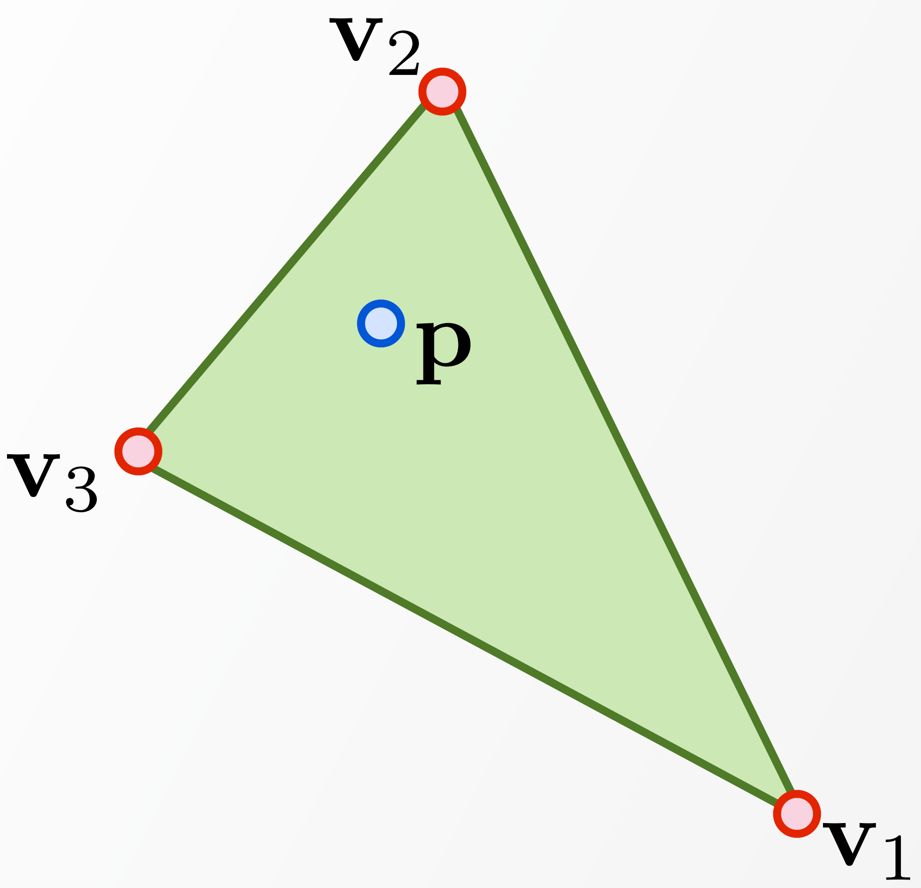
Shape Modeling and Geometry Processing

Derivation of the cotangent formula

INTERACTIVE GEOMETRY LAB

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Barycentric Coordinates



$$\mathbf{p} = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + w_3\mathbf{v}_3$$

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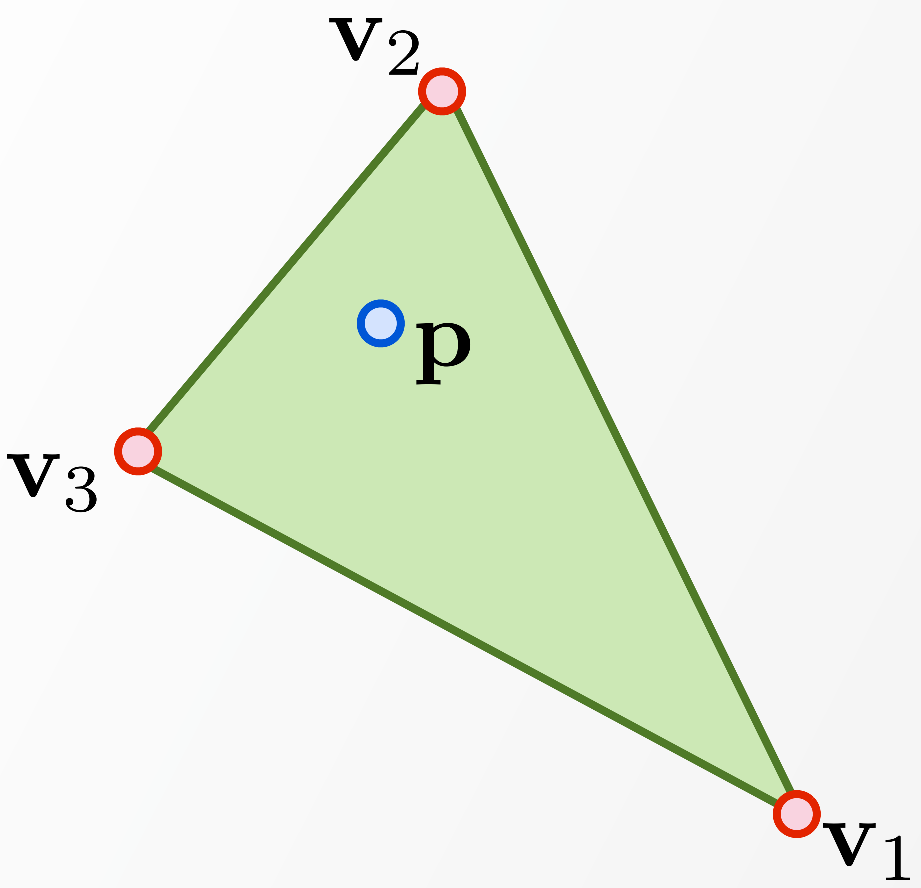

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Partition of unity: $w_1 + w_2 + w_3 = 1$

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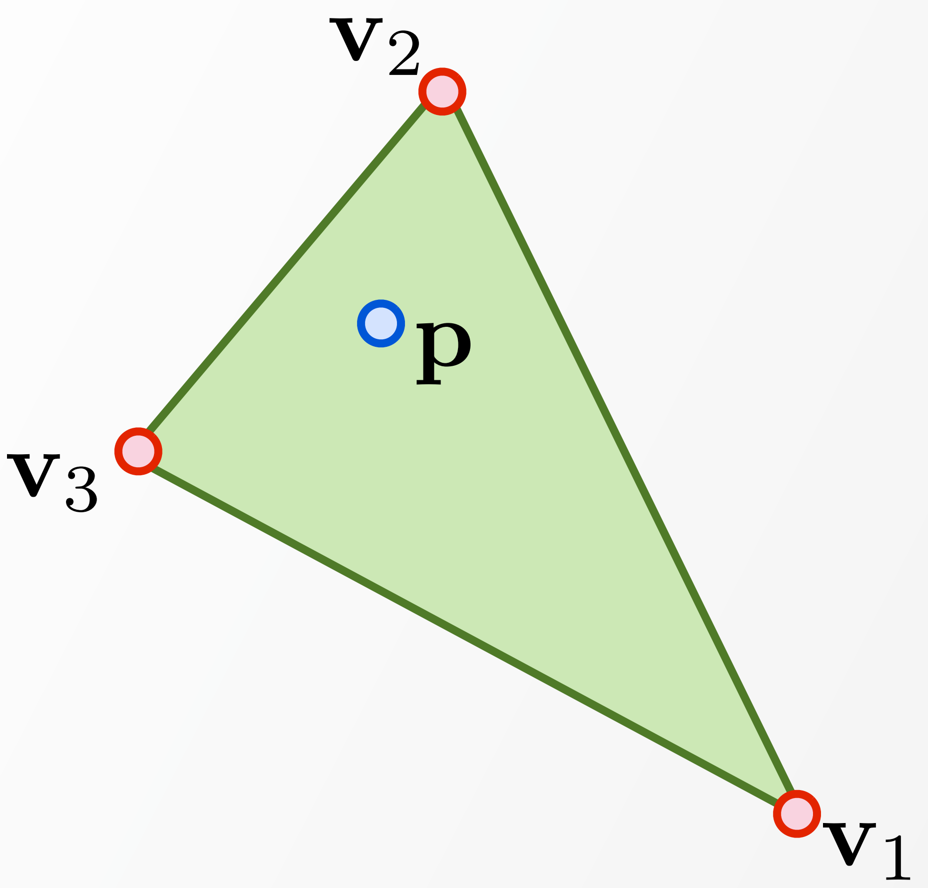

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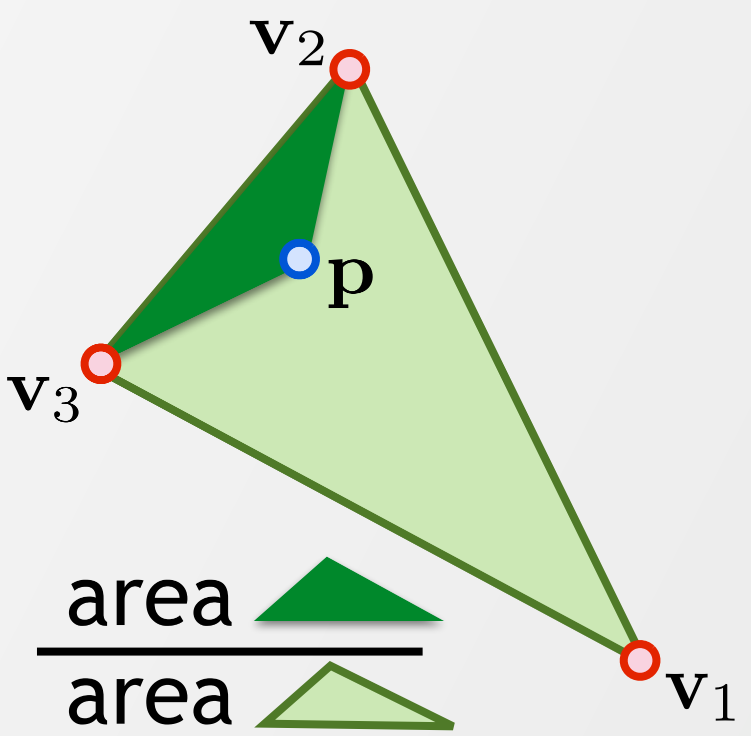

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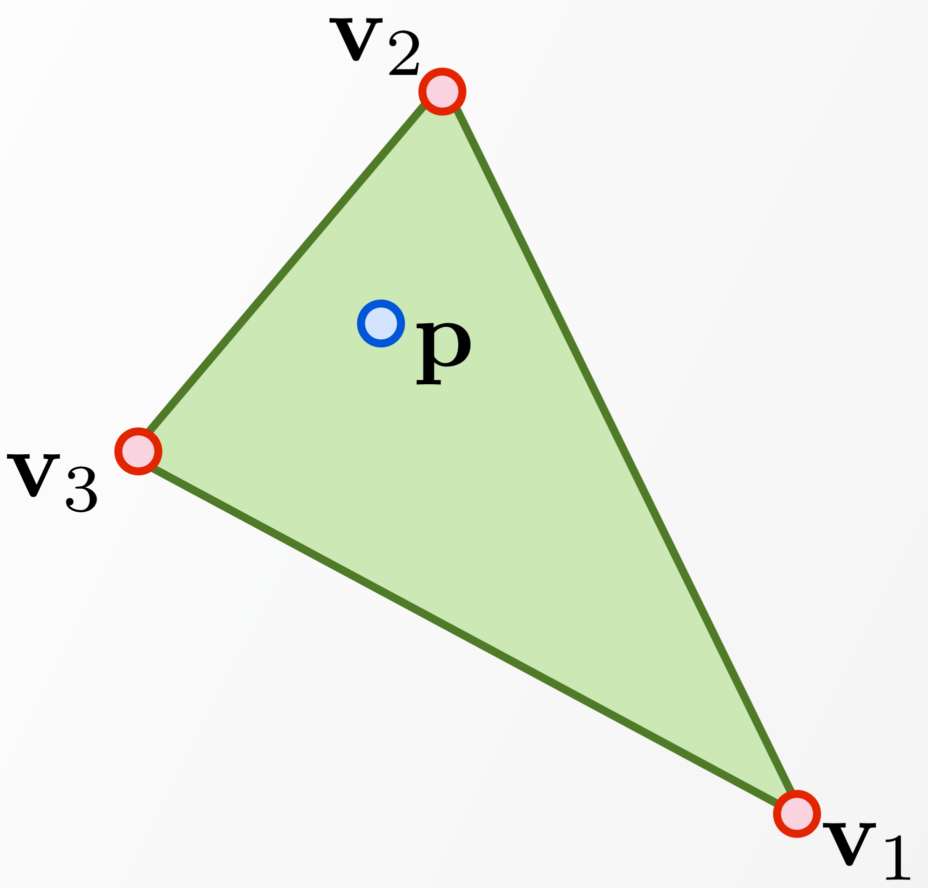

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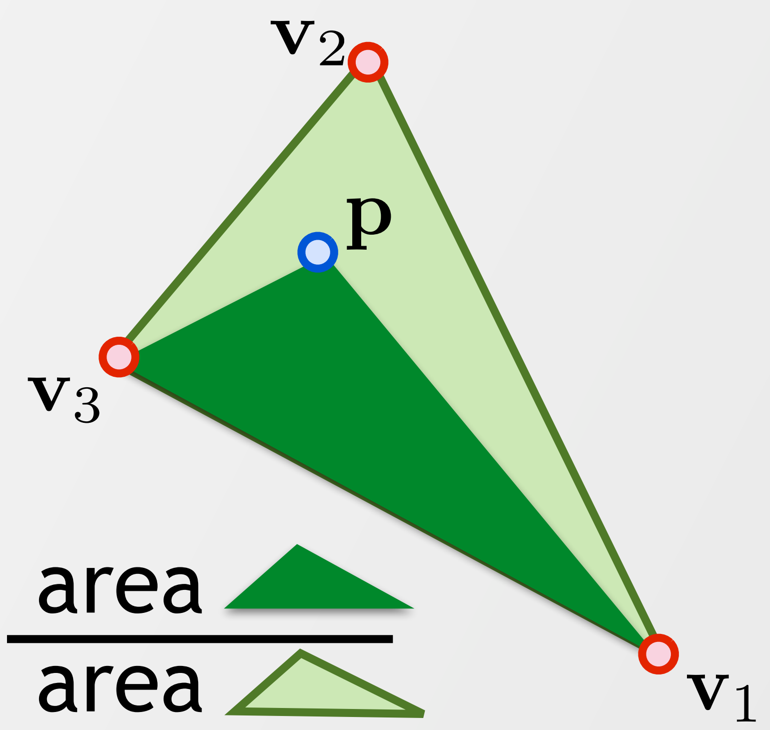

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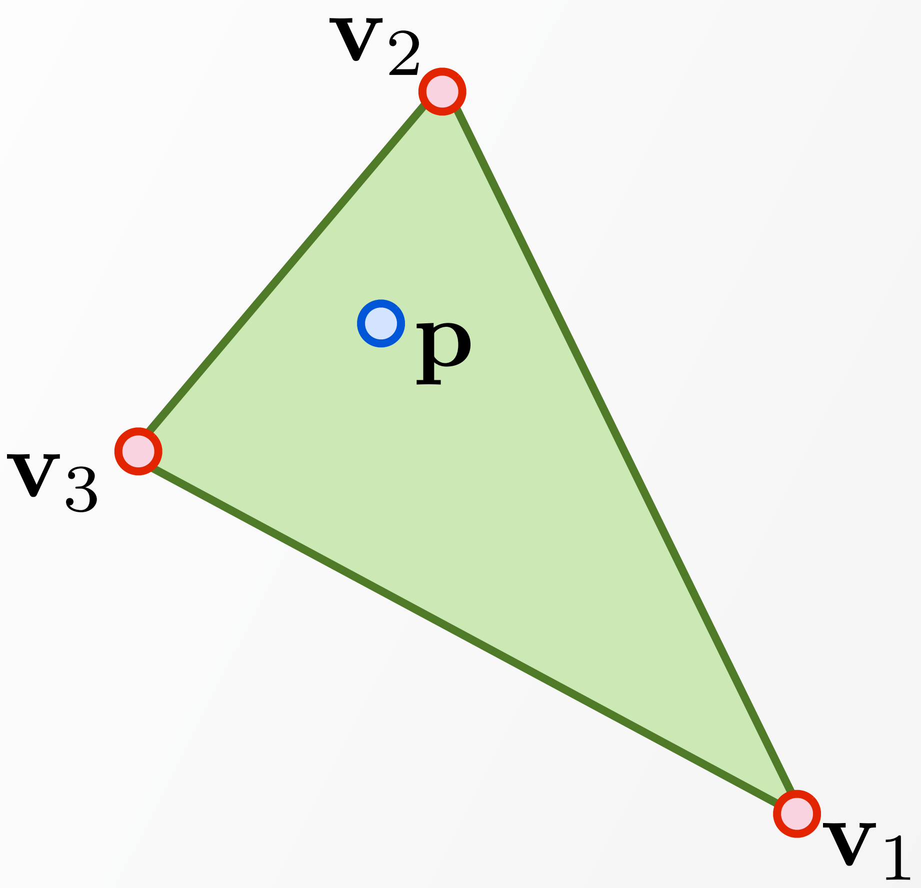

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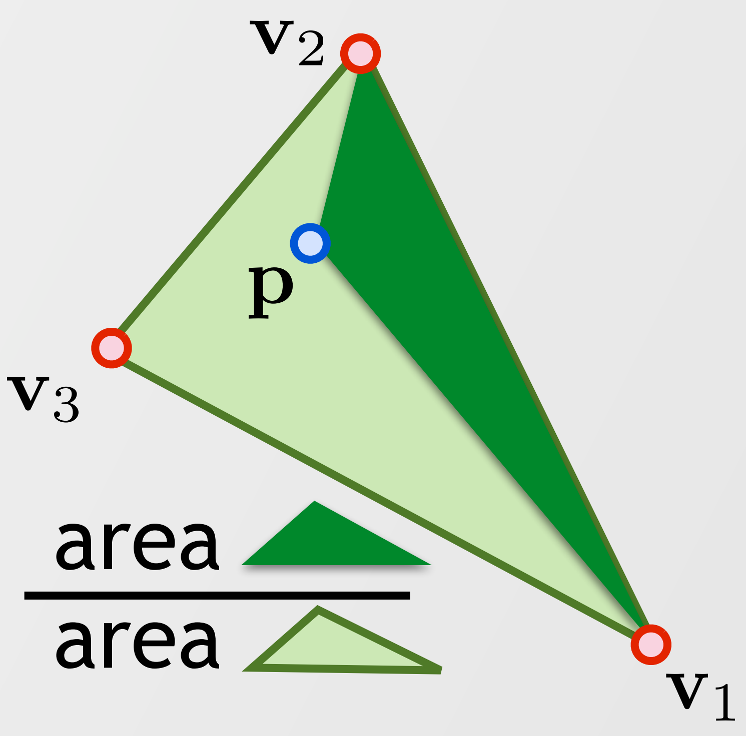

Piecewise linear functions on meshes

Hat functions and PL interpolation

$$f(\mathbf{p}) = B_i(\mathbf{p}) f_i + B_j(\mathbf{p}) f_j + B_k(\mathbf{p}) f_k$$

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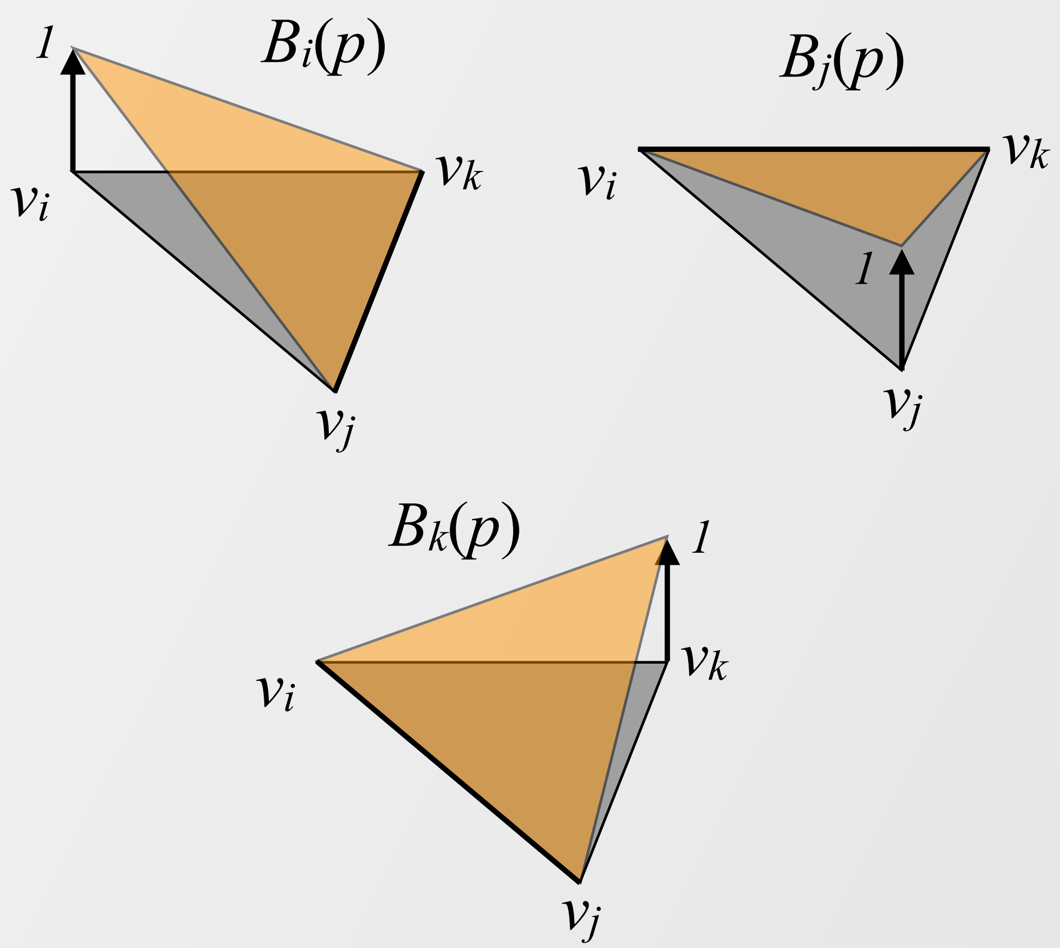

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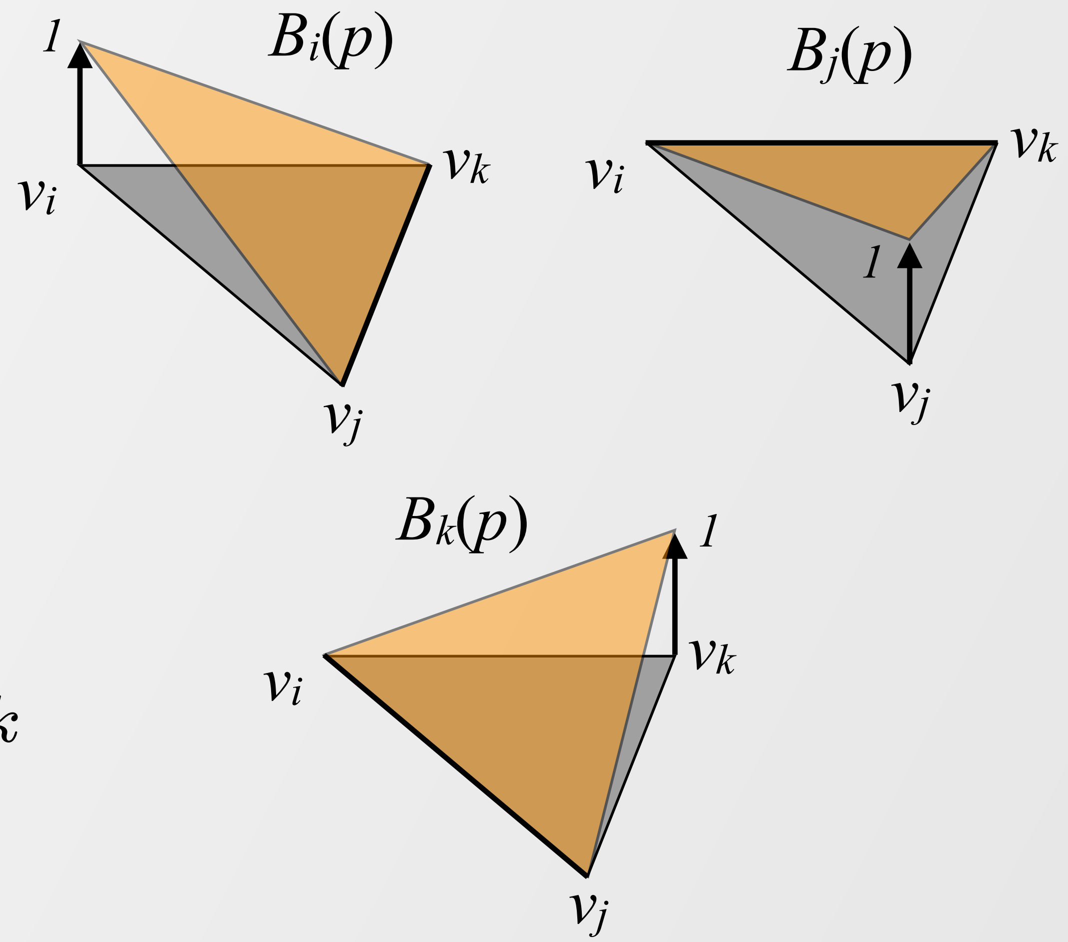

Gradients

$$\nabla f(\mathbf{p}) = \nabla B_i(\mathbf{p})f_i + \nabla B_j(\mathbf{p})f_j + \nabla B_k(\mathbf{p})f_k$$

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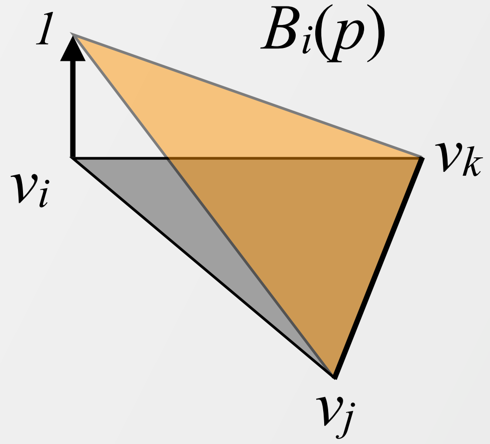

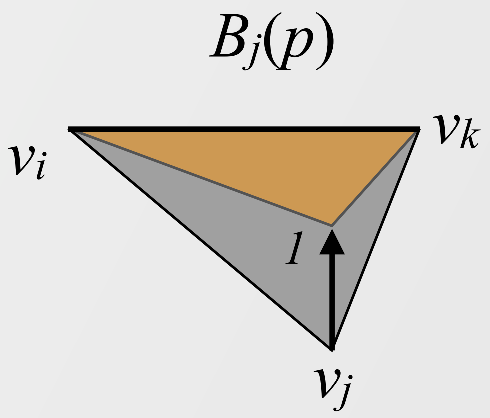

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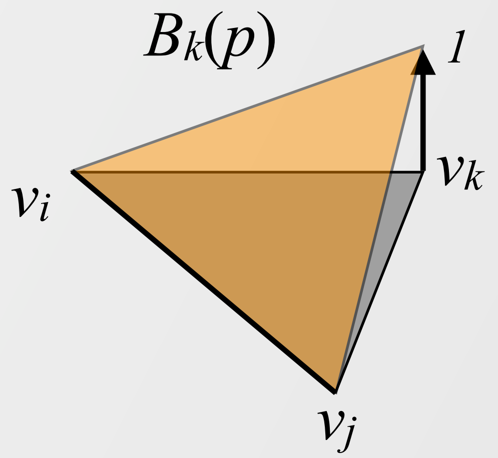
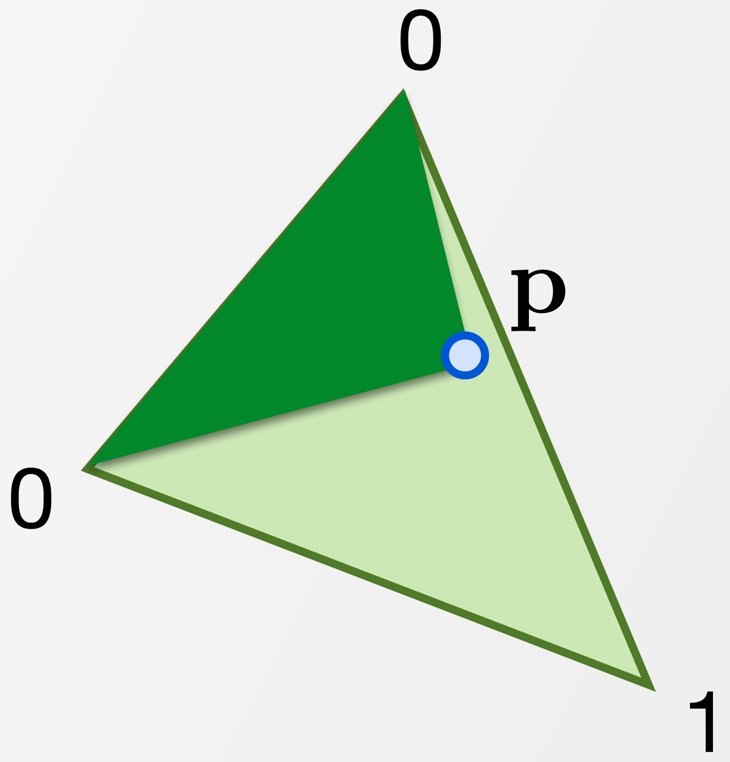
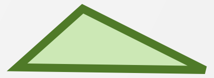


$$\nabla f(\mathbf{p}) = (f_j - f_i)\nabla B_j(\mathbf{p}) + (f_k - f_i)\nabla B_k(\mathbf{p})$$

The hat function



$B(\mathbf{p}) = \dfrac{\text{area}}{\text{area}}$  

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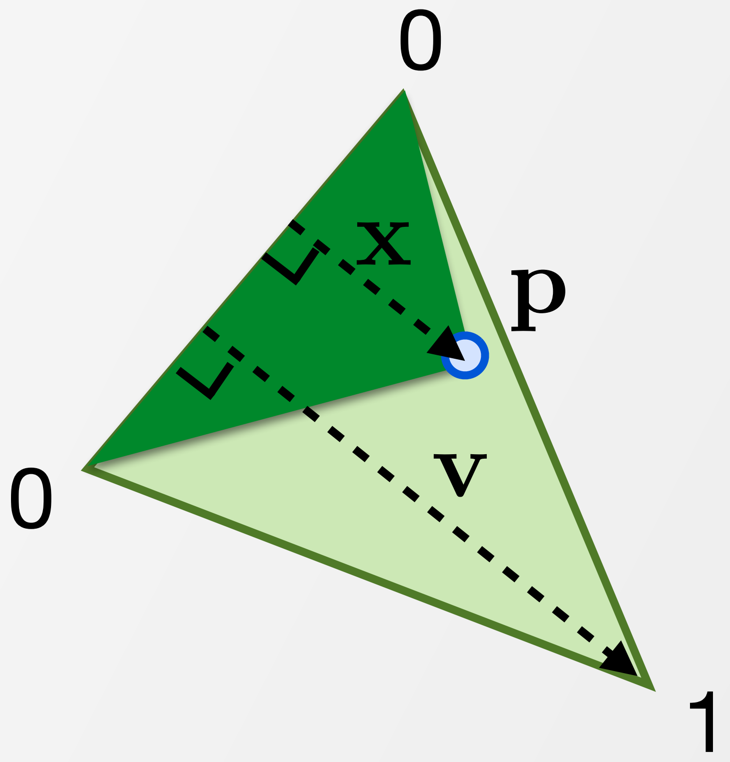

$$B(\mathbf{p}) = \frac{\text{area } \blacktriangle}{\text{area } \triangle} = \frac{\|\mathbf{x}\|}{\|\mathbf{v}\|}$$

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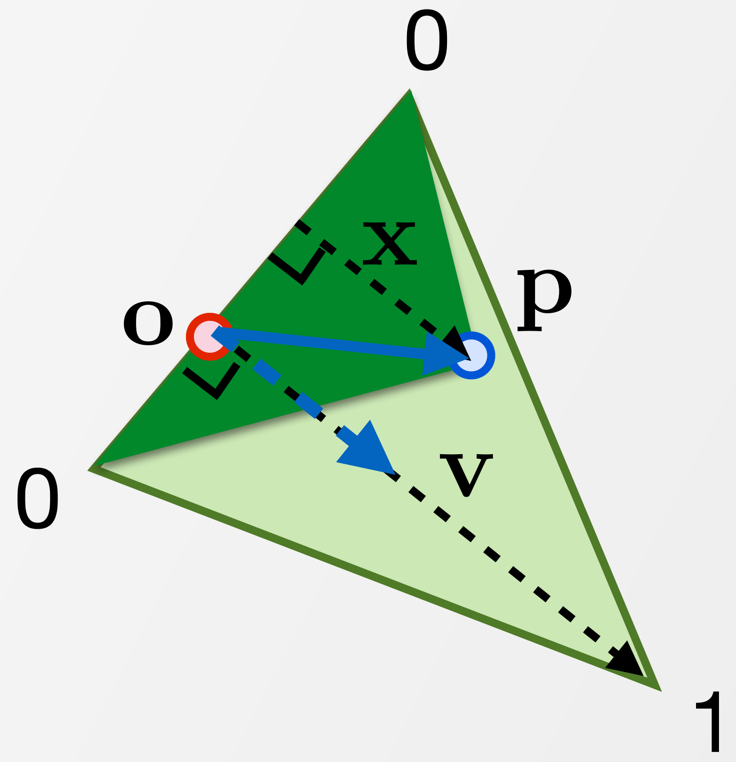

$$B(\mathbf{p}) = \frac{\text{area } \blacktriangle}{\text{area } \triangle} = \frac{\|\mathbf{x}\|}{\|\mathbf{v}\|} = \frac{(\mathbf{p}-\mathbf{o}) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}}{\|\mathbf{v}\|}$$

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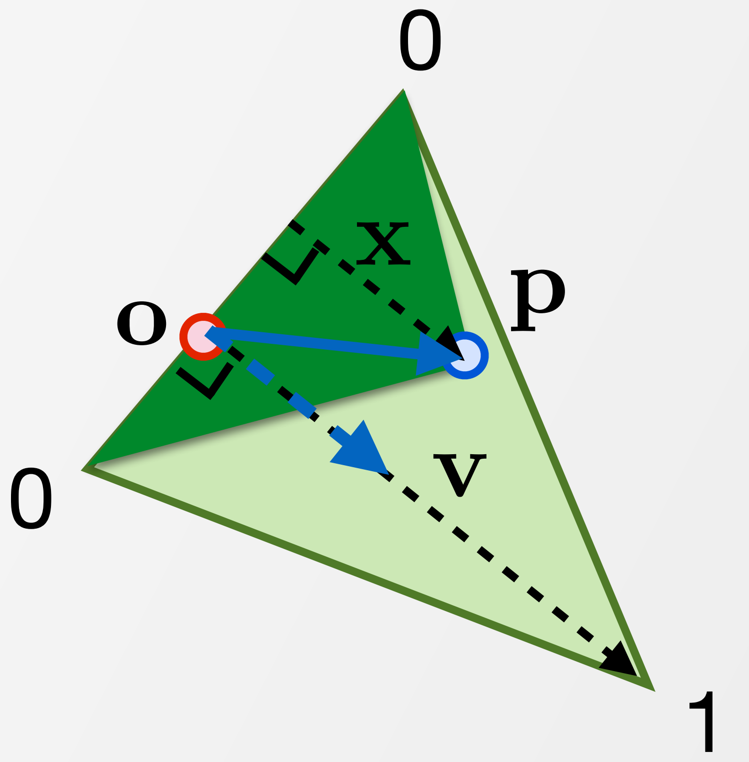

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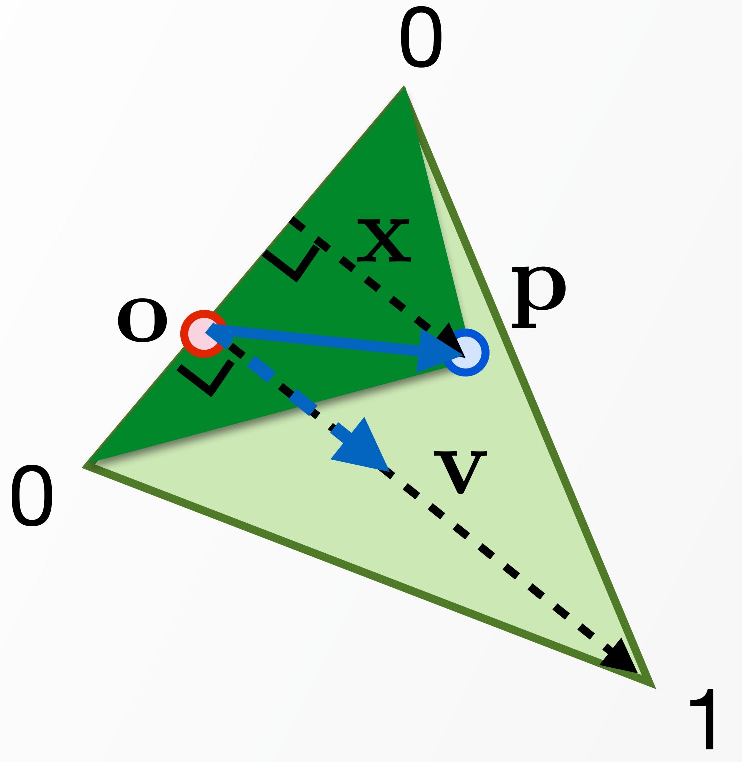
Gradient of the hat function



$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

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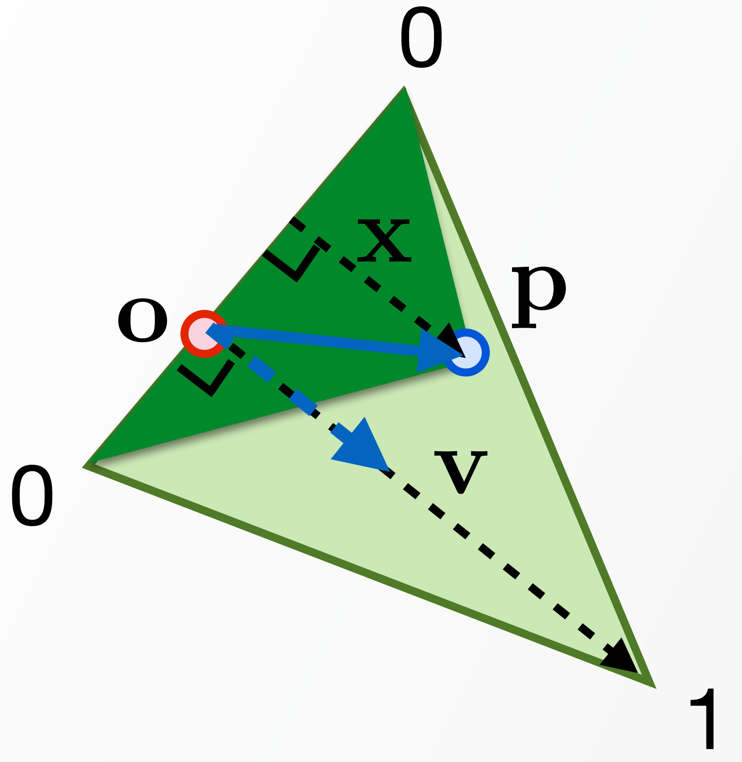

$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{v}\|}$$

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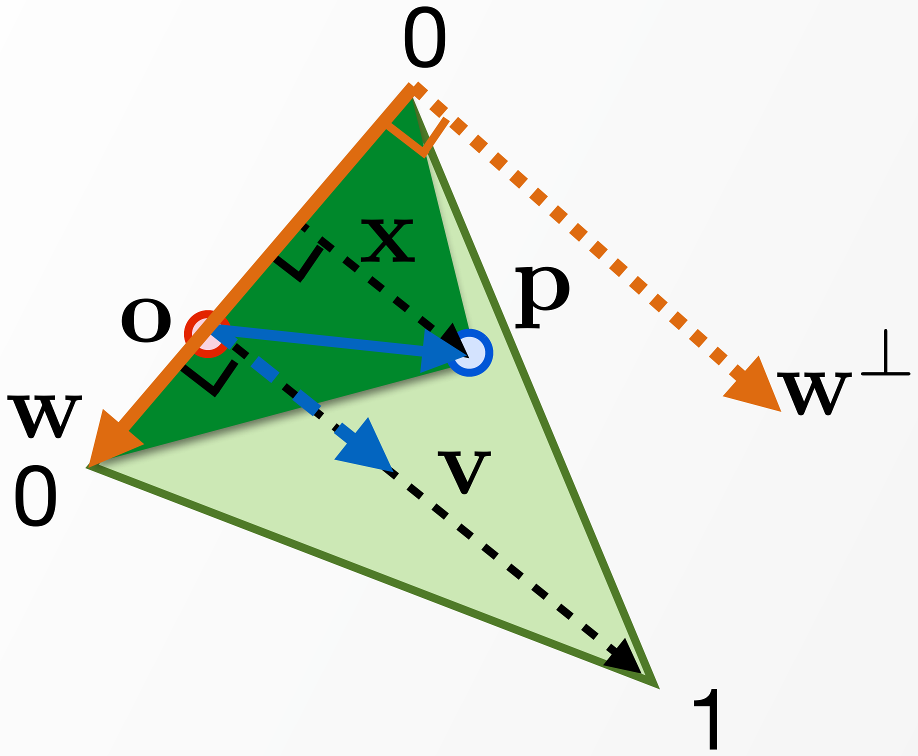

$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{v}\|}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\|}$$

$$B(\mathbf{p}) = \frac{(\mathbf{p}-\mathbf{o})\cdot\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{v}\|}$$

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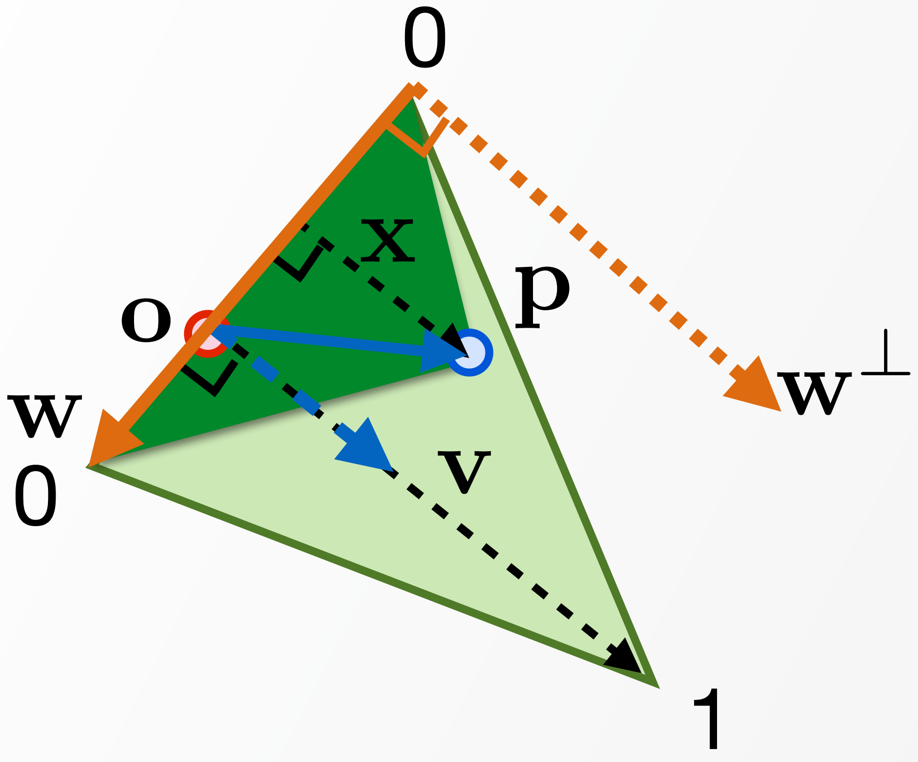

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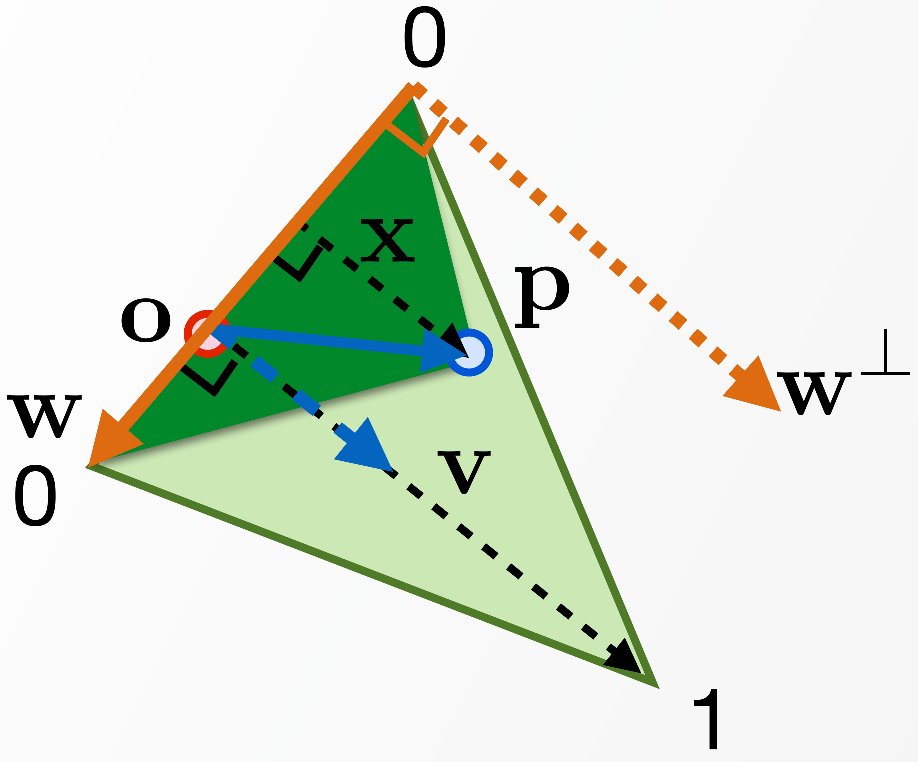

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$$\|\mathbf{w}^\perp\| = \|\mathbf{w}\|$$

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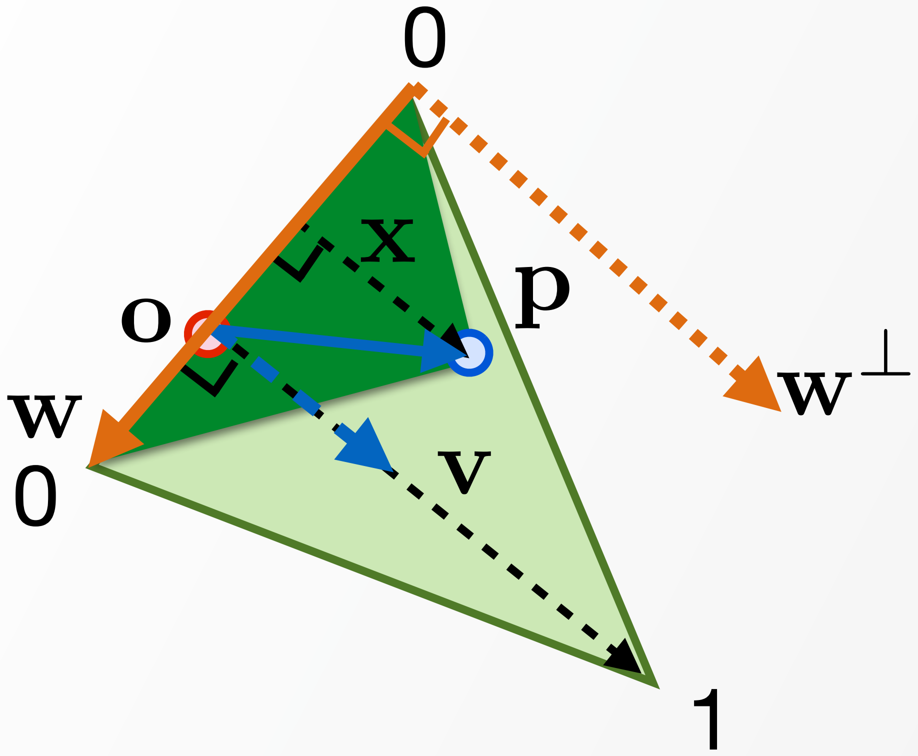

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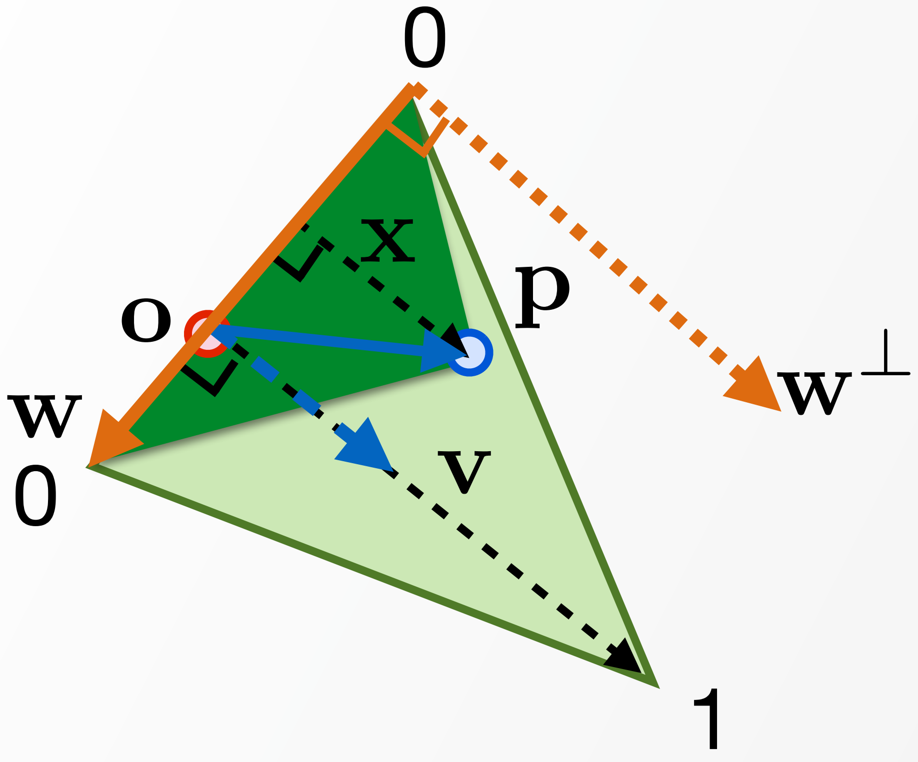

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$$A = \frac{\|\mathbf{v}\|\|\mathbf{w}\|}{2}$$

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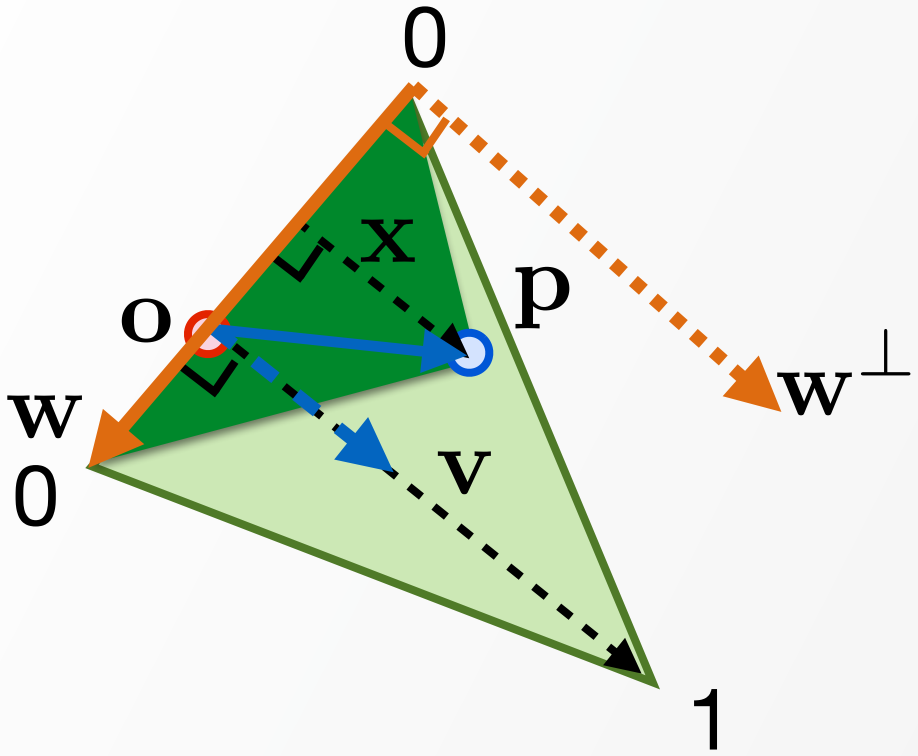

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$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{2A}$$

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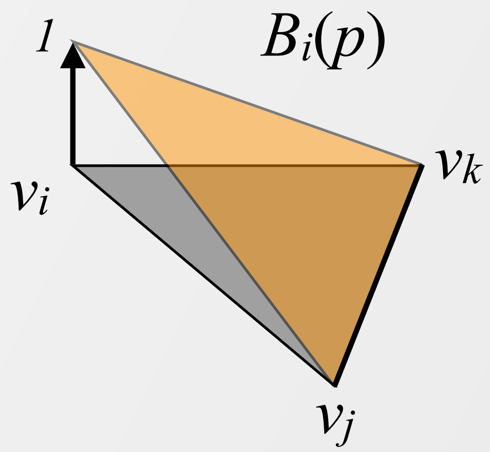

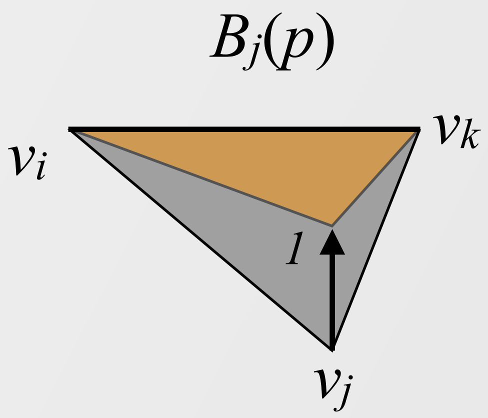

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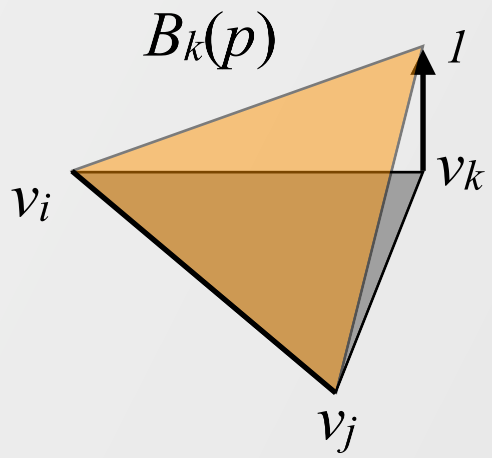

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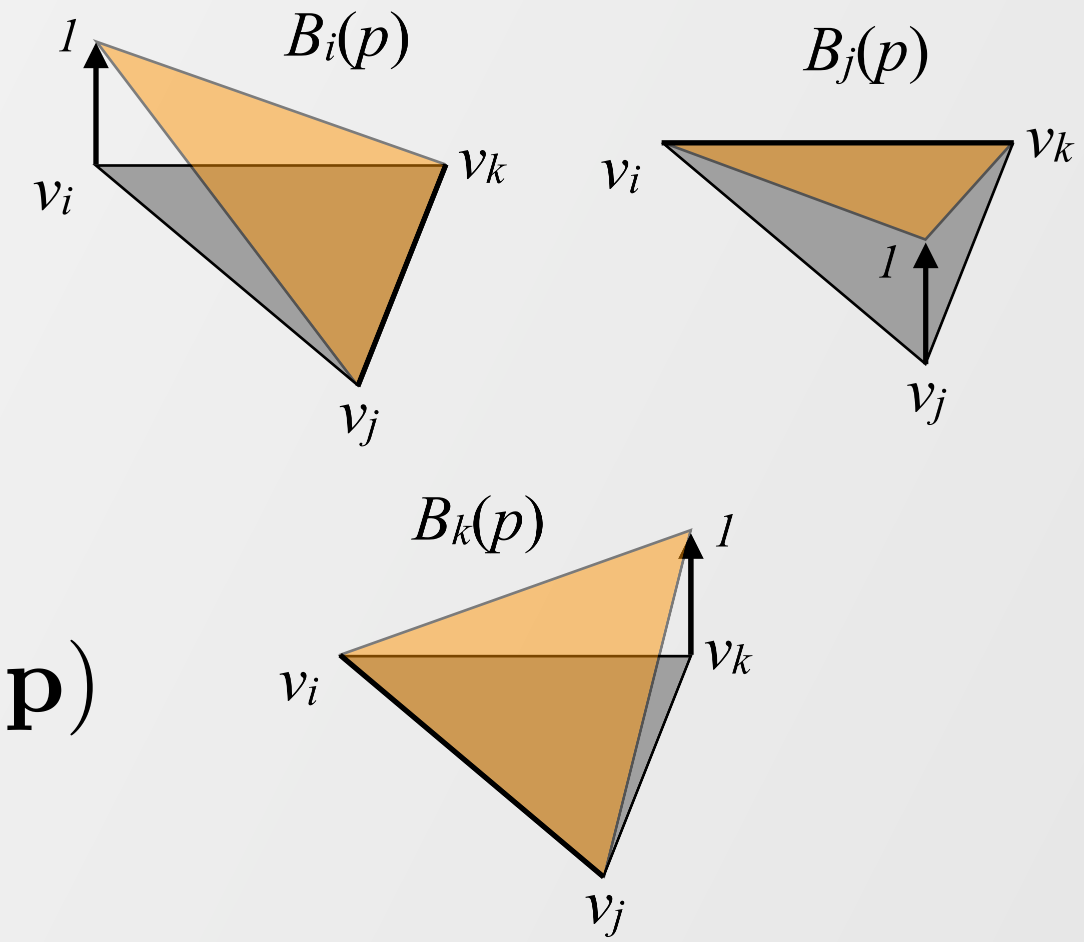

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$$\nabla f(\mathbf{p}) = (f_j - f_i) \frac{(\mathbf{v_i} - \mathbf{v_k})^{\perp}}{2A} + (f_k - f_i) \frac{(\mathbf{v_j} - \mathbf{v_i})^{\perp}}{2A}$$

Laplacian operator

- The Laplace operator is defined as the **divergence** of the **gradient** of a function:

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = f_{uu} + f_{vv}$$

- A function is **harmonic** if its Laplacian is null everywhere
- Its generalization to surfaces is called Laplace-Beltrami

$$\Delta_S \mathbf{V} = -2H\mathbf{n}$$

where **V** are the coordinates of the mesh vertices, *H* is the mean curvature and **n** is the normal

Discretization of the Laplacian

$$\Delta f(\mathbf{u}) = \frac{1}{A(\mathbf{u})} \int_{A(\mathbf{u})} \Delta f(\mathbf{u}) dA$$

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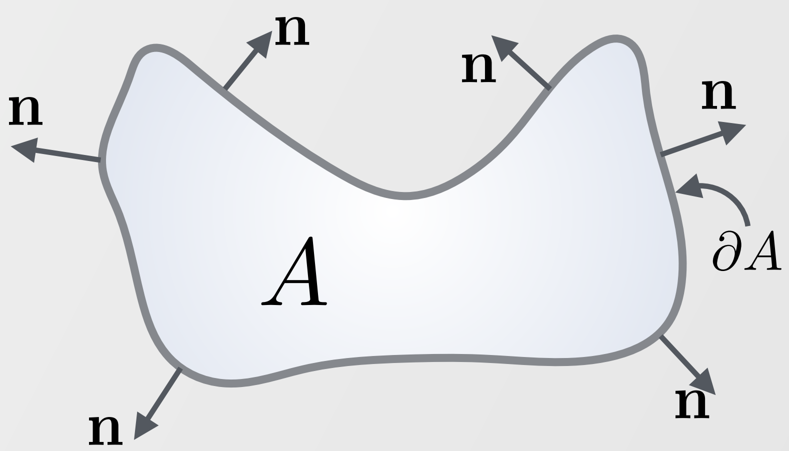

Divergence theorem:

$$\int_A \mathrm{div}\mathbf{F}(\mathbf{u}) dA = \oint_{\partial A} \mathbf{F}(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds$$

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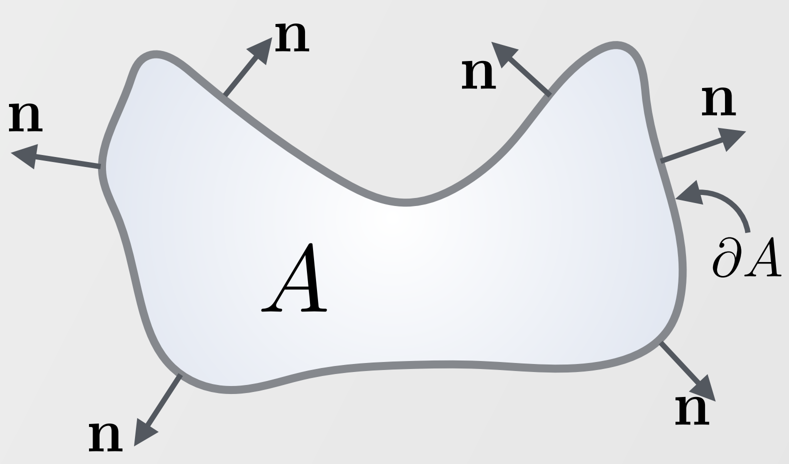

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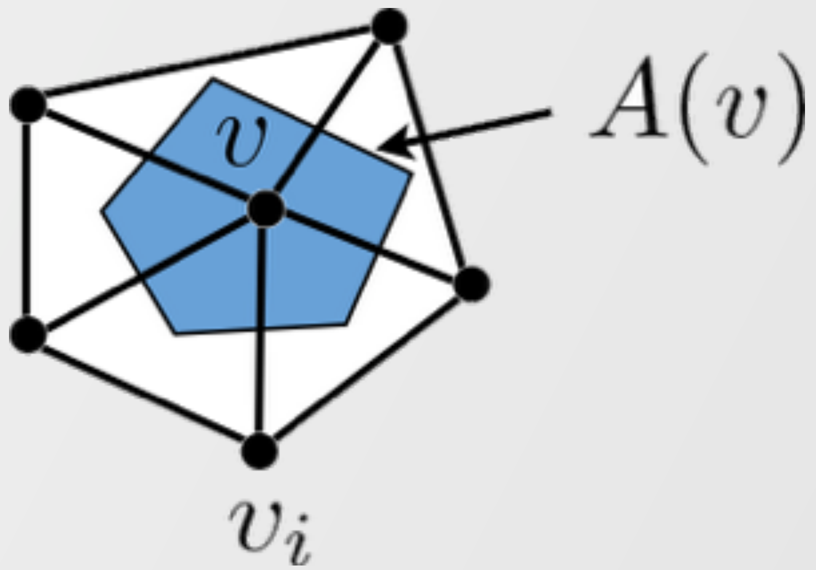

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$$= \sum_{t \in S(\mathbf{u})} \int_{\partial A(\mathbf{u}) \cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds$$

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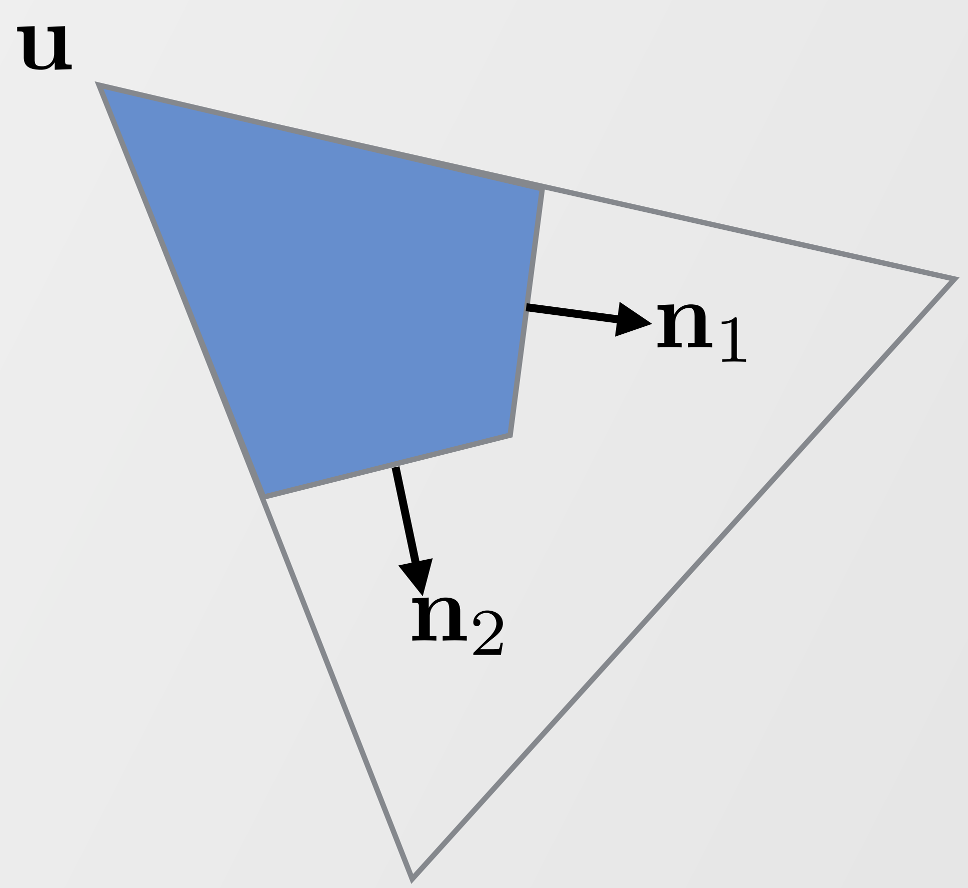

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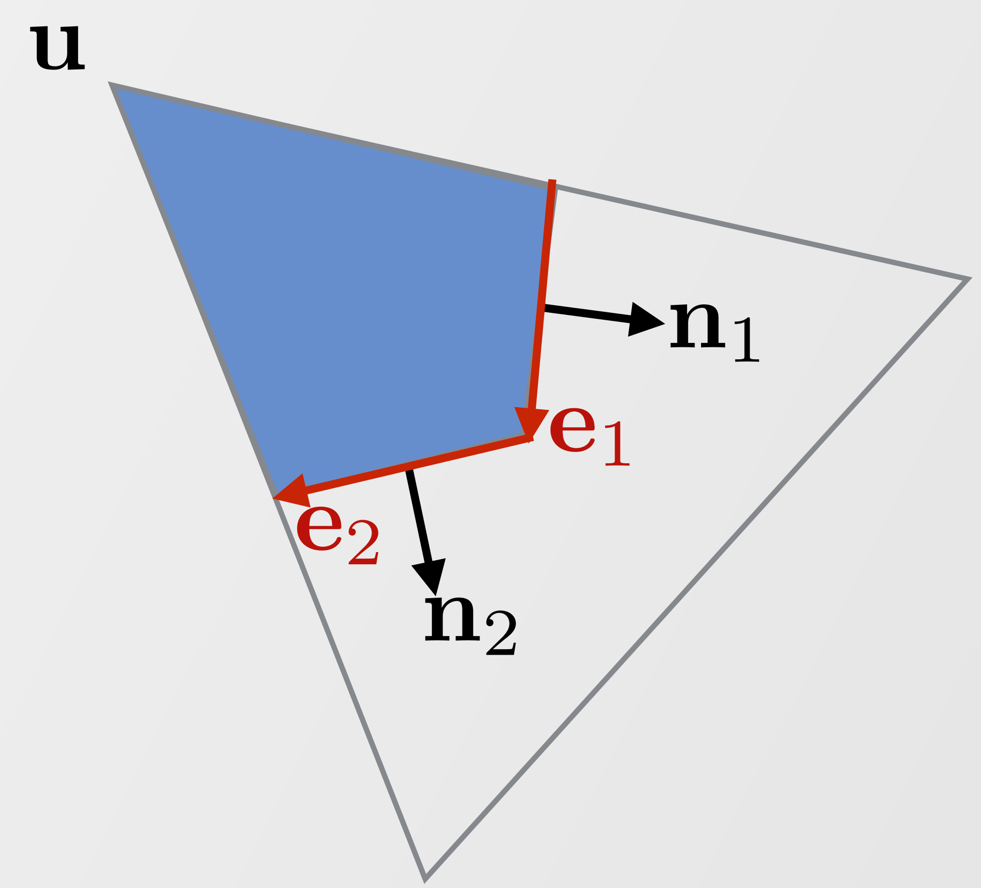

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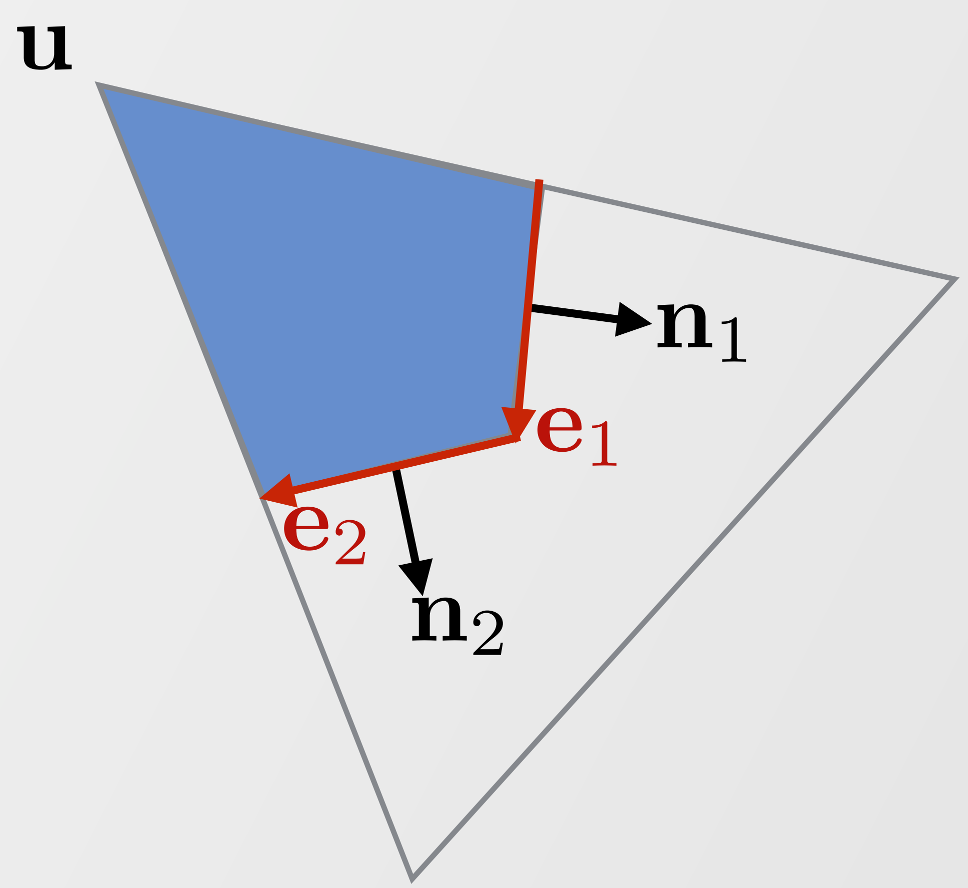
$$= \nabla f(\mathbf{u}) \cdot (\|\mathbf{e}_1\| \mathbf{n}_1 + \|\mathbf{e}_2\| \mathbf{n}_2)$$



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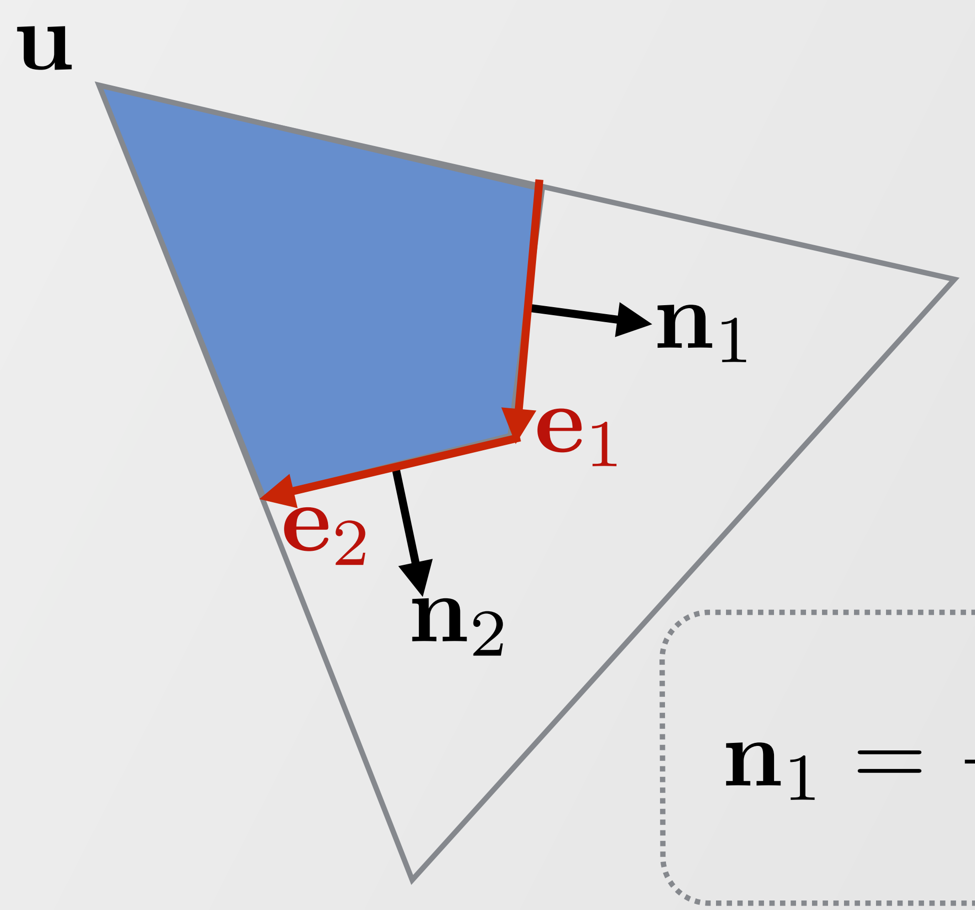
$$= \|\mathbf{e}_1\| \nabla f(\mathbf{u}) \cdot \mathbf{n}_1 + \|\mathbf{e}_2\| \nabla f(\mathbf{u}) \cdot \mathbf{n}_2$$

$$= \nabla f(\mathbf{u}) \cdot (\|\mathbf{e}_1\| \mathbf{n}_1 + \|\mathbf{e}_2\| \mathbf{n}_2)$$



$$\mathbf{n}_1 = \frac{\mathbf{e}_1^{\perp}}{\|\mathbf{e}_1^{\perp}\|}$$

Discretization of the Laplacian

$$\Delta f(\mathbf{u}) = \frac{1}{A(\mathbf{u})}\int_{A(\mathbf{u})} \Delta f(\mathbf{u}) dA = \frac{1}{A(\mathbf{u})}\sum_{t \in S(\mathbf{u})} \int_{\partial A(\mathbf{u}) \cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds$$

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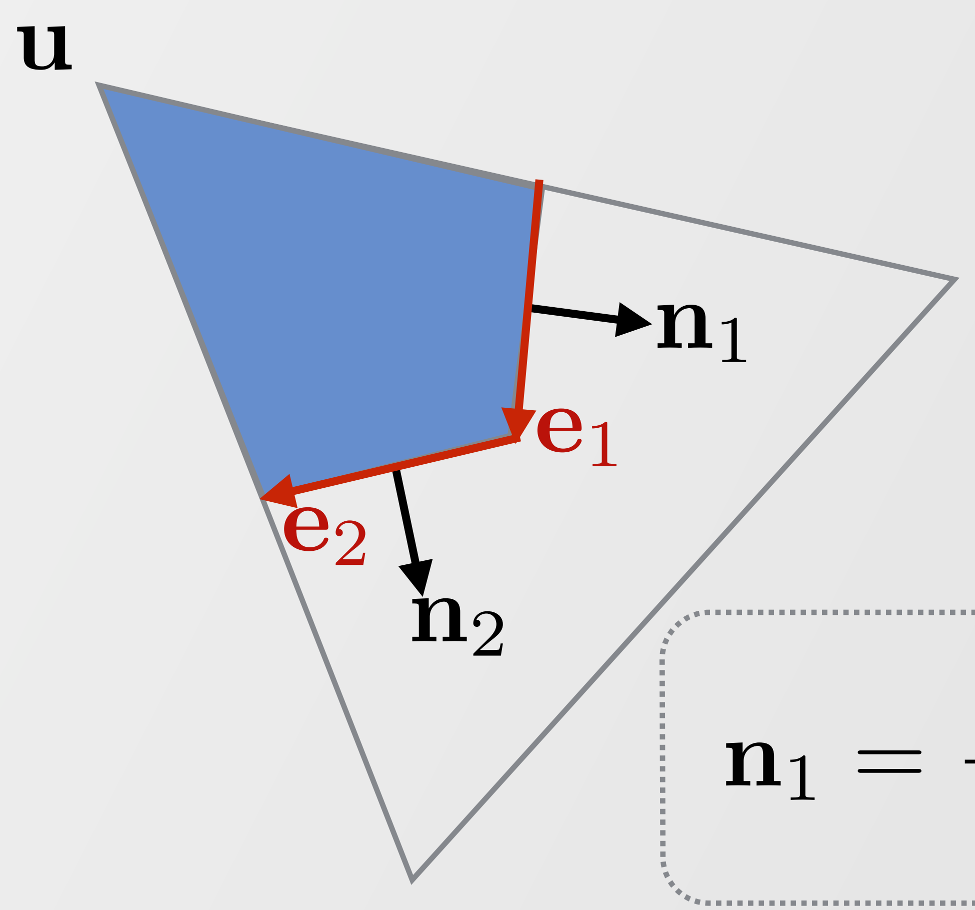

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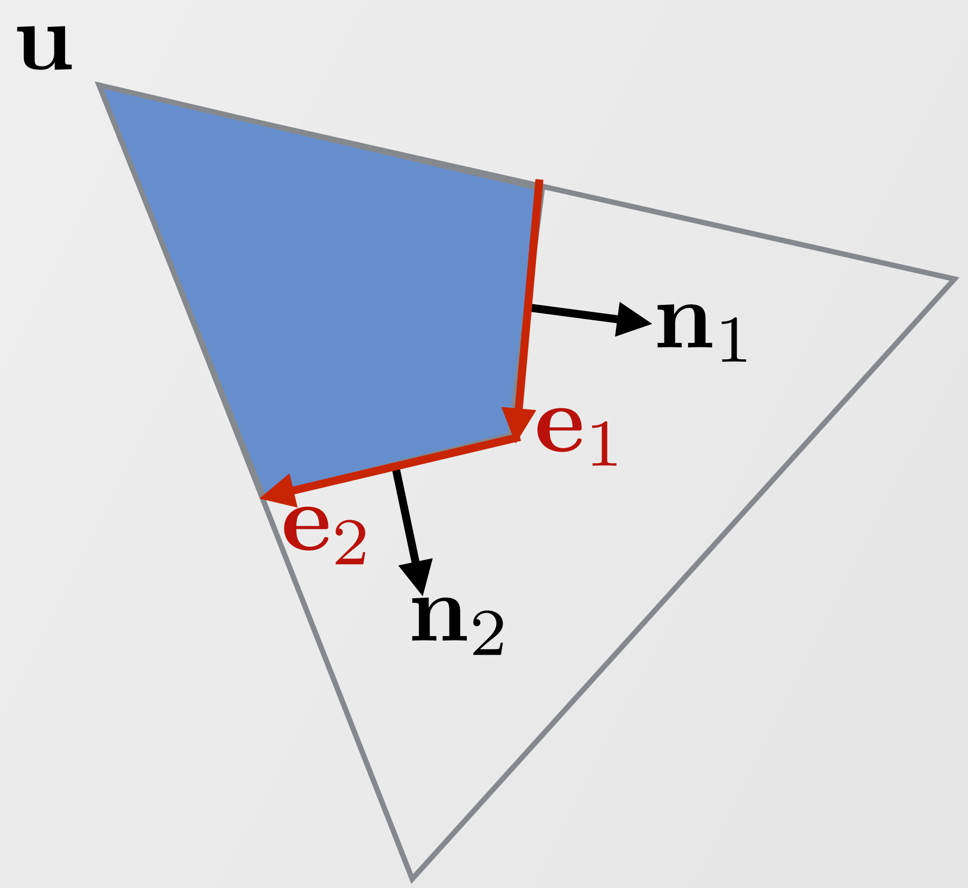

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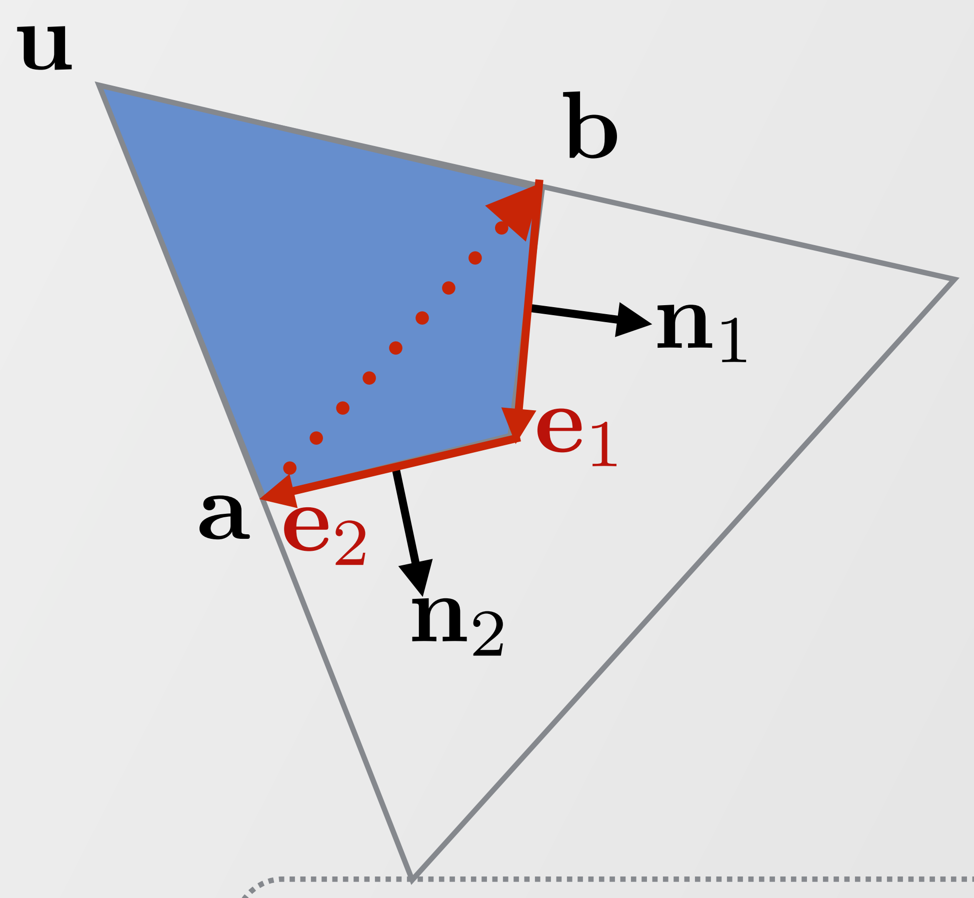

$$\mathbf{e}_1 + \mathbf{e}_2 + (\mathbf{b} - \mathbf{a}) = 0$$

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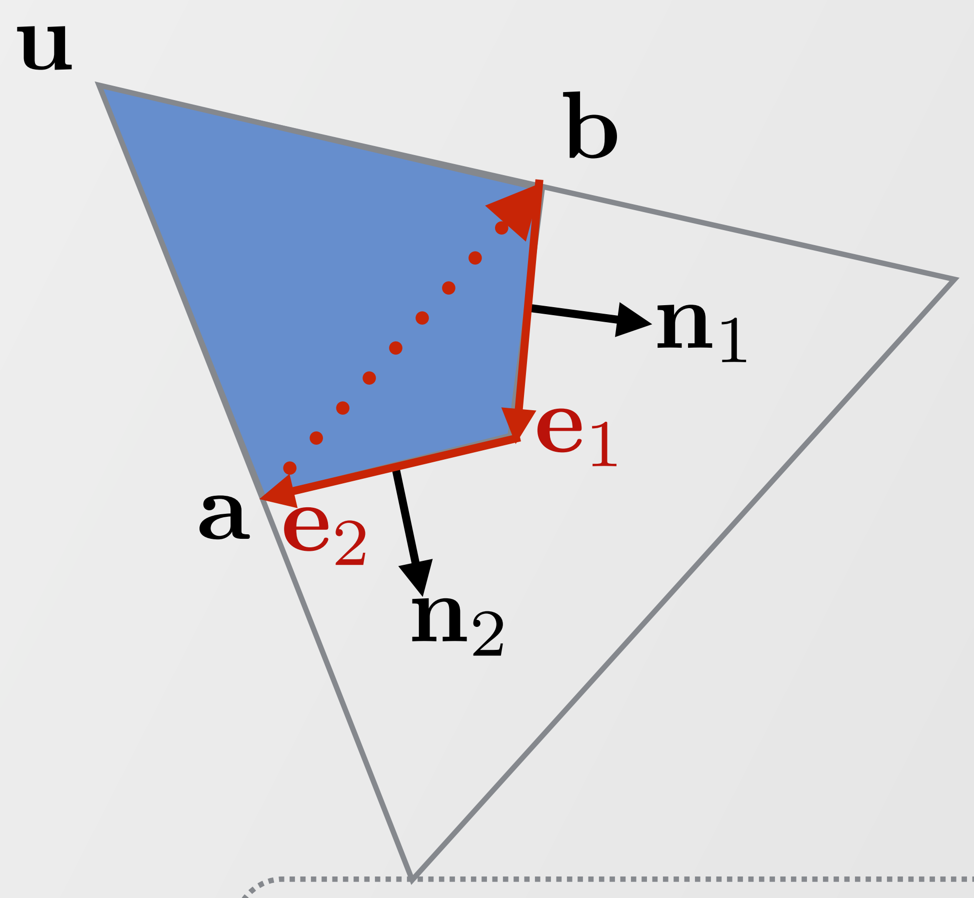
$$= \nabla f(\mathbf{u})\cdot (\mathbf{e}_1 + \mathbf{e}_2)^\perp$$

$$= \nabla f(\mathbf{u})\cdot (\mathbf{a} - \mathbf{b})^\perp$$



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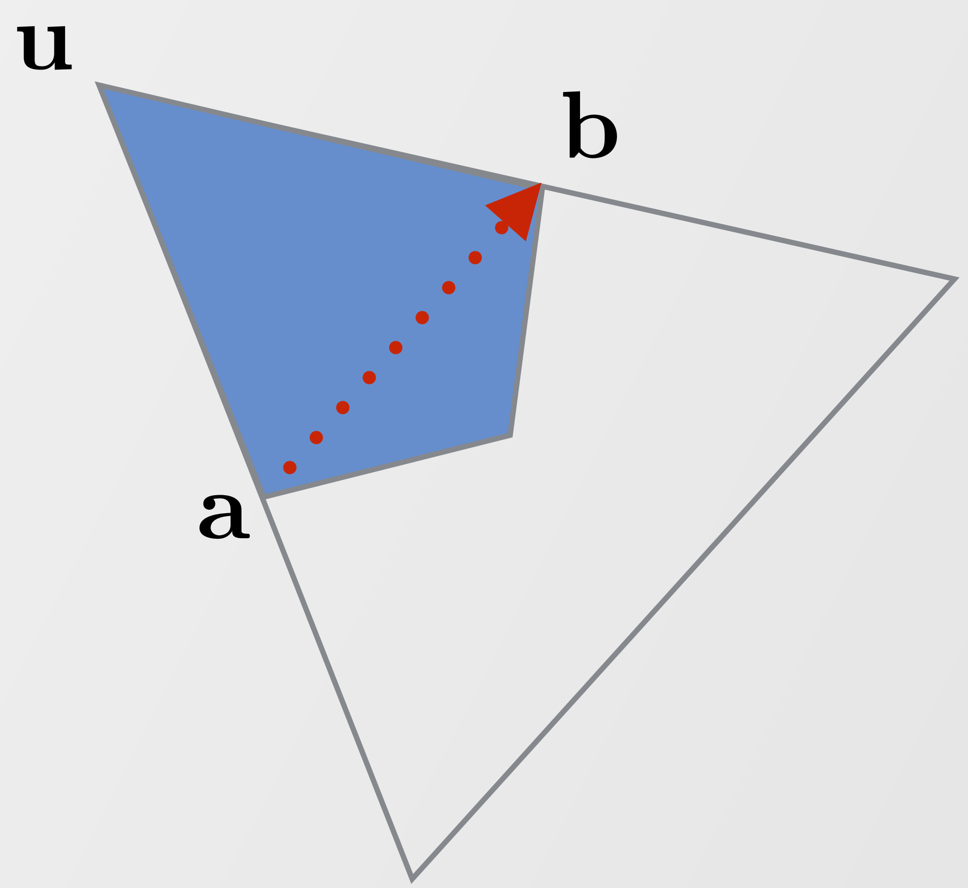

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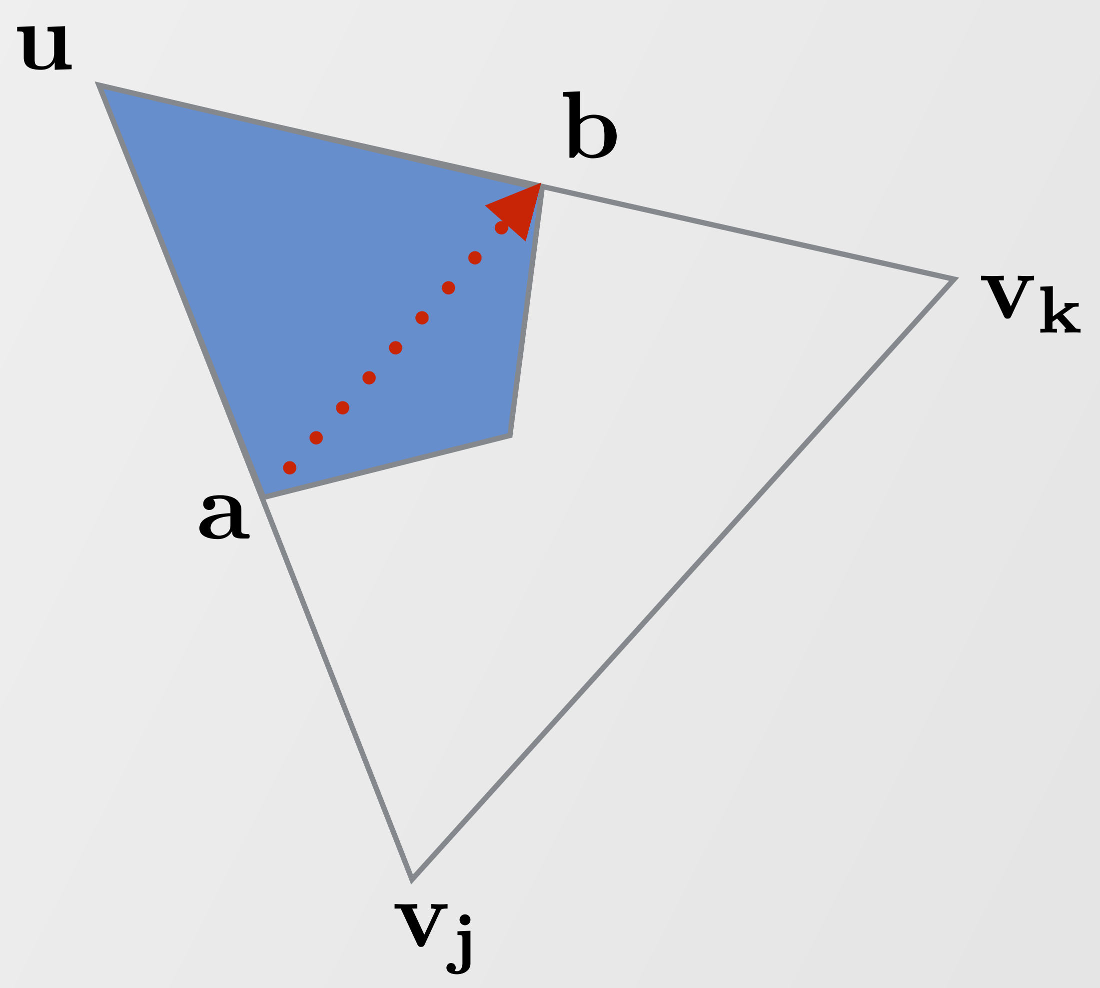
$$= \frac{1}{2}\nabla f(\mathbf{u}) \cdot (\mathbf{v_j}-\mathbf{v_k})^{\perp}$$



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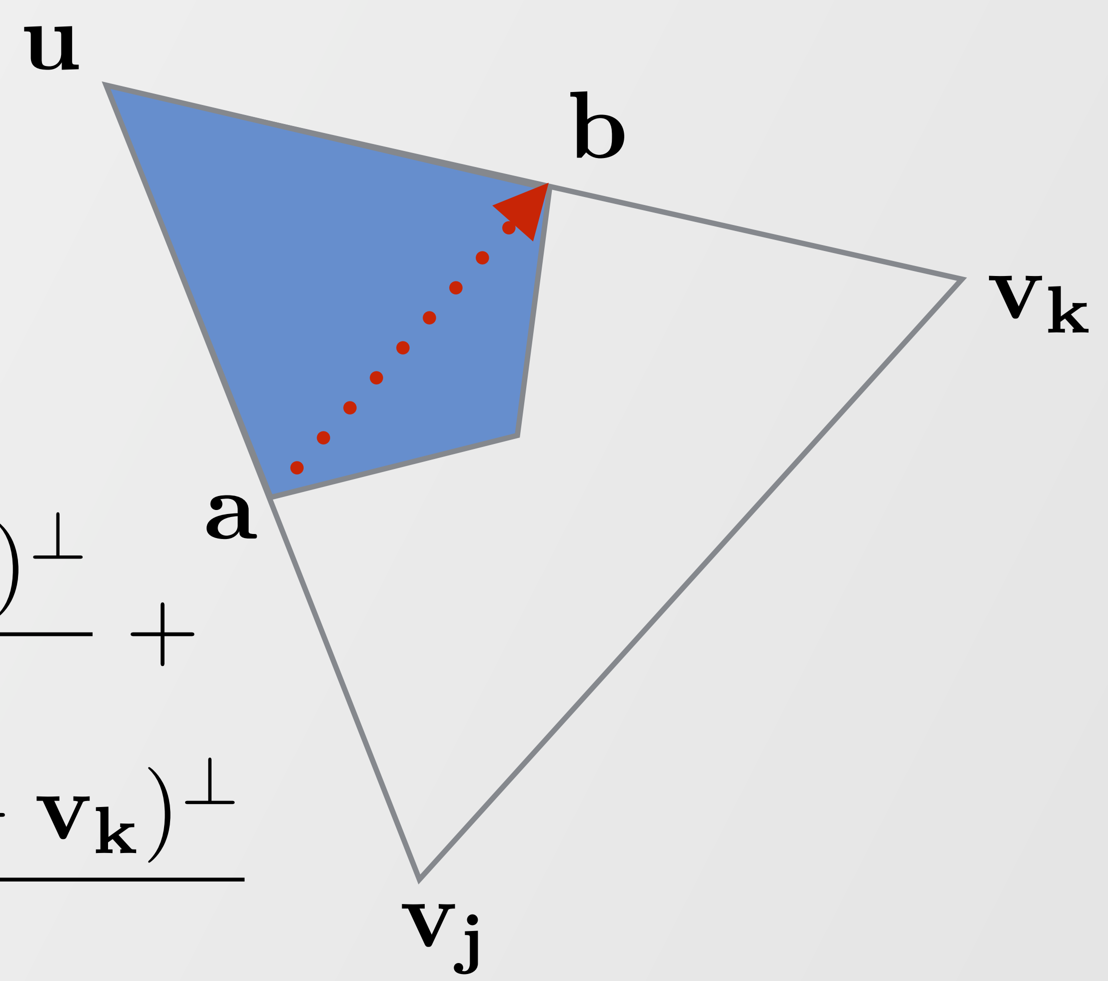
$$= \frac{1}{2}\nabla f(\mathbf{u})\cdot(\mathbf{v_j}-\mathbf{v_k})^{\perp}$$

$$= (f(\mathbf{v_j}) - f(\mathbf{u}))\frac{(\mathbf{u}-\mathbf{v_k})^{\perp}\cdot(\mathbf{v_j}-\mathbf{v_k})^{\perp}}{4A_t} +$$

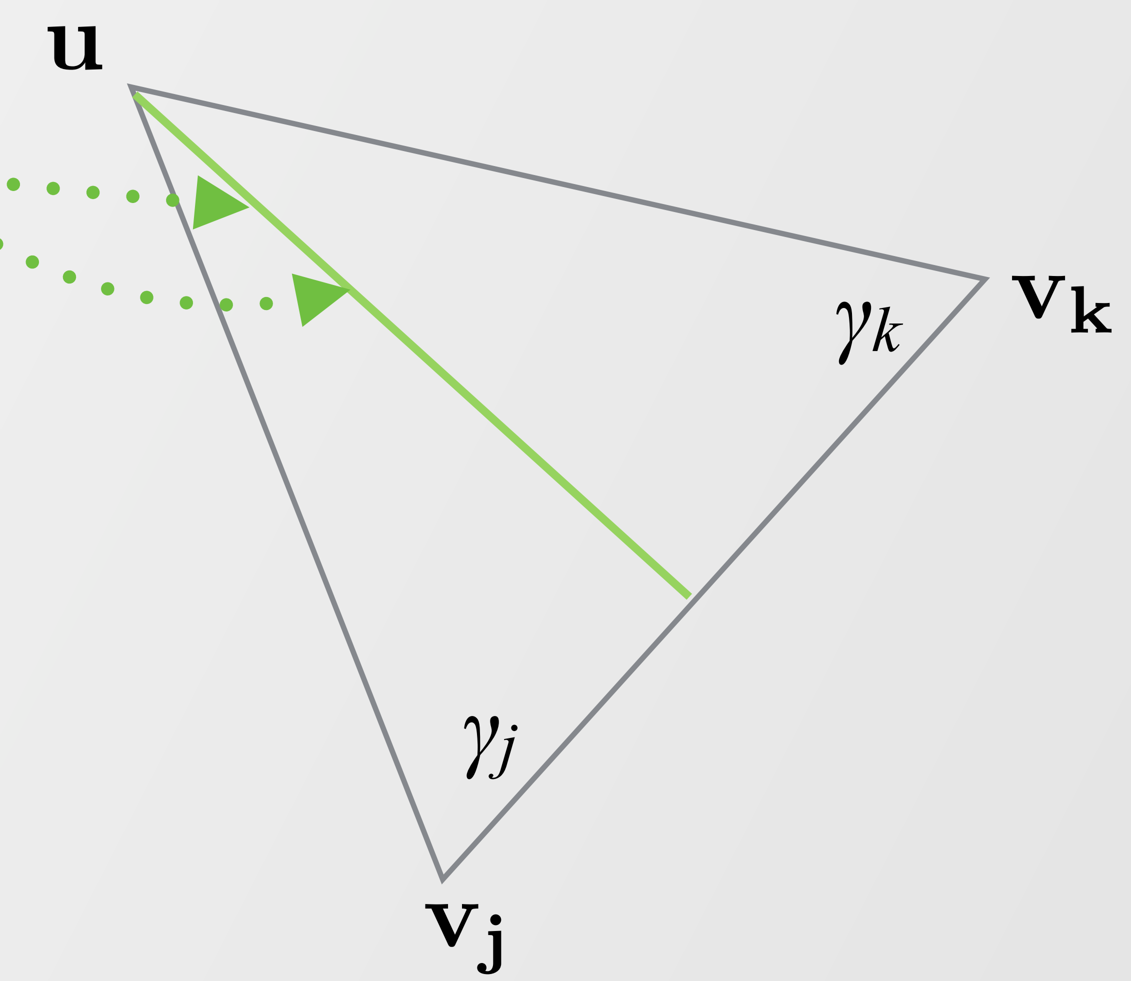
$$+ (f(\mathbf{v_k}) - f(\mathbf{u}))\frac{(\mathbf{v_j}-\mathbf{u})^{\perp}\cdot(\mathbf{v_j}-\mathbf{v_k})^{\perp}}{4A_t}$$



Useful trigonometric formulas

Triangle area using trigonometry:

$$A_t = \frac{1}{2} \sin\gamma_j \|\mathbf{v_j} - \mathbf{u}\| \|\mathbf{v_j} - \mathbf{v_k}\| = \frac{1}{2} \sin\gamma_k \|\mathbf{u} - \mathbf{v_k}\| \|\mathbf{v_j} - \mathbf{v_k}\|$$



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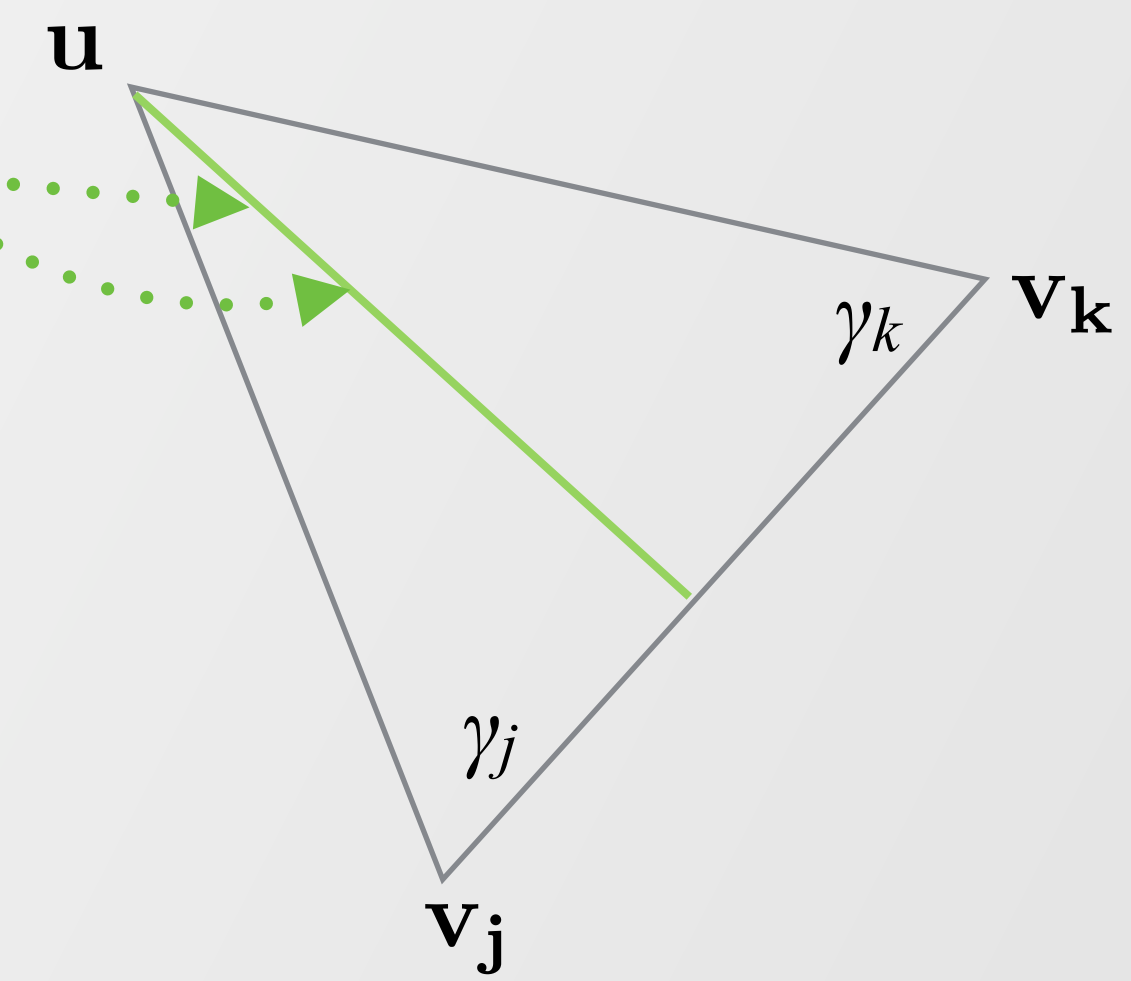
Law of cosine:

$$\cos\gamma_j = \frac{(\mathbf{v_j} - \mathbf{u}) \cdot (\mathbf{v_j} - \mathbf{v_k})}{\|\mathbf{v_j} - \mathbf{u}\| \|\mathbf{v_j} - \mathbf{v_k}\|}$$

$$\cos\gamma_k = \frac{(\mathbf{u} - \mathbf{v_k}) \cdot (\mathbf{v_j} - \mathbf{v_k})}{\|\mathbf{u} - \mathbf{v_k}\| \|\mathbf{v_j} - \mathbf{v_k}\|}$$

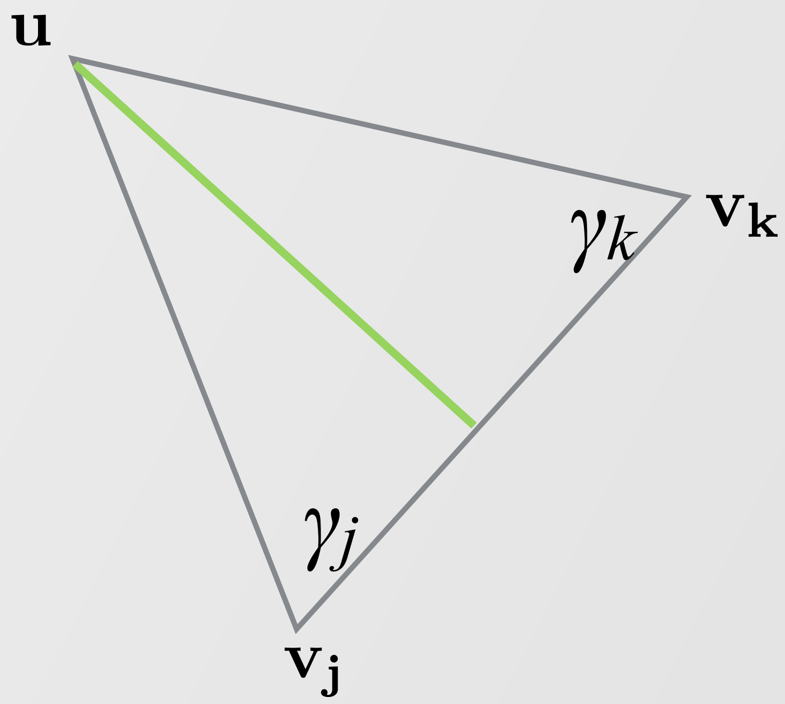


Substitute

$$\int_{\partial A(\mathbf{u})\cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds =$$

$$= (f(\mathbf{v_j}) - f(\mathbf{u})) \frac{(\mathbf{u} - \mathbf{v_k})^{\perp} \cdot (\mathbf{v_j} - \mathbf{v_k})^{\perp}}{4A_t} +$$

$$+ (f(\mathbf{v_k}) - f(\mathbf{u})) \frac{(\mathbf{v_j} - \mathbf{u})^{\perp} \cdot (\mathbf{v_j} - \mathbf{v_k})^{\perp}}{4A_t}$$



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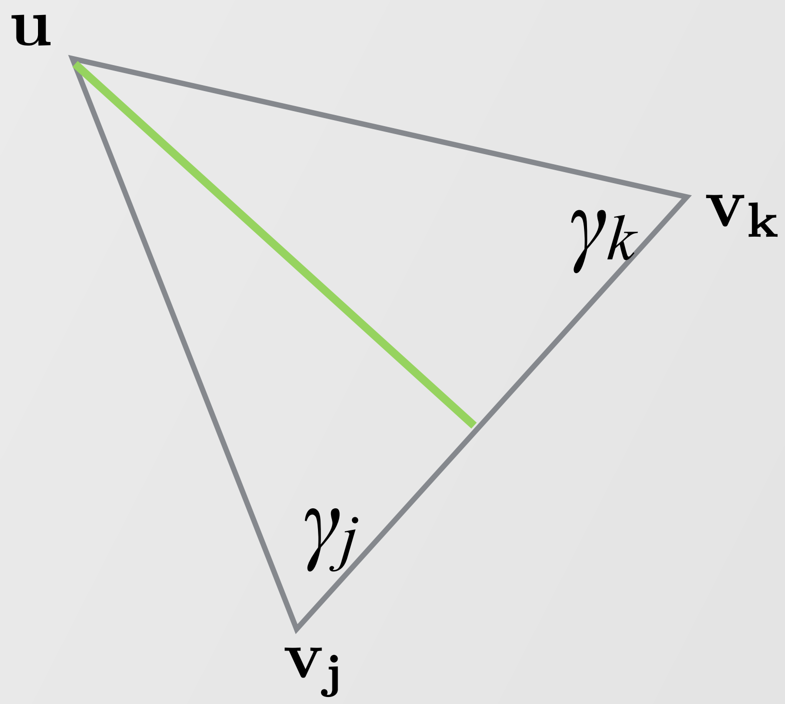
$$+ (f(\mathbf{v_k}) - f(\mathbf{u})) \frac{(\mathbf{v_j} - \mathbf{u})^\perp \cdot (\mathbf{v_j} - \mathbf{v_k})^\perp}{4A_t}$$

$$\cos \gamma_k = \frac{(\mathbf{u} - \mathbf{v_k}) \cdot (\mathbf{v_j} - \mathbf{v_k})}{\|\mathbf{u} - \mathbf{v_k}\| \|\mathbf{v_j} - \mathbf{v_k}\|}$$

$$\cos \gamma_j = \frac{(\mathbf{v_j} - \mathbf{u}) \cdot (\mathbf{v_j} - \mathbf{v_k})}{\|\mathbf{v_j} - \mathbf{u}\| \|\mathbf{v_j} - \mathbf{v_k}\|}$$



Substitute

$$\int_{\partial A(\mathbf{u})\cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds =$$

$$= (f(\mathbf{v_j}) - f(\mathbf{u}))\frac{(\mathbf{u}-\mathbf{v_k})^\perp \cdot (\mathbf{v_j}-\mathbf{v_k})^\perp}{4A_t} + $$

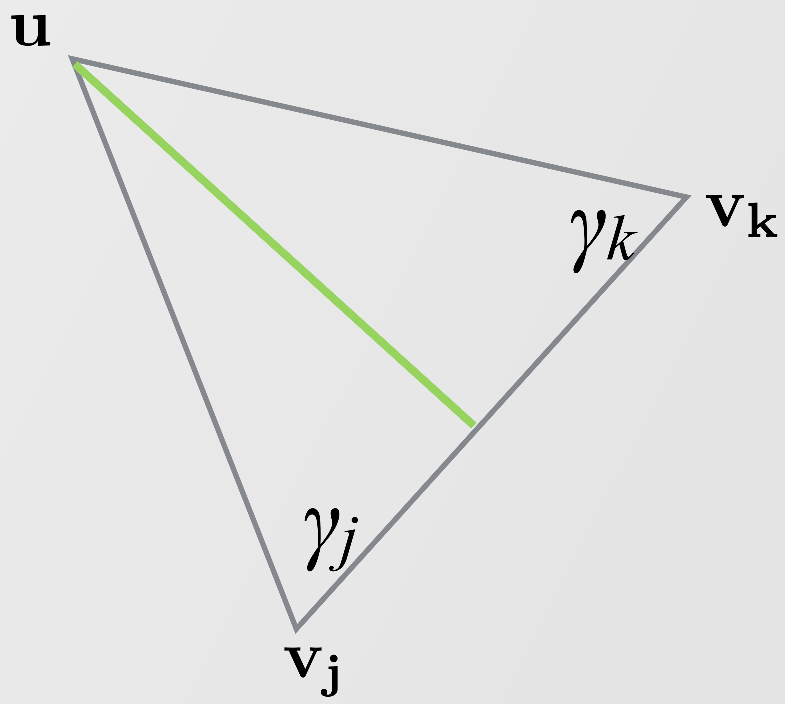
$$+ (f(\mathbf{v_k}) - f(\mathbf{u}))\frac{(\mathbf{v_j}-\mathbf{u})^\perp \cdot (\mathbf{v_j}-\mathbf{v_k})^\perp}{4A_t}$$

$$\frac{1}{2}\sin\gamma_k \|\mathbf{u}-\mathbf{v_k}\|\|\mathbf{v_j}-\mathbf{v_k}\|$$

$$\frac{1}{2}\sin\gamma_j \|\mathbf{v_j}-\mathbf{u}\|\|\mathbf{v_j}-\mathbf{v_k}\|$$

$$\cos\gamma_k = \frac{(\mathbf{u}-\mathbf{v_k})\cdot(\mathbf{v_j}-\mathbf{v_k})}{\|\mathbf{u}-\mathbf{v_k}\|\|\mathbf{v_j}-\mathbf{v_k}\|}$$

$$\cos\gamma_j = \frac{(\mathbf{v_j}-\mathbf{u})\cdot(\mathbf{v_j}-\mathbf{v_k})}{\|\mathbf{v_j}-\mathbf{u}\|\|\mathbf{v_j}-\mathbf{v_k}\|}$$



Substitute

$$\int_{\partial A(\mathbf{u})\cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds =$$

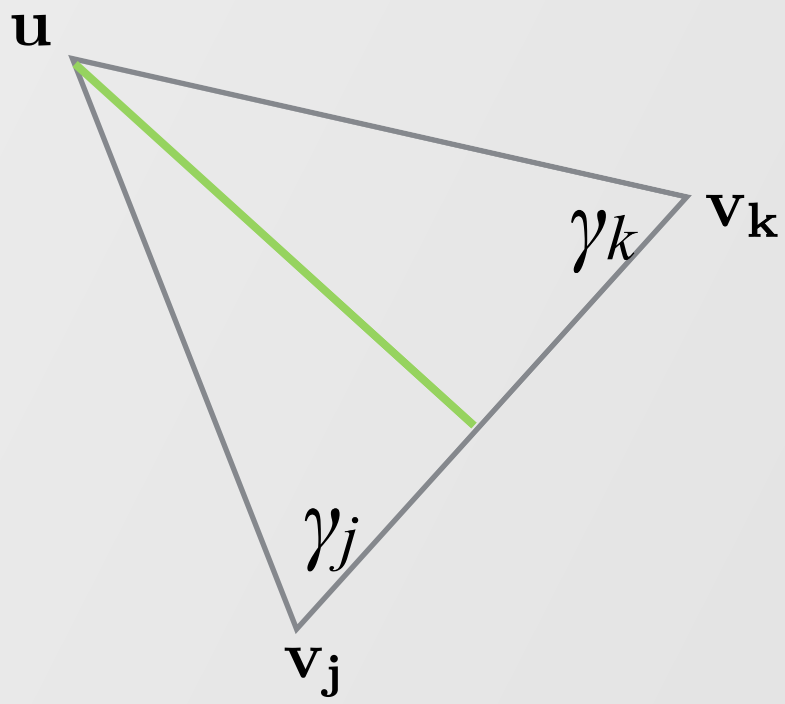
$$= (f(\mathbf{v_j}) - f(\mathbf{u})) \frac{(\mathbf{u} - \mathbf{v_k})^\perp \cdot (\mathbf{v_j} - \mathbf{v_k})^\perp}{4A_t} + \frac{1}{2} \sin\gamma_k \|\mathbf{u} - \mathbf{v_k}\| \|\mathbf{v_j} - \mathbf{v_k}\|$$

$$+ (f(\mathbf{v_k}) - f(\mathbf{u})) \frac{(\mathbf{v_j} - \mathbf{u})^\perp \cdot (\mathbf{v_j} - \mathbf{v_k})^\perp}{4A_t} \quad \frac{1}{2} \sin\gamma_j \|\mathbf{v_j} - \mathbf{u}\| \|\mathbf{v_j} - \mathbf{v_k}\|$$

$$\cos\gamma_k = \frac{(\mathbf{u} - \mathbf{v_k}) \cdot (\mathbf{v_j} - \mathbf{v_k})}{\|\mathbf{u} - \mathbf{v_k}\| \|\mathbf{v_j} - \mathbf{v_k}\|}$$

$$\cos\gamma_j = \frac{(\mathbf{v_j} - \mathbf{u}) \cdot (\mathbf{v_j} - \mathbf{v_k})}{\|\mathbf{v_j} - \mathbf{u}\| \|\mathbf{v_j} - \mathbf{v_k}\|}$$

$$= \frac{\cot\gamma_k \,(f(\mathbf{v_j}) - f(\mathbf{u})) + \cot\gamma_j \,(f(\mathbf{v_k}) - f(\mathbf{u}))}{2}$$

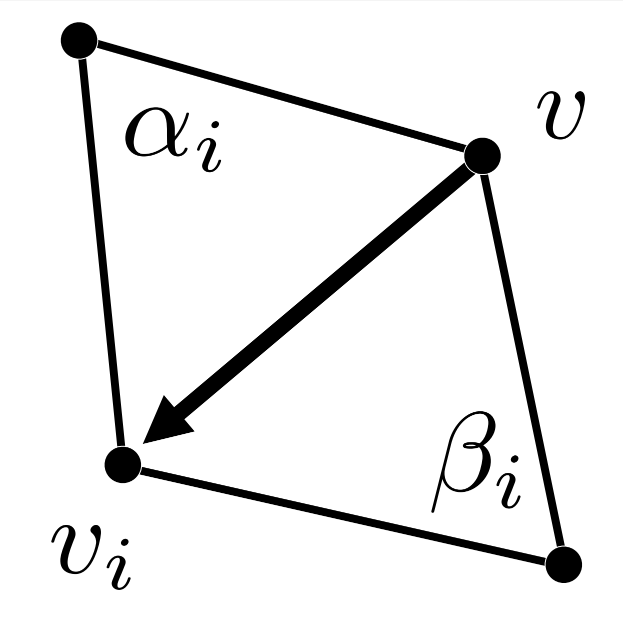


We are done, sum on all triangles

$$\int_{\partial A(\mathbf{u})\cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds = \frac{\cot\gamma_k \left(f(\mathbf{v_j}) - f(\mathbf{u})\right) + \cot\gamma_j \left(f(\mathbf{v_k}) - f(\mathbf{u})\right)}{2}$$

$$\Delta f(\mathbf{u}) = \frac{1}{A(\mathbf{u})} \int_{A(\mathbf{u})} \Delta f(\mathbf{u}) dA = \frac{1}{A(\mathbf{u})} \sum_{t \in S(\mathbf{u})} \int_{\partial A(\mathbf{u})\cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds$$

$$= \frac{1}{2A(\mathbf{u})} \sum_{\mathbf{v_i} \in N(\mathbf{u})} \left(\cot\alpha_i + \cot\beta_i\right) \left(f(\mathbf{v_i}) - f(\mathbf{u})\right)$$



Linear Precision

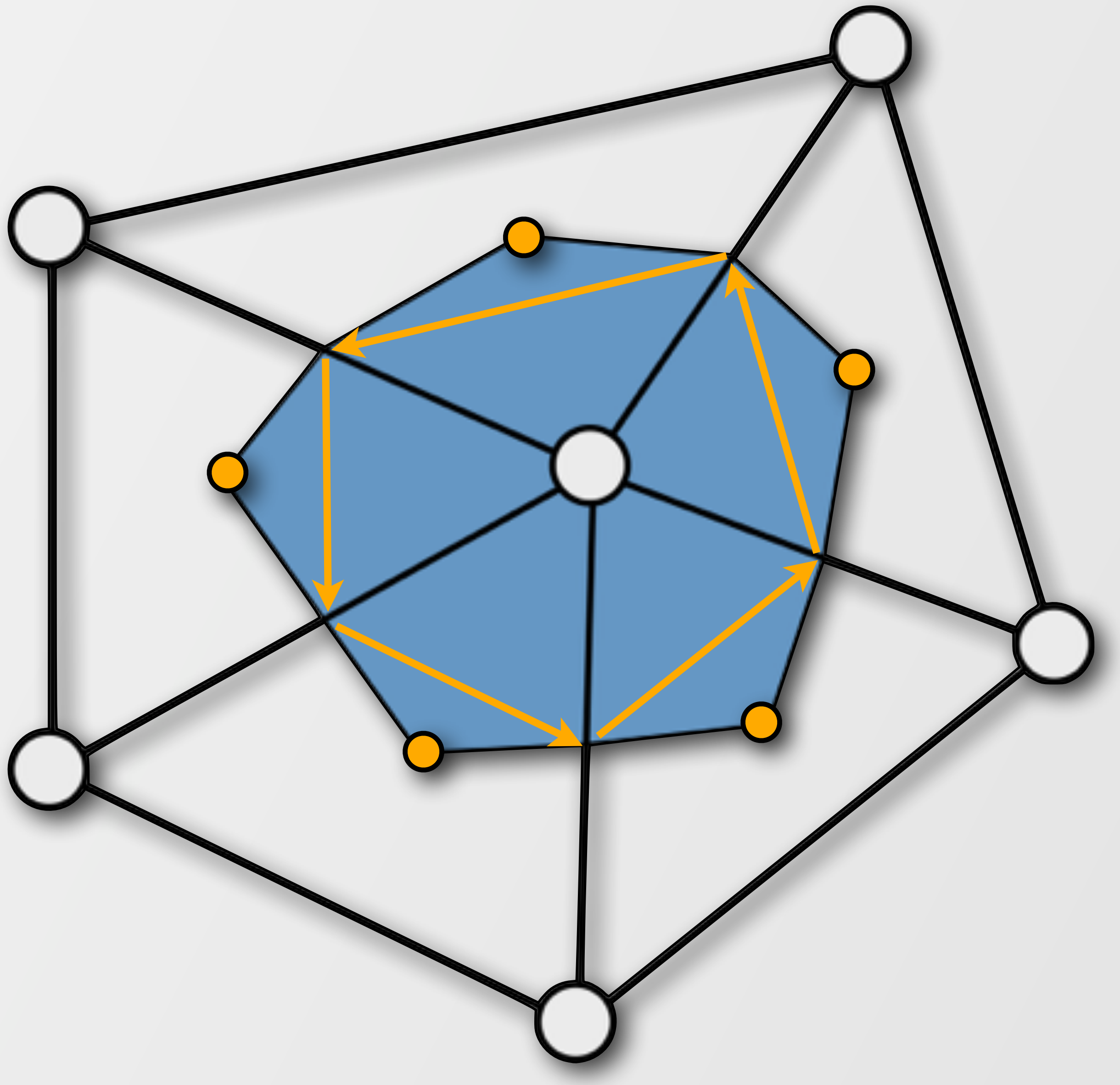
$\Delta v = 0$ on the Euclidean plane

$$\Delta f(\mathbf{u}) = \frac{1}{A(\mathbf{u})} \int_{A(\mathbf{u})} \Delta f(\mathbf{u}) dA$$

$$= \frac{1}{A(\mathbf{u})} \sum_{t \in S(\mathbf{u})} \int_{\partial A(\mathbf{u}) \cap t} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds$$

$\nabla f(\mathbf{u}) = I$ and $\sum_{t \in S(v)} n(\mathbf{u})^{\perp} = 0$ $\longrightarrow$ $\Delta v = 0$

on the Euclidean plane



The End

Thank you!