252-0538-00L, Spring 2017

Shape Modeling and Geometry Processing

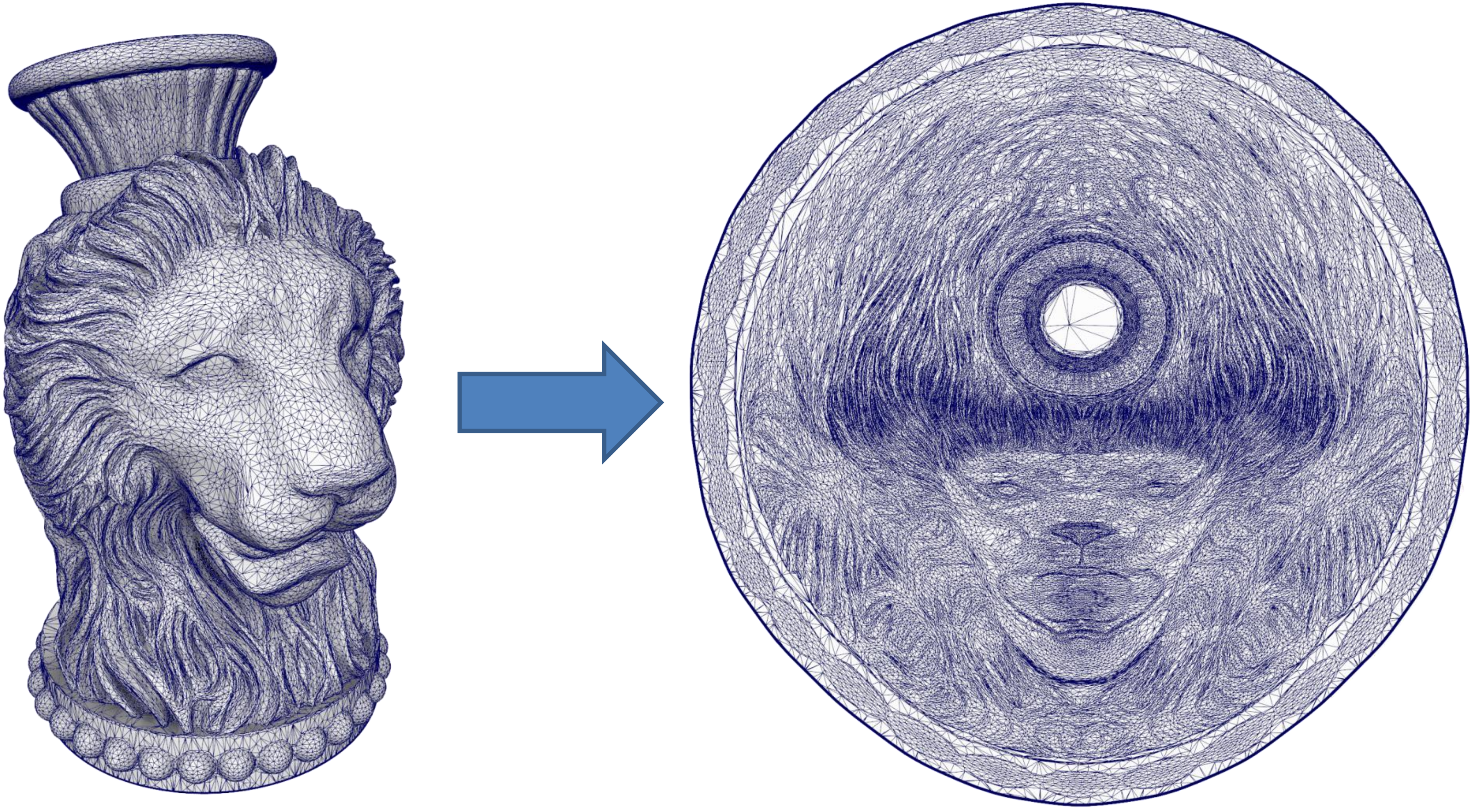
Exercise 4: Mesh Parameterization

INTERACTIVE GEOMETRY LAB

April 7, 2017

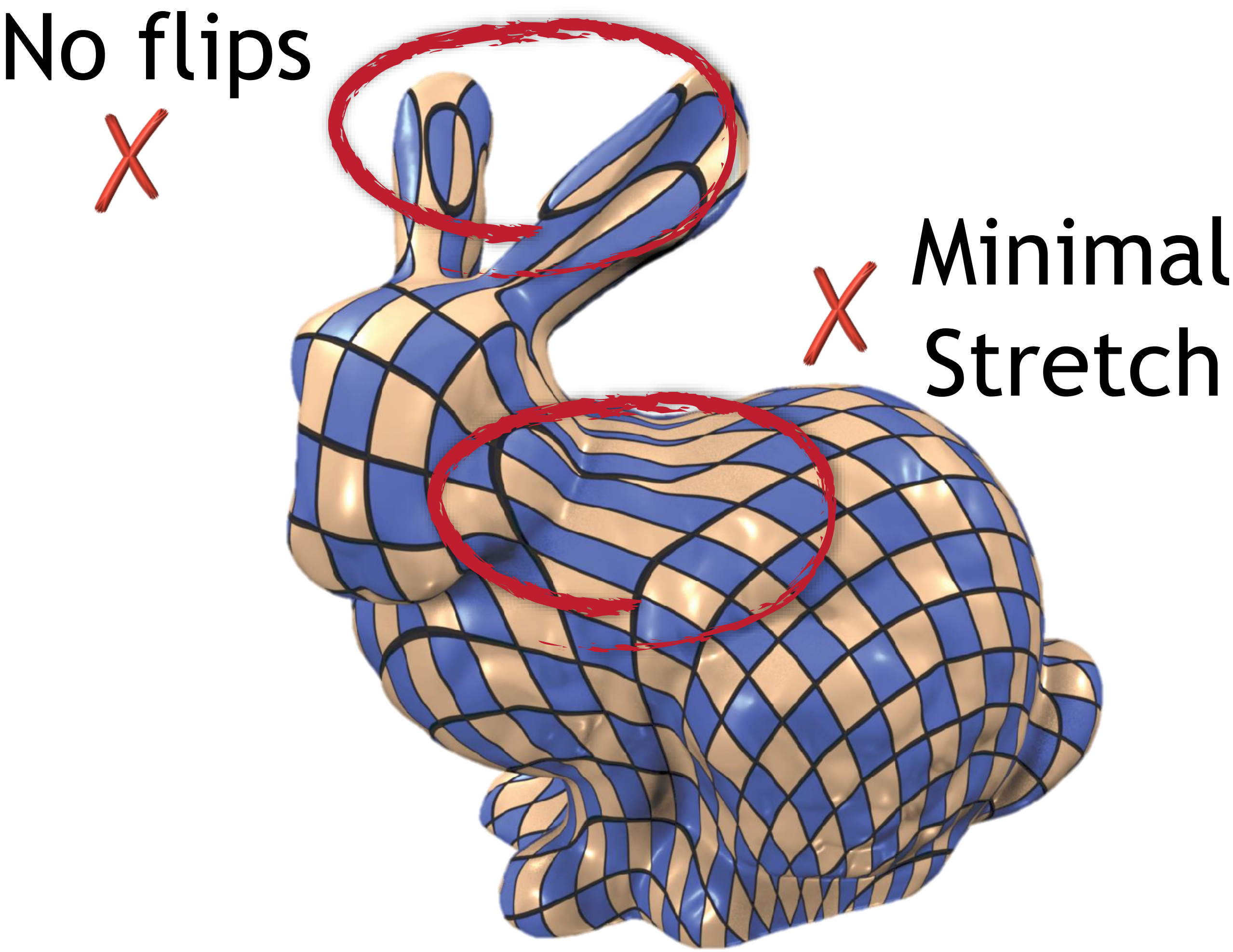
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

What Is a Parameterization?

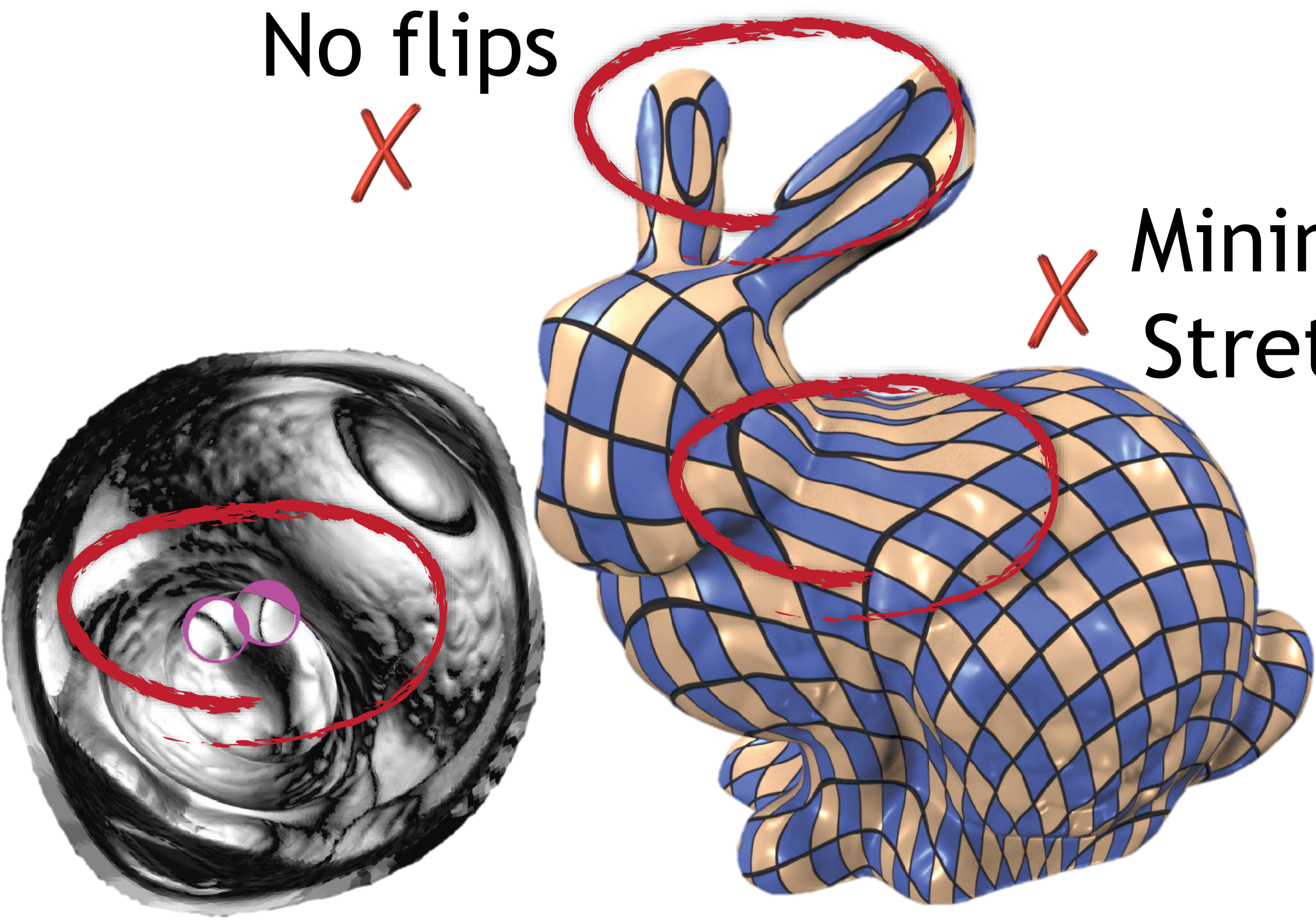


What Is a Parameterization?

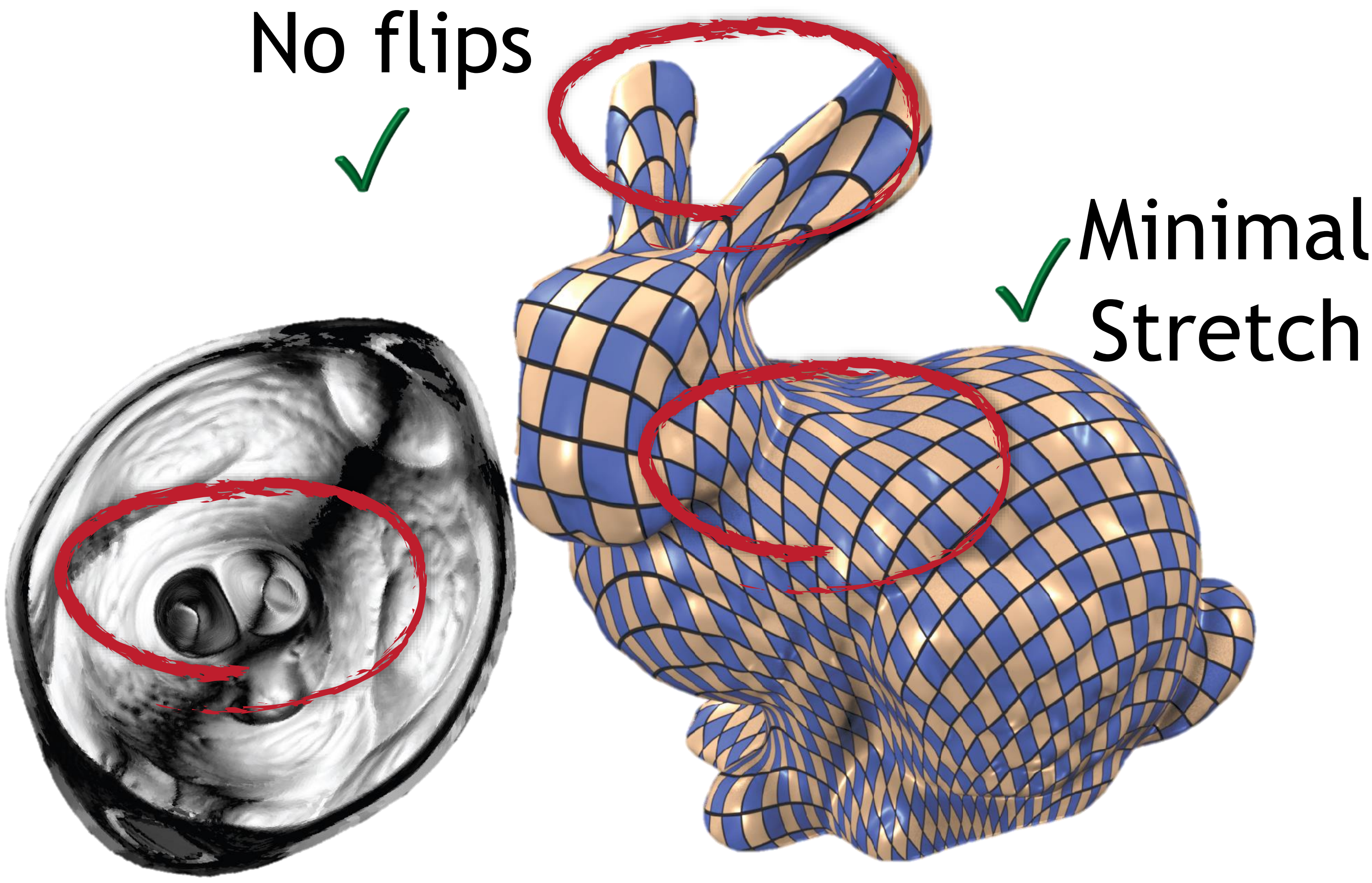
Good Parameterization



Good Parameterization



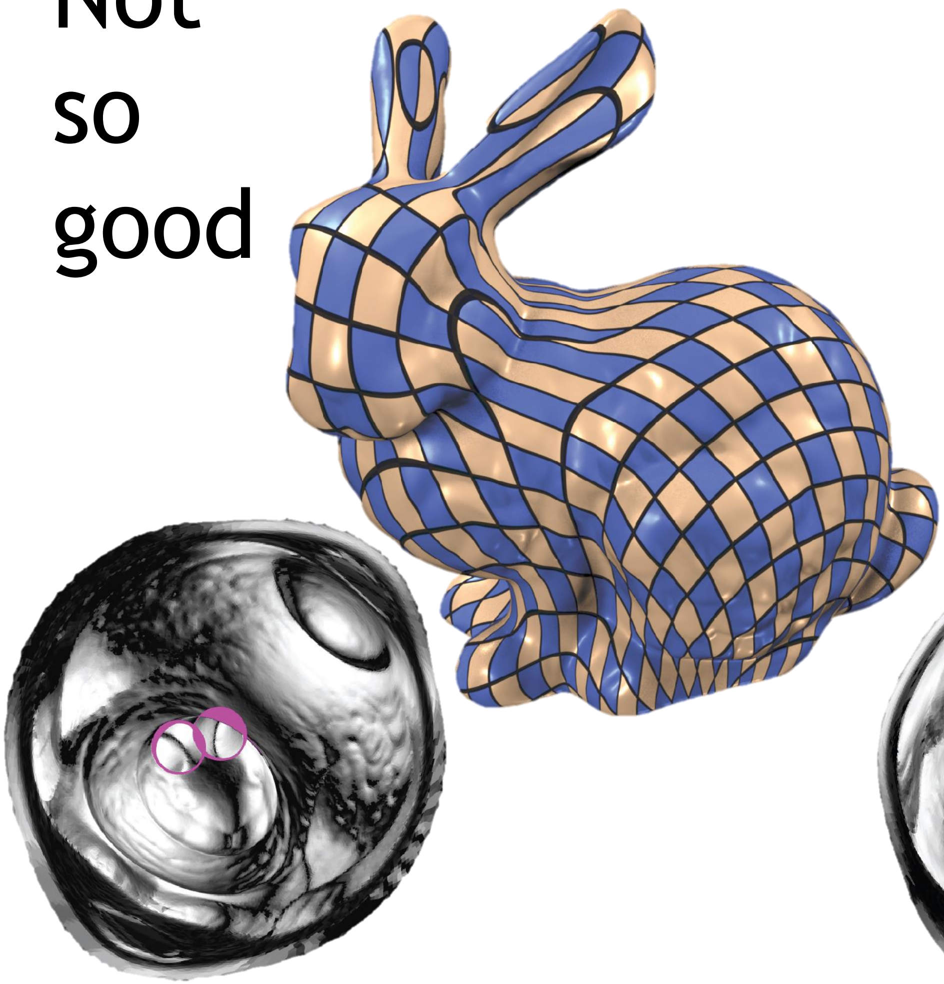
Good Parameterization

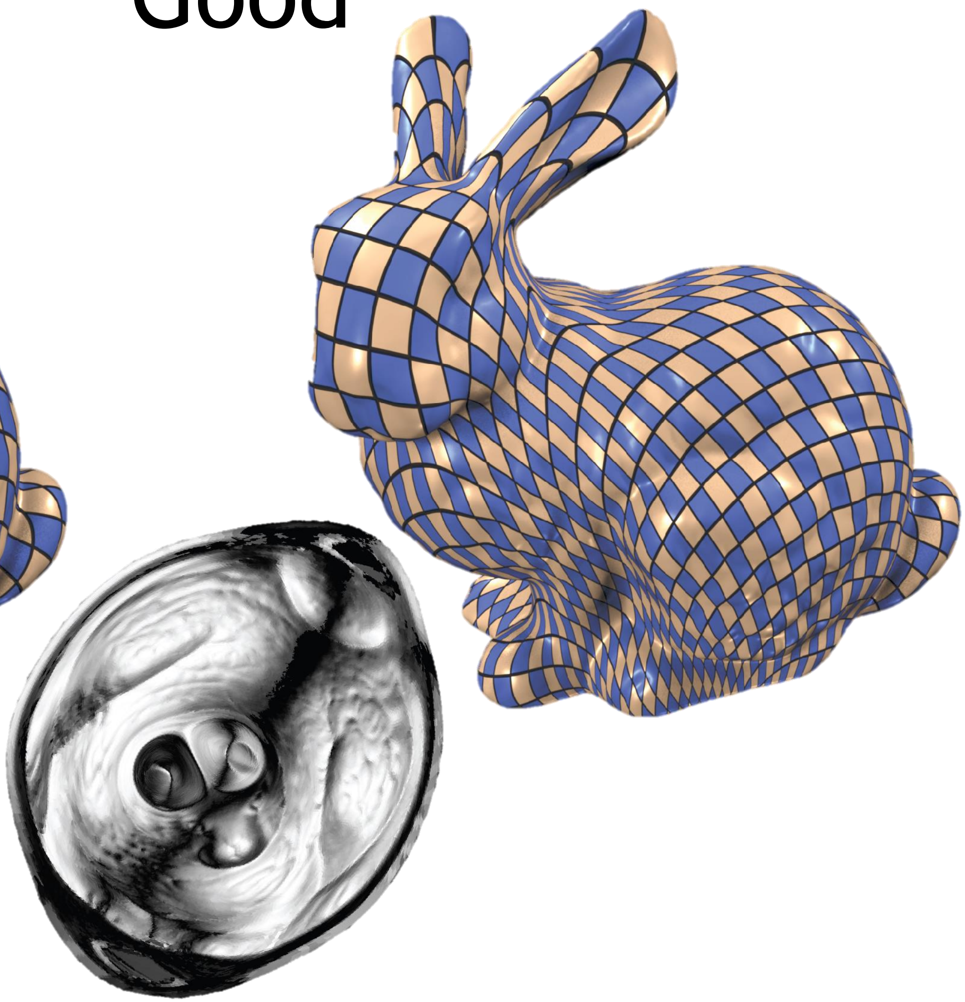


Good Parameterization

Not so good

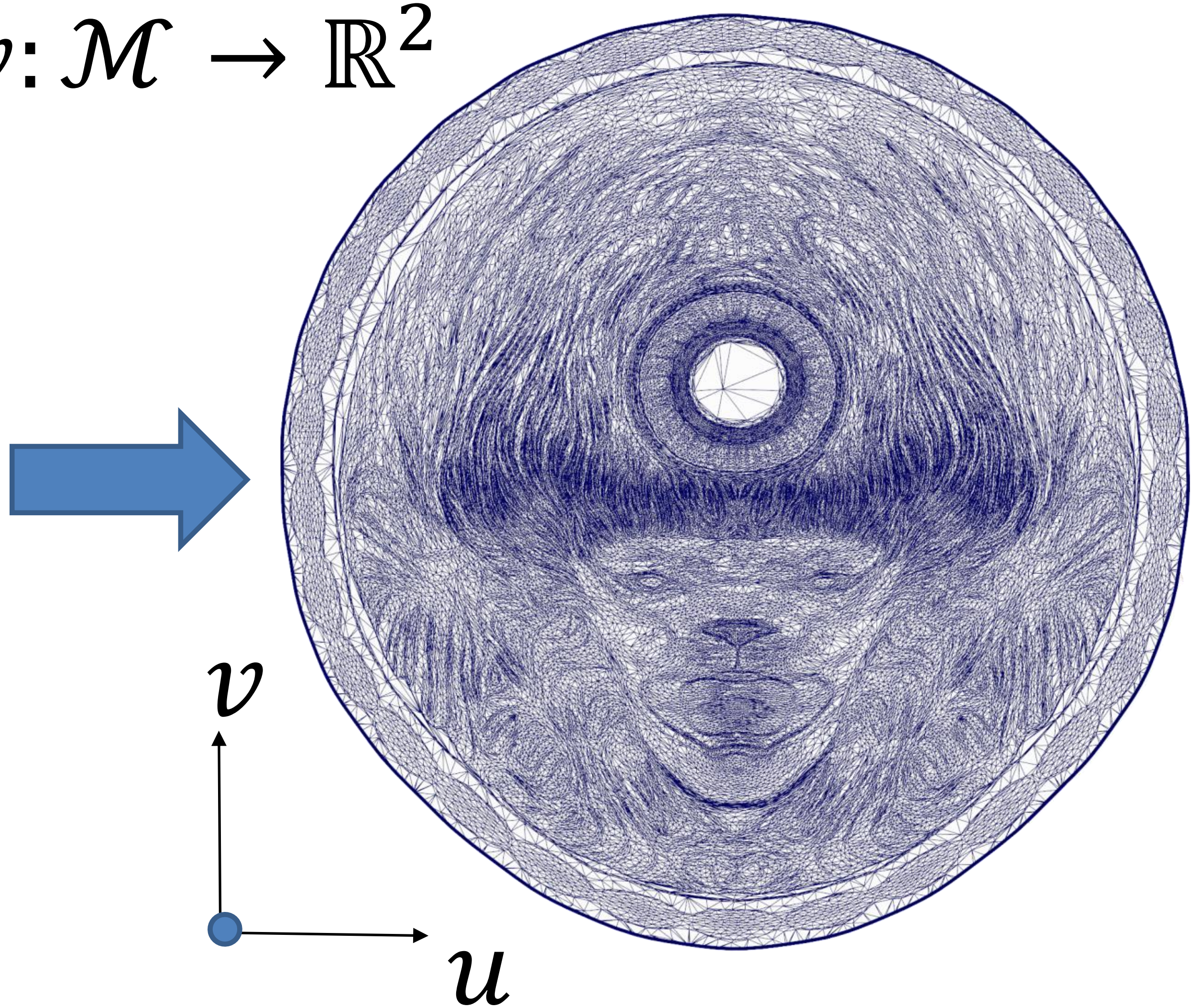
Good





What Is a Parameterization?

$$u, v: \mathcal{M} \rightarrow \mathbb{R}^2$$



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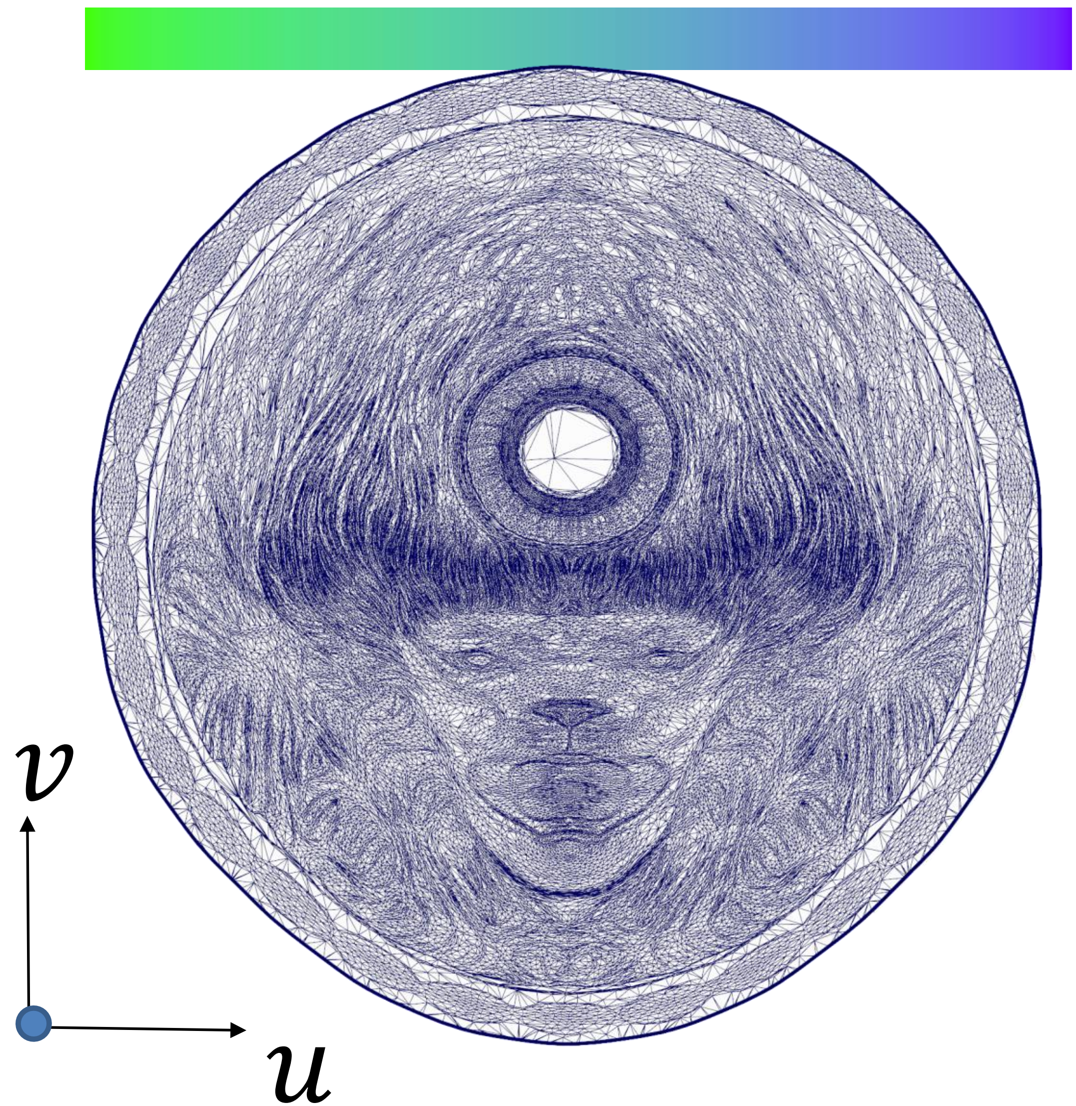
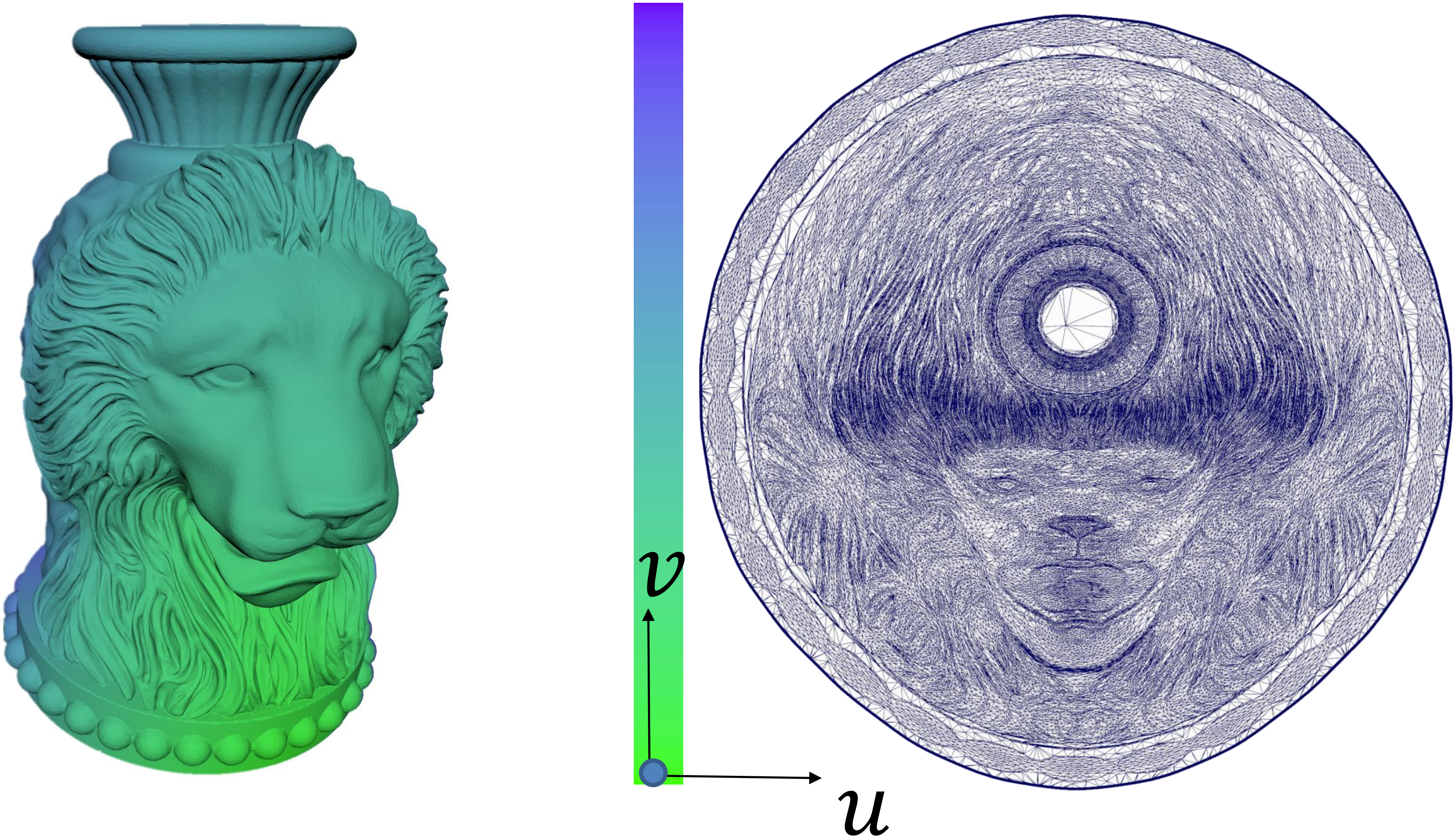


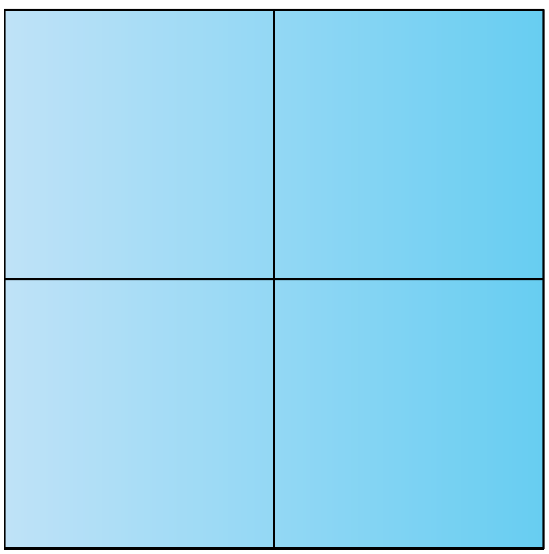
What Is a Parameterization?



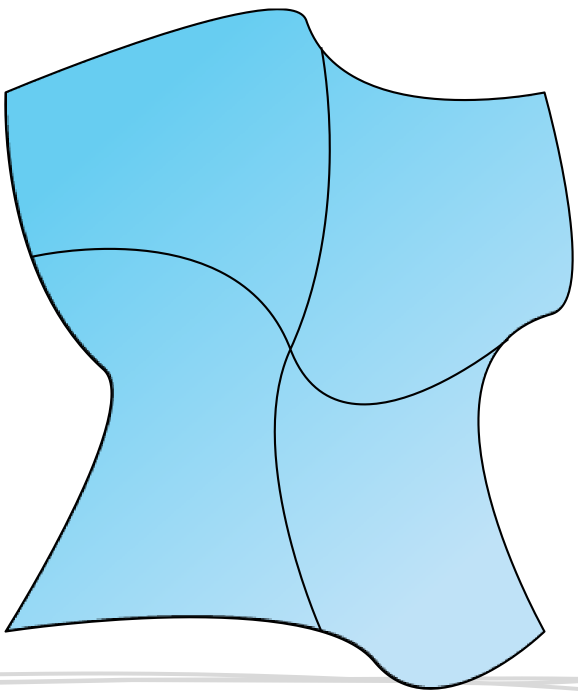
The Jacobian

- A mapping $\mathbf{f}(x,y)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by two functions:

$$u(x,y) \qquad v(x,y)$$



$$\mathbf{f}(x,y) = (u,v)$$



The Jacobian

- A mapping $\mathbf{f}(x, y) \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by two functions:

$$u(x, y) \qquad v(x, y)$$

- The Jabocian $\mathbf{J}$ is defined by

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}$$

The Jacobian

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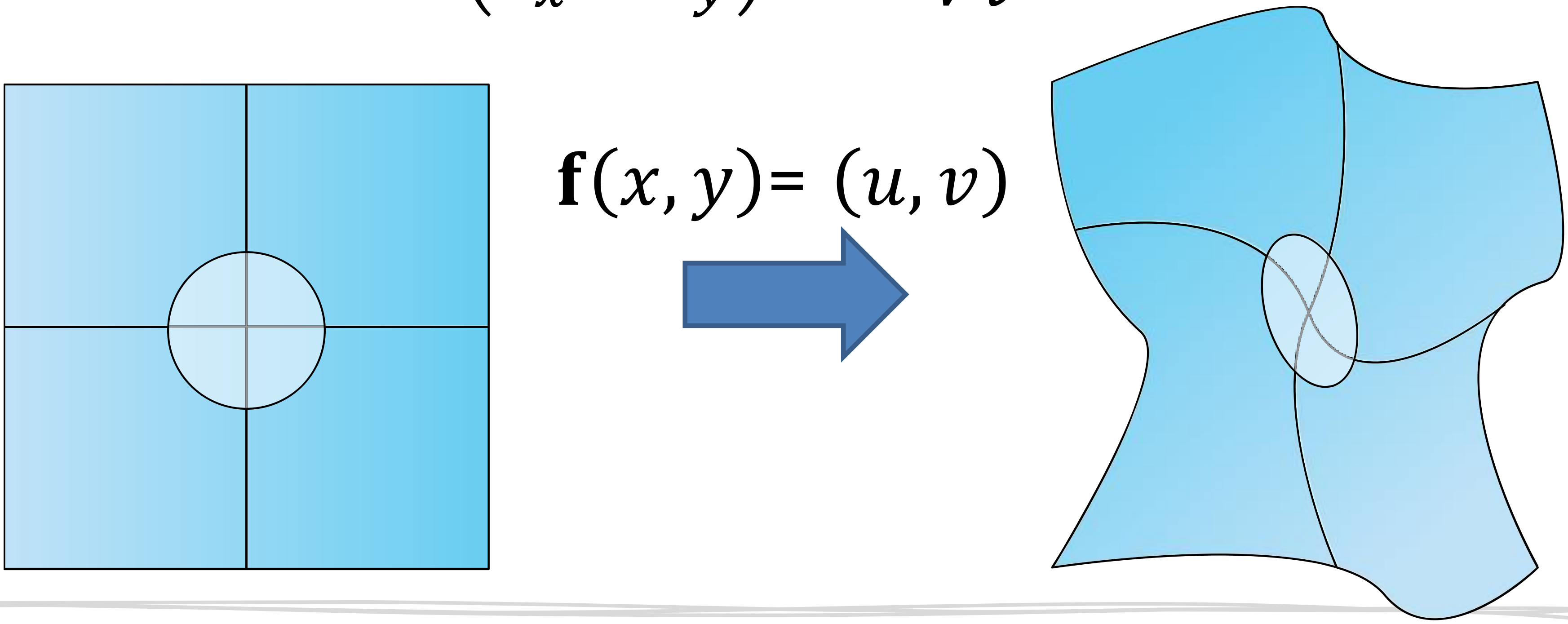
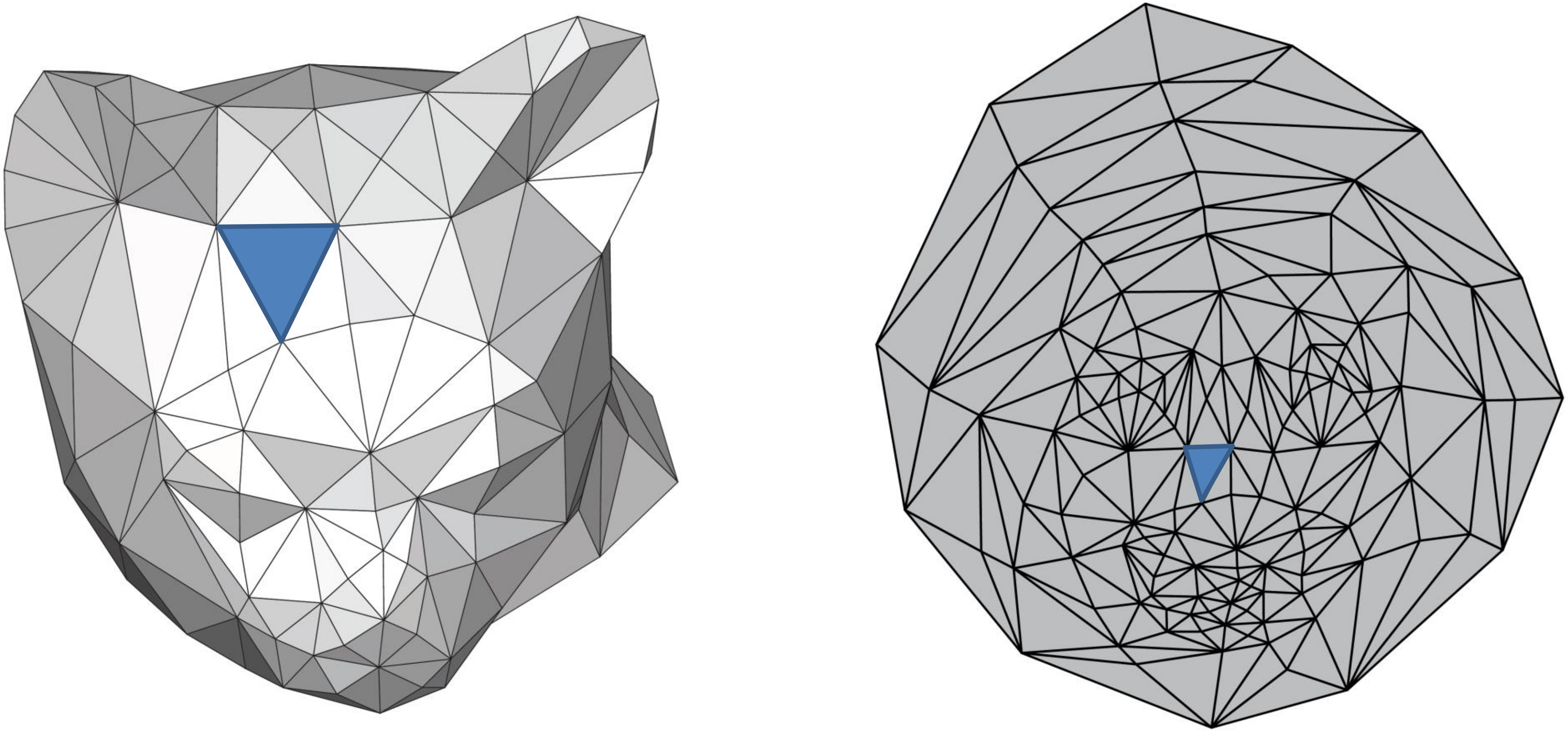
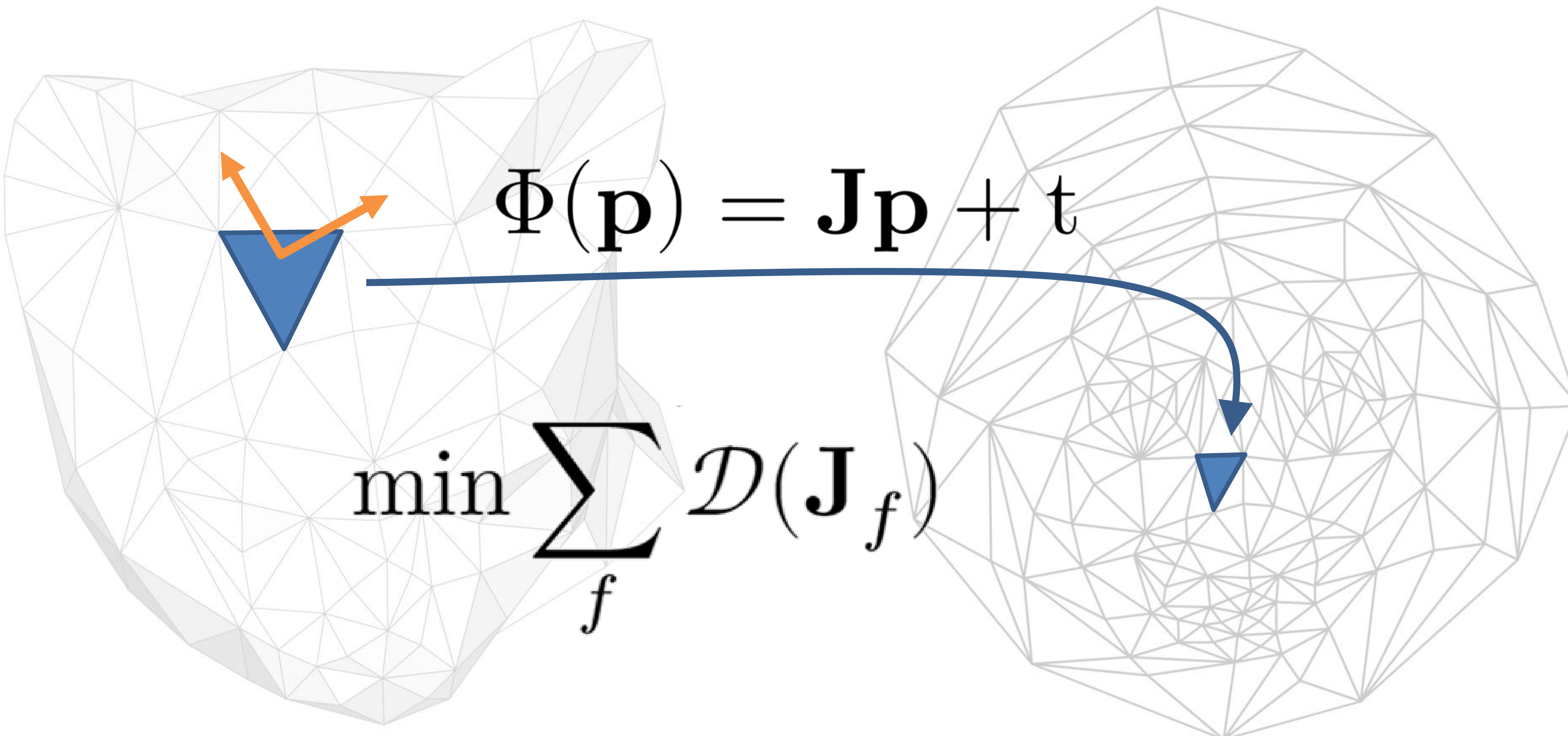


Distortion Measures



Distortion Measures



$$\Phi(\mathbf{p}) = \mathbf{J}\mathbf{p} + \mathrm{t}$$

$$\min \sum_{f} \mathcal{D}(\mathbf{J}_f)$$

Distortion types

- Conformal – angle preserving

$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} + \mathbf{J}^{\mathrm{T}} - (tr\,\mathbf{J})\,\mathbf{I} \right\|_F^2$$

- Isometric – length preserving

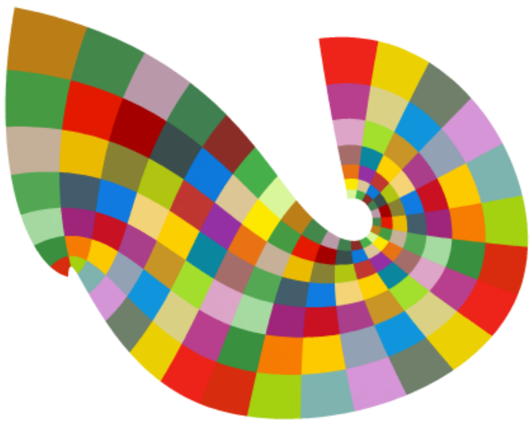
$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} - \mathrm{R} \right\|_F^2$$

- Authalic - area presereing

$$\mathcal{D}(\mathbf{J}) = (\det \mathbf{J})^2$$

Distortion types

Conformal



Isometric

Distortion types

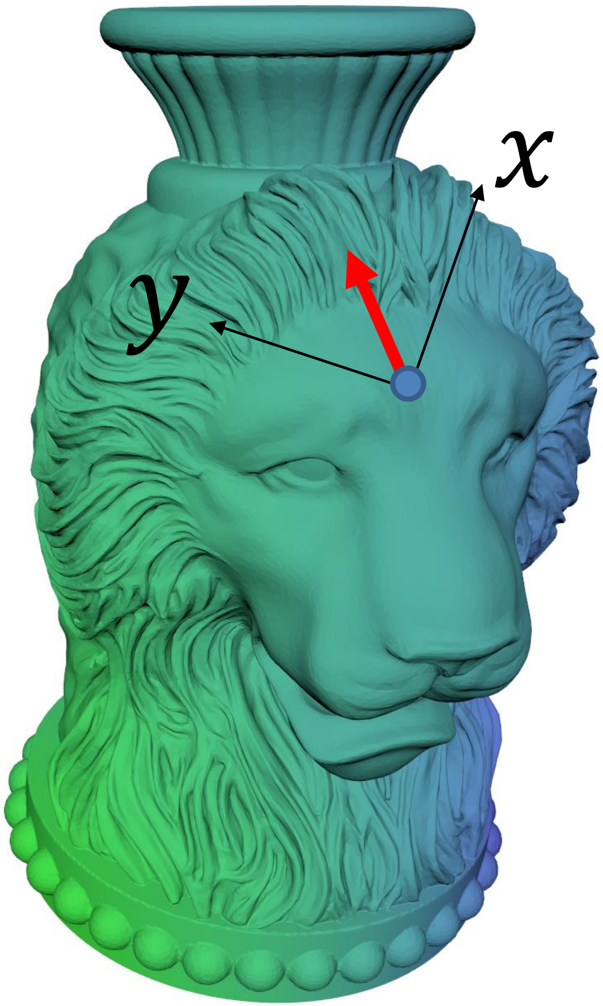
Conformal

Isometric

Gradients on surfaces

- Like the Euclidean gradient
- Arrow pointing in the steepest direction
- libIGL tutorial 204

Gradients on surfaces

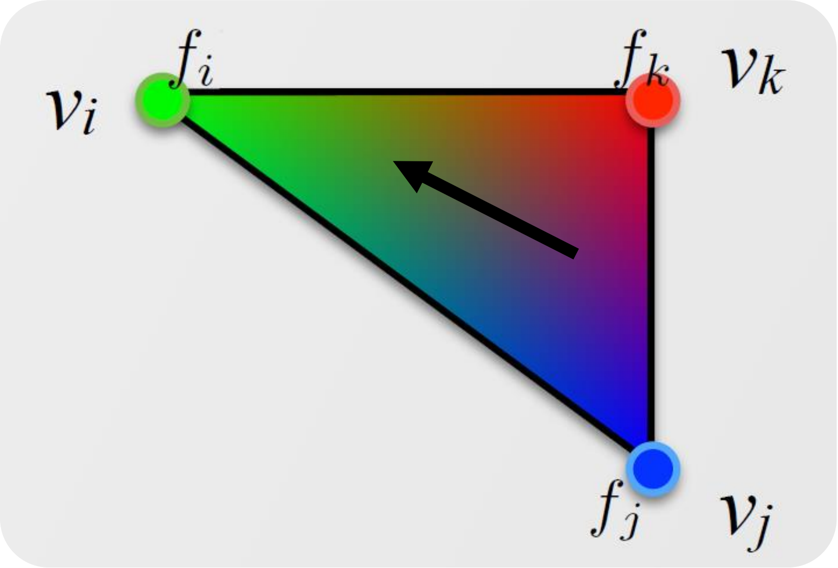


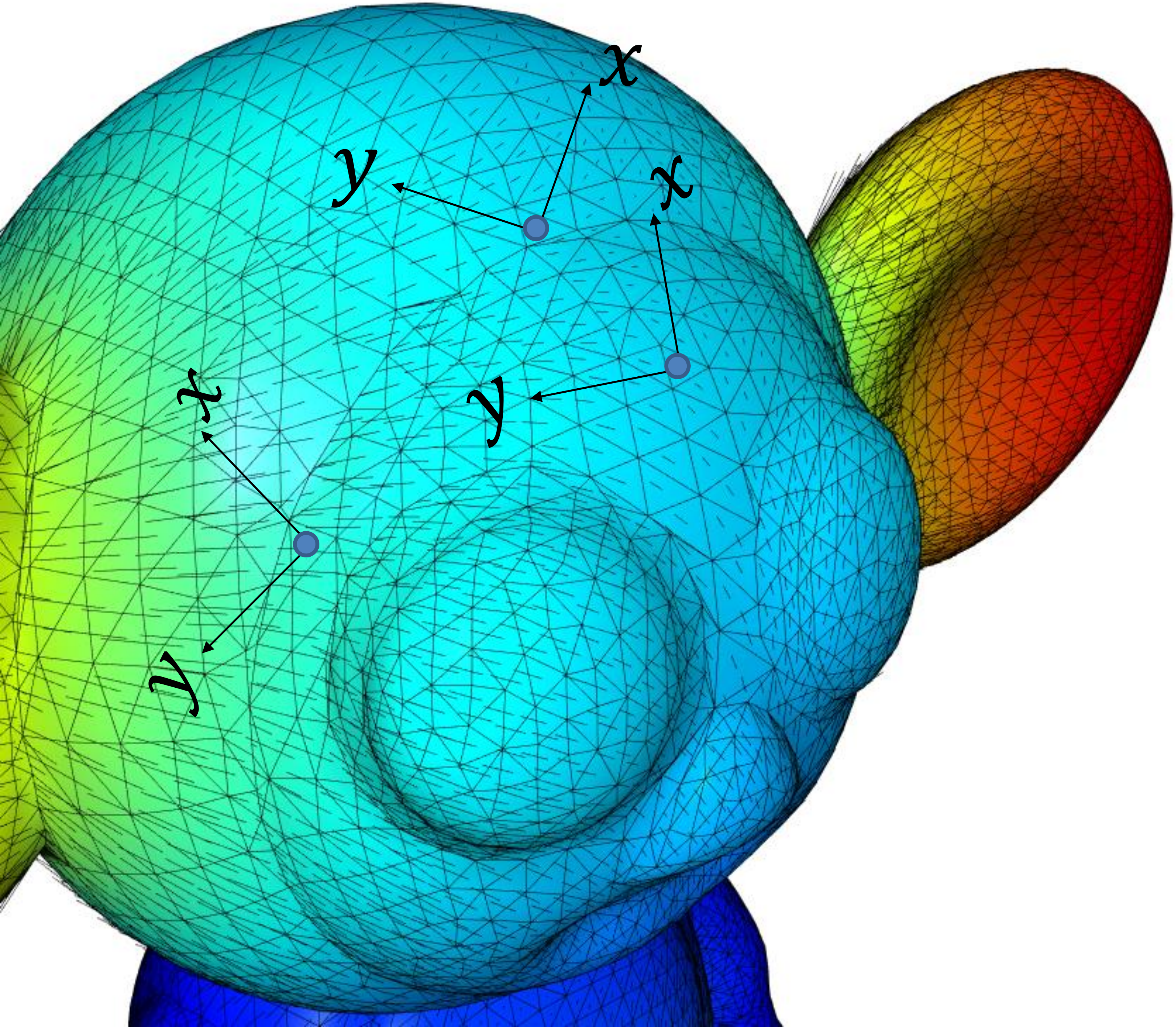
Arrows are tangent to the surface

described in *local frames*

Discrete Gradients

- Functions are interpolated linearly on each face
- The gradient is an arrow on a face





x
y
x
y
x
y

The Gradient Matrices

$$D_x \qquad D_y$$

Given a local frame per face compute the directional derivative w.r.t. the frames

$$\underbrace{\nabla \overbrace{f}^{\#V \times 1}}_{\#F \times 2} = \underbrace{\left(D_x f, \overbrace{D_y}^{\#F \times \#V} f\right)}_{\#F \times 2}$$

Example: Dirichlet energy

$$\min \sum_f \mathcal{D}(\mathbf{J}_f) \qquad \mathcal{D}(\mathbf{J}) = A_f \|\mathbf{J}\|_F^2$$

$$\sum_f A_f \|\mathbf{J}\|_F^2 =$$

$$= \Big\| \underbrace{A^{0.5}\big(D_x u, D_y u, D_x v, D_y v\big)}_{\#F \times 4} \Big\|_F^2$$

$$A = \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{\#F} \end{bmatrix}$$

Example: Dirichlet energy

$$= \left\| A^{0.5}\left(D_x u, D_y u, D_x v, D_y v\right) \right\|_F^2$$

$$= \left\| \begin{pmatrix} A^{0.5} D_x & 0 \\ A^{0.5} D_y & 0 \\ 0 & A^{0.5} D_x \\ 0 & A^{0.5} D_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2$$

$$= \left\| \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2$$

Example: Dirichlet energy

$$\left\|\mathcal{A}\begin{pmatrix}u\\v\end{pmatrix}\right\|^2$$

$$\min_{u,v}\left\|\mathcal{A}\begin{pmatrix}u\\v\end{pmatrix}\right\|^2 \rightarrow \text{solve } \mathcal{A}^{\mathrm{T}}\mathcal{A}\begin{pmatrix}u\\v\end{pmatrix} = 0$$

$$\mathcal{A}^{\mathrm{T}}\mathcal{A} = \begin{pmatrix} D_x^T A D_x + D_y^T A D_y & 0 \\ 0 & D_x^T A D_x + D_y^T A D_y \end{pmatrix}$$

Example: Dirichlet energy

$$\mathcal{A}^{\mathrm{T}}\mathcal{A} = \begin{pmatrix} D_x^T A D_x + D_y^T A D_y & 0 \\ 0 & D_x^T A D_x + D_y^T A D_y \end{pmatrix}$$

$$\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$$

Cotangent Laplacian

In the assignment

- Conformal – angle preserving

$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} + \mathbf{J}^{\mathrm{T}} - (tr\,\mathbf{J})\,\mathbf{I} \right\|_F^2$$

- Isometric – length preserving

$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} - \mathrm{R} \right\|_F^2$$

Bonus

- ~~Authalic - area presereing~~

$$\cancel{\mathcal{D}(\mathbf{J}) = (\det \mathbf{J})^2}$$

Boundary Conditions

In the assignment you are asked to fixed the boundary of the parameterization to a disc, or to fixed two vertices. These constraints can be specified by a linear system

$$C \begin{pmatrix} u \\ v \end{pmatrix} = d$$

Boundary Conditions

$$\min_{u,v} \left\| \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} - b \right\|^2$$

$$S.T. \quad C \begin{pmatrix} u \\ v \end{pmatrix} = d$$

$$\text{solve} \quad \begin{pmatrix} \mathcal{A}^{\mathrm{T}}\mathcal{A} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathcal{A}^{\mathrm{T}} b \\ d \end{pmatrix}$$

Lagrange multipliers
Can be ignored

Thank You

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