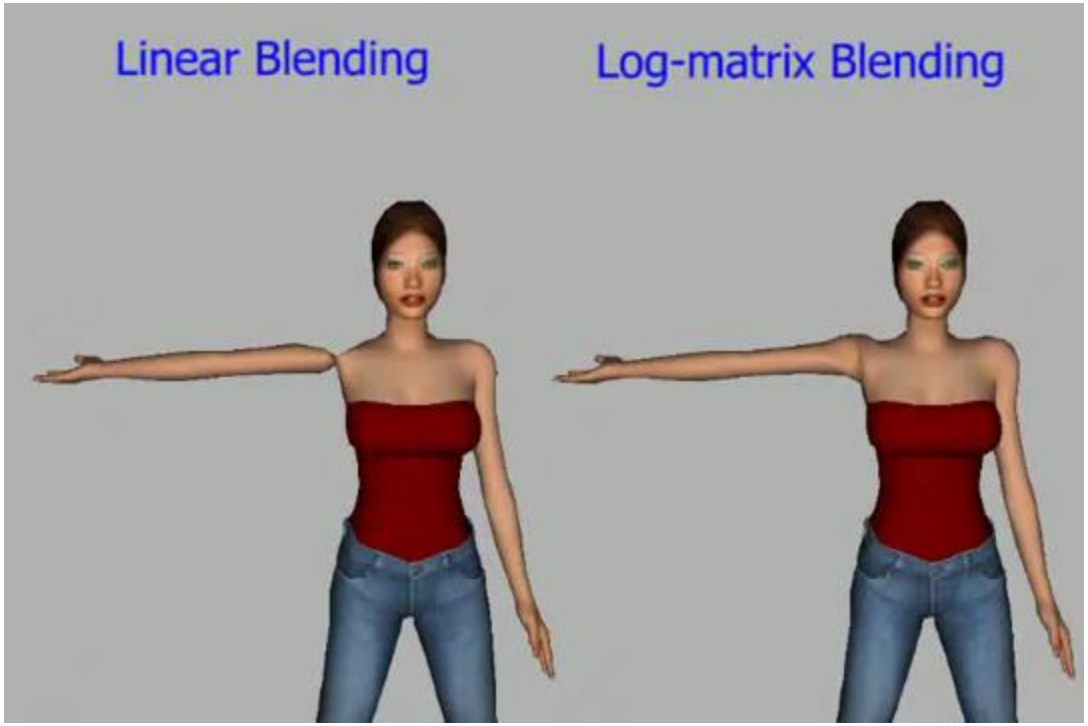






ETH zürich

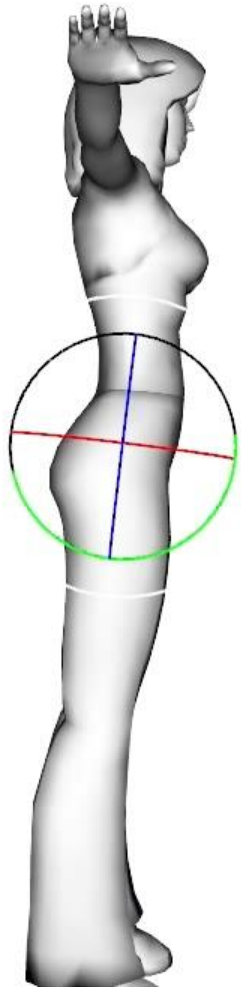
Challenges with LBS



ETH zürich

Challenges with LBS

Problem of standard skinning methods



ETH zürich

Alec Jacobson, Ilya Baran, Jovan Popović, S
ACM SIGGRAPH 2011; selected for Research Highlights in CACM (2014)

Bounded Biharmonic Weights

igl

ETH zürich

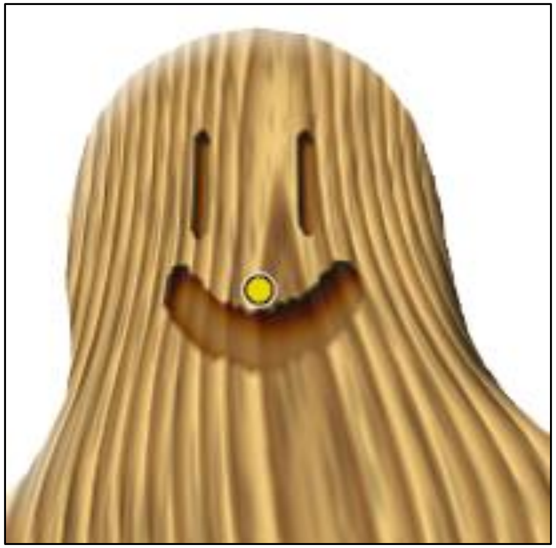
Weights must be smooth everywhere, *especially* at handles

Bounded Biharmonic Weights

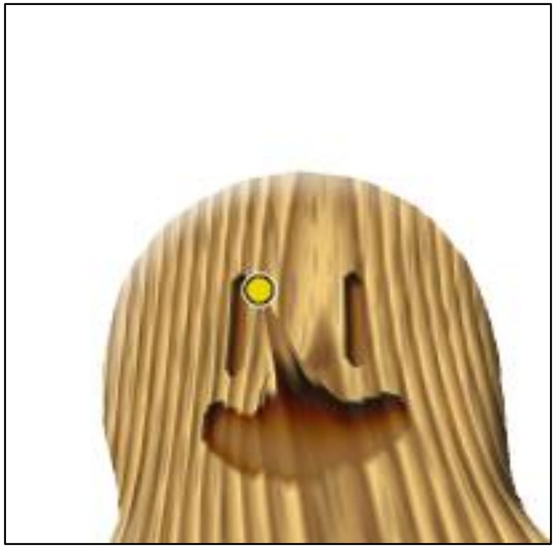
Extension of Harmonic Coordinates
[Joshi et al. 2005]

ETH zürich

Weights must be smooth everywhere, *especially* at handles



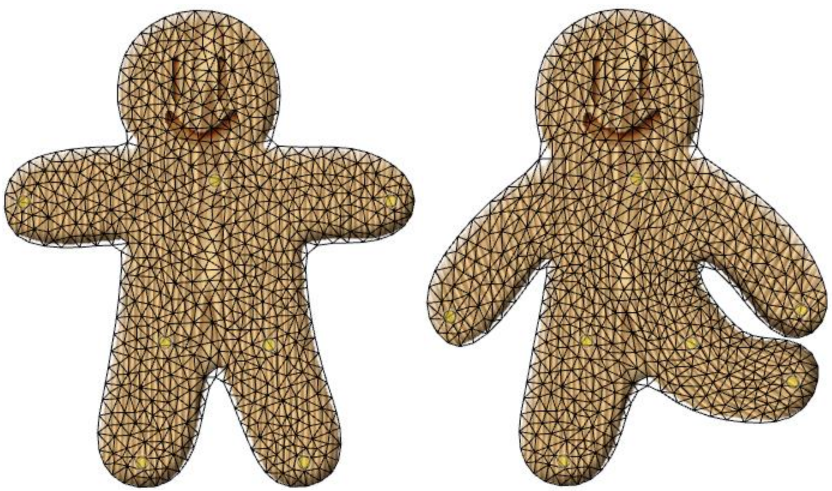
Bounded Biharmonic Weights



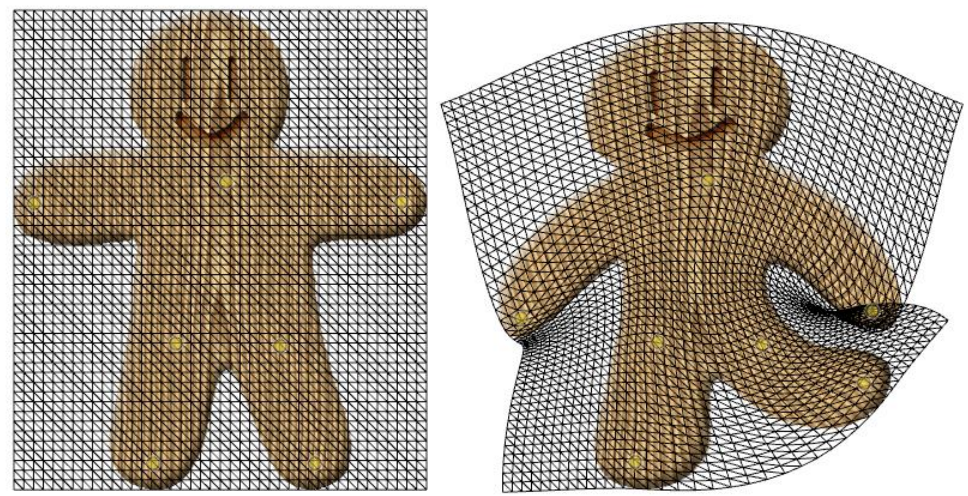
Extension of Harmonic Coordinates
[Joshi et al. 2005]

ETH zürich

Shape-awareness ensures respect of domain's features



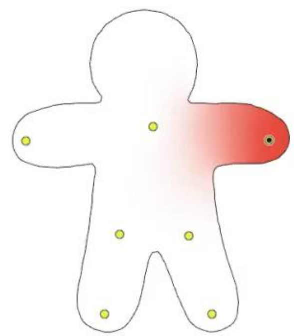
Bounded Biharmonic Weights



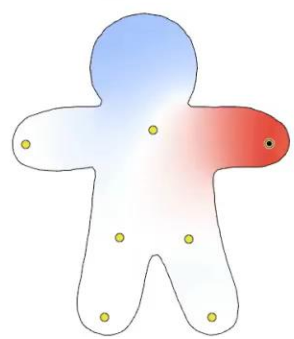
Non-shape-aware methods
e.g. [Schaefer et al. 2006]

Non-negative weights are necessary for intuitive response

Bounded Biharmonic Weights



Unconstrained biharmonic
[Botsch and Kobbelt 2004]



ETH zürich

Weights must maintain other simple, but important properties

$$\sum_{j \in H} w_j(\mathbf{x}^0) = 1$$

Partition of unity

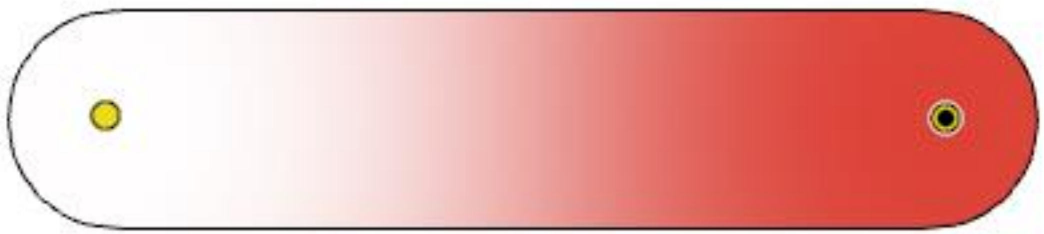
Handle vertices

$$w_j \Big|_{H_k} = \delta_{jk}$$

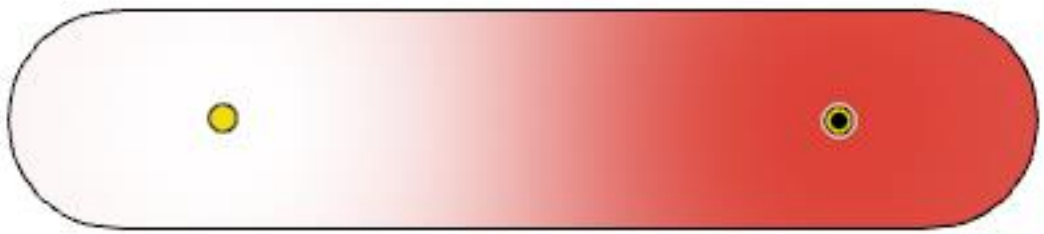
w_j is linear along cage faces

Interpolation of handles

How about $w_j(\mathbf{x}^0) = d(\mathbf{x}^0, H_j)^{-1}$?

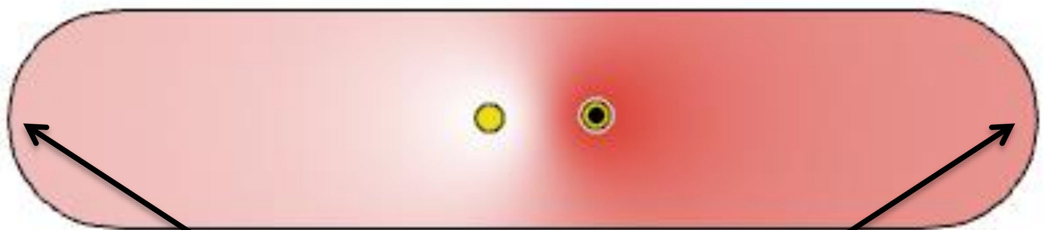


Inverse distance methods inherently suffer from *fall-off effect*



Inverse distance methods inherently suffer from *fall-off effect*

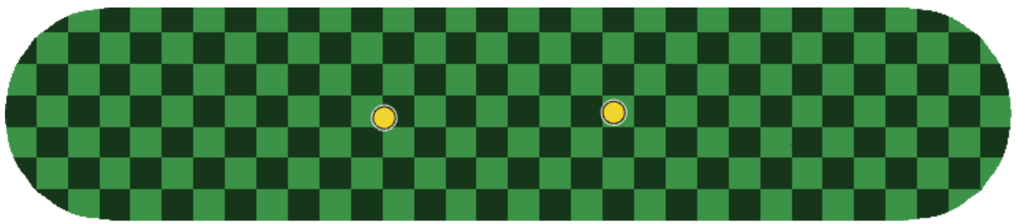
Inverse distance methods inherently suffer from *fall-off effect*



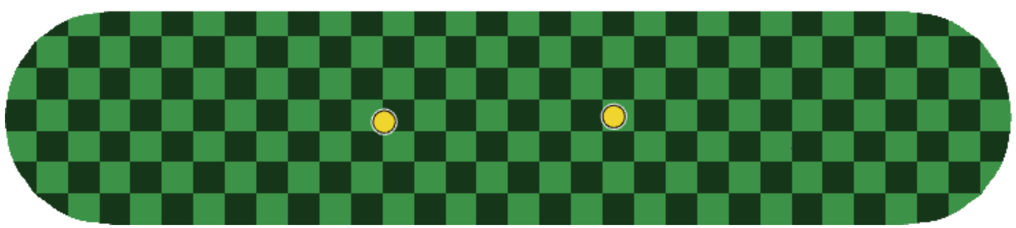
Approaching 0.5

Inverse distance methods inherently suffer from *fall-off effect*

Inverse-distance weights



BBW



Bounded biharmonic weights enforce properties as constraints to minimization

$$\arg\min_{w_j} \frac{1}{2}\int_\Omega |\Delta w_j|^2 dV$$

$$w_j\Big|_{H_k} = \delta_{jk}$$

w_j is linear along cage faces

Bounded biharmonic weights enforce properties as constraints to minimization

$$\arg\min_{w_j} \frac{1}{2}\int_\Omega |\Delta w_j|^2 dV$$

$$w_j\Big|_{H_k} = \delta_{jk}$$

w_j is linear along cage faces

Constant inequality constraints

$$0 \leq w_j(\mathbf{x}^0) \leq 1$$

Partition of unity

$$\sum_{j\in H} w_j(\mathbf{x}^0) = 1$$

ETH zürich

Bounded biharmonic weights enforce properties as constraints to minimization

$$\arg\min_{w_j} \frac{1}{2}\int_\Omega |\Delta w_j|^2 dV$$

$$w_j\Big|_{H_k} = \delta_{jk}$$

w_j is linear along cage faces

Constant inequality constraints

$$0 \leq w_j(\mathbf{x}^0) \leq 1$$

Solve independently and normalize

$$w_j(\mathbf{x}^0) = \frac{w_j(\mathbf{x}^0)}{\sum_{i\in H} w_i(\mathbf{x}^0)}$$

Weights optimized as precomputation at bind-time

$$\arg\min_{w_j} \frac{1}{2}\int_\Omega |\Delta w_j|^2 dV$$

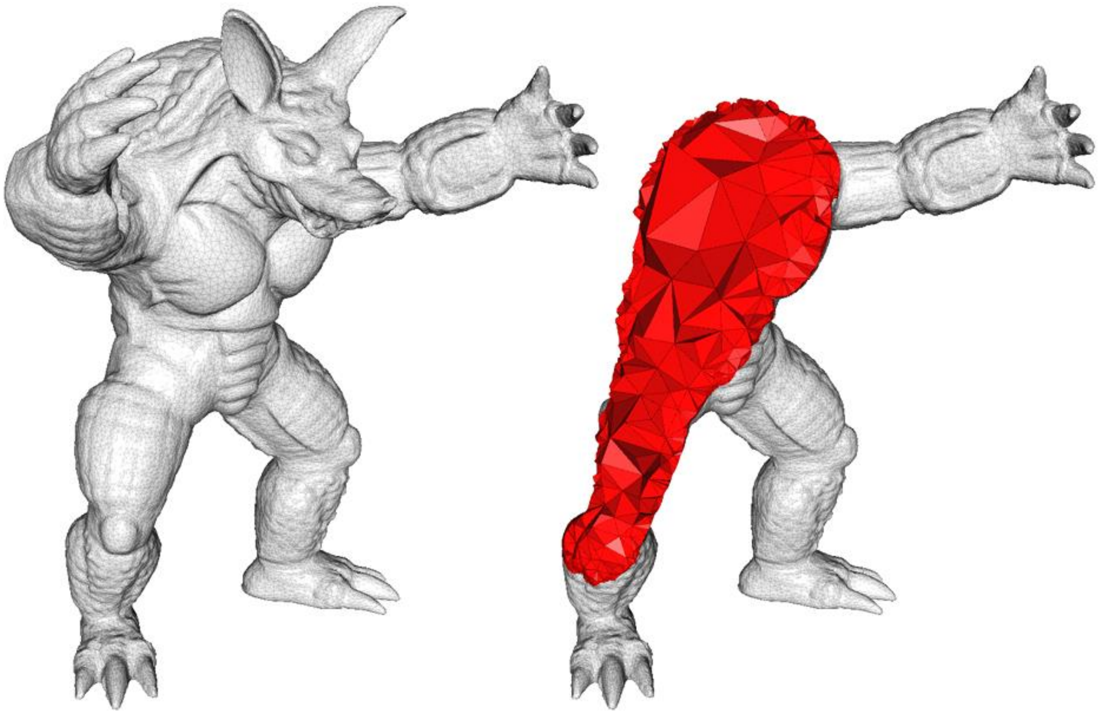
$$w_j\big|_{H_k} = \delta_{jk}$$

w_j is linear along cage faces

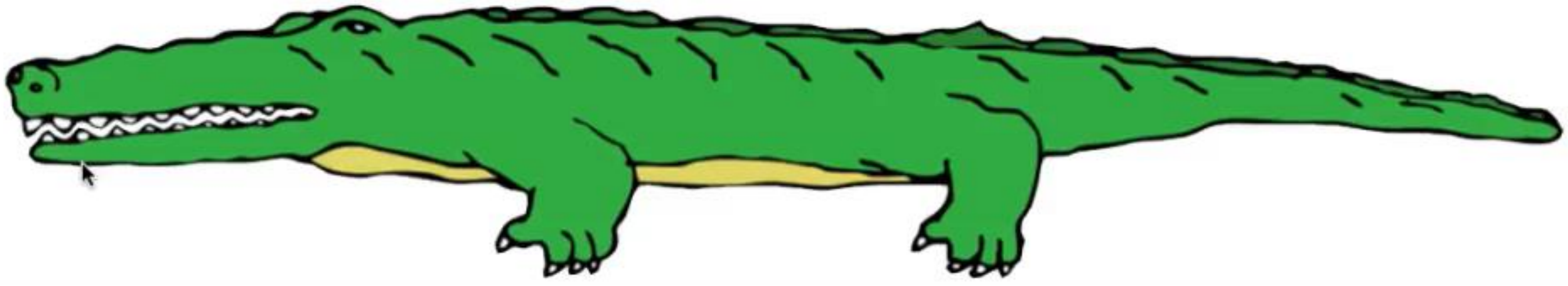
$$0 \leq w_j(\mathbf{x}^0) \leq 1$$

FEM discretization

- 2D → Triangle mesh
- 3D → Tet mesh

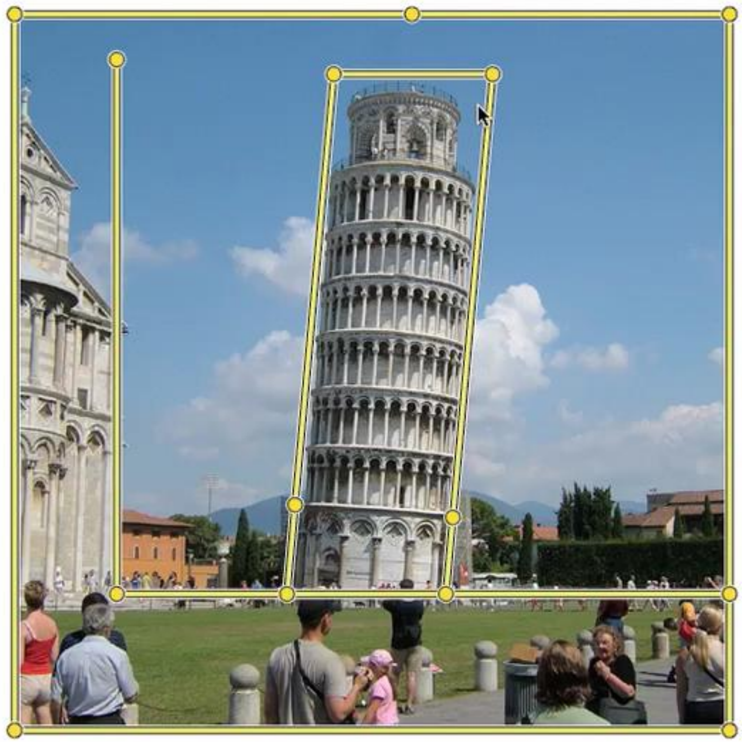


Some examples of BBW in action



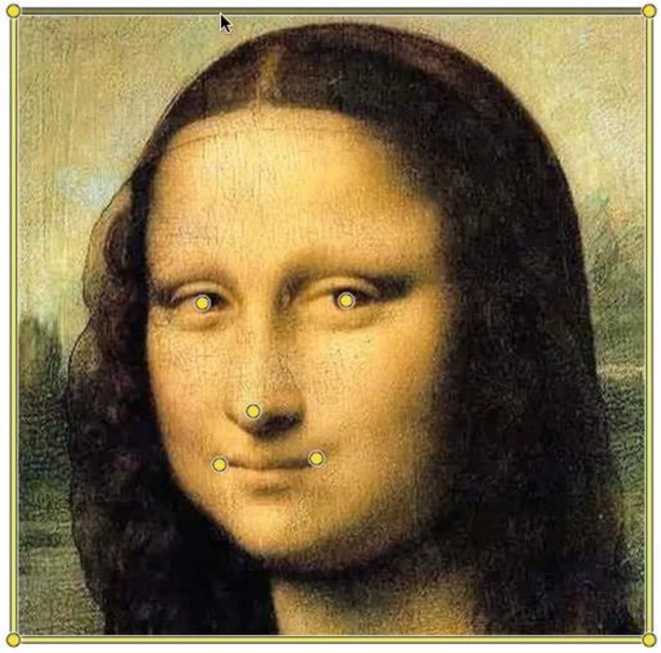
ETH zürich

Some examples of BBW in action

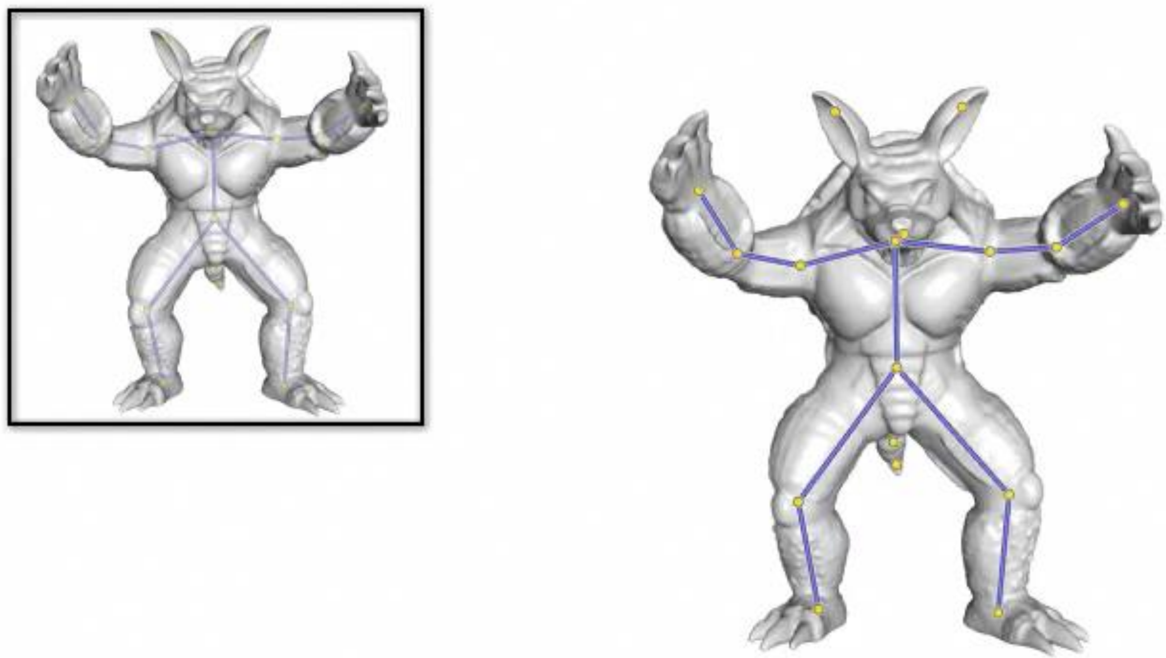


ETH zürich

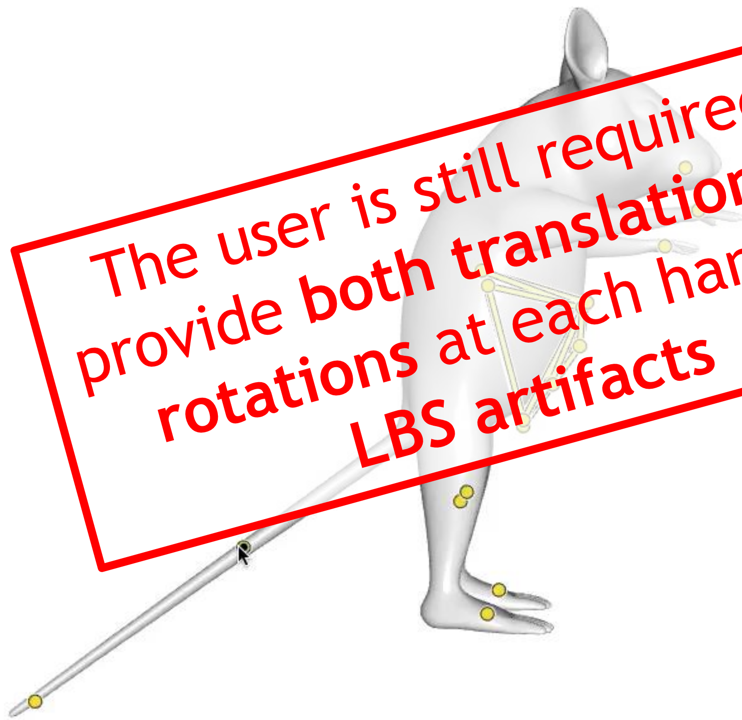
Some examples of BBW in action



3D Characters



Mixing different handle types



The user is still required to provide **both translations** and **rotations** at each handle; **LBS artifacts**

Thank You!