252-0538-00L, Spring 2018

Shape Modeling and Geometry Processing

Mappings

Representation and Distortion

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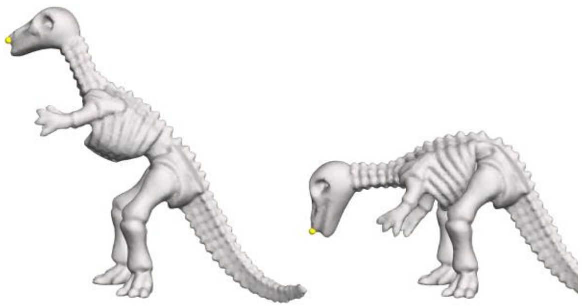
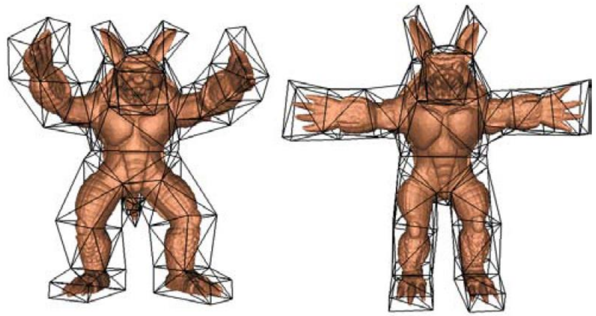
Mappings
Representation and Distortion

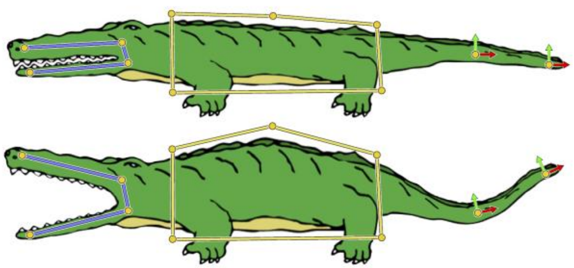


[Sorkine & Alexa 07]



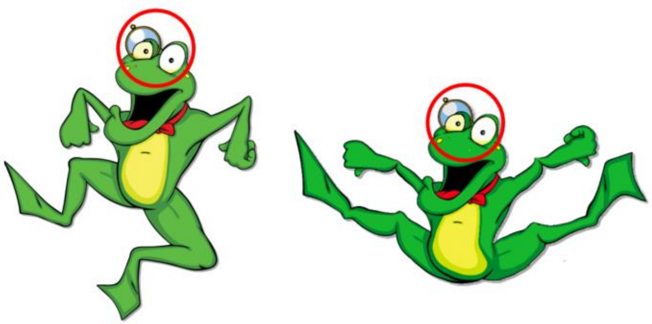
[Ju et al. 05]

[Poranne & Lipman 14]



[Jacobson 07]

[Weber et al. 09]



[Lipman et al. 07]

Mappings
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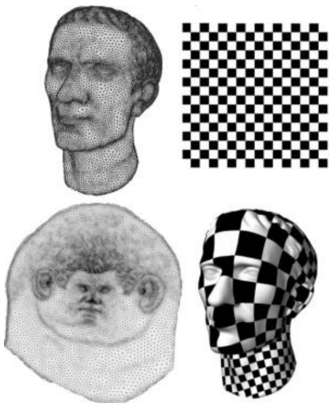
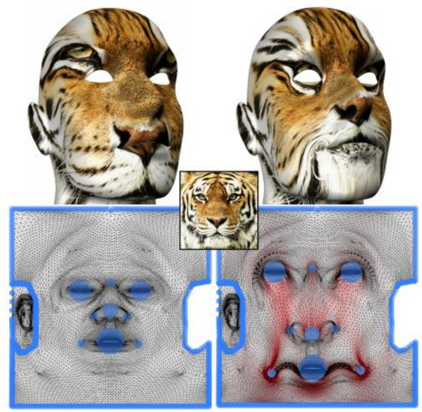
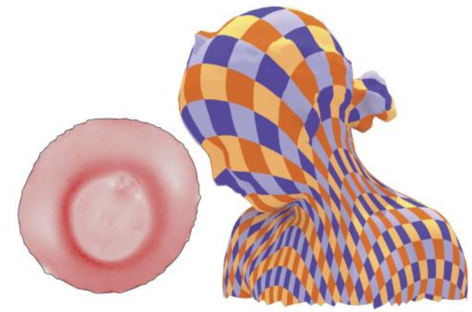


[Lévy et al. 02]

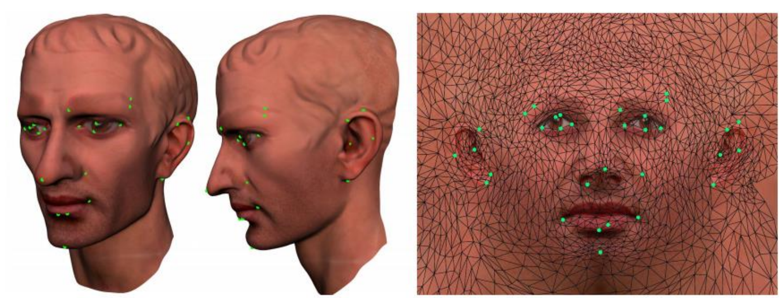


[Schuler et al. 13]



[Fu et al. 15]

[Mullen et al. 08]



[Weber et al. 12]

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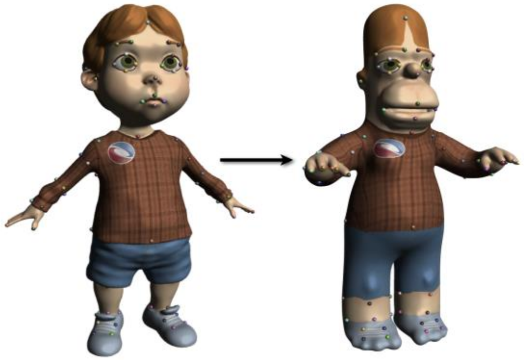
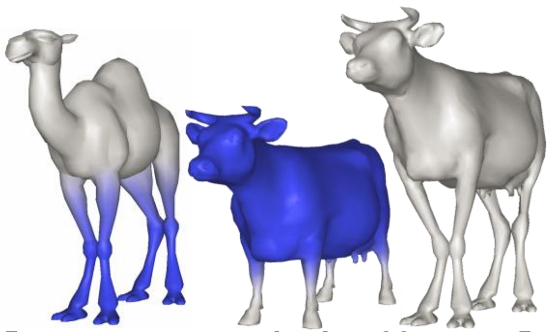
Mappings
Representation and Distortion

[Kim et al. 11]

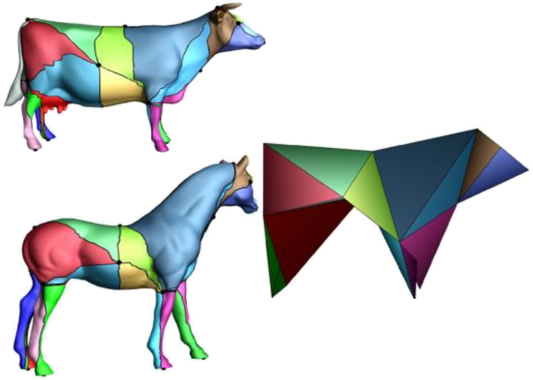
[Ovsjanikov et al. 12]



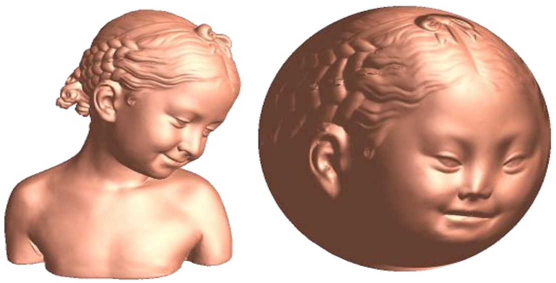
[Panozzo et al. 13]



[Kraevoy and Sheffer 04]



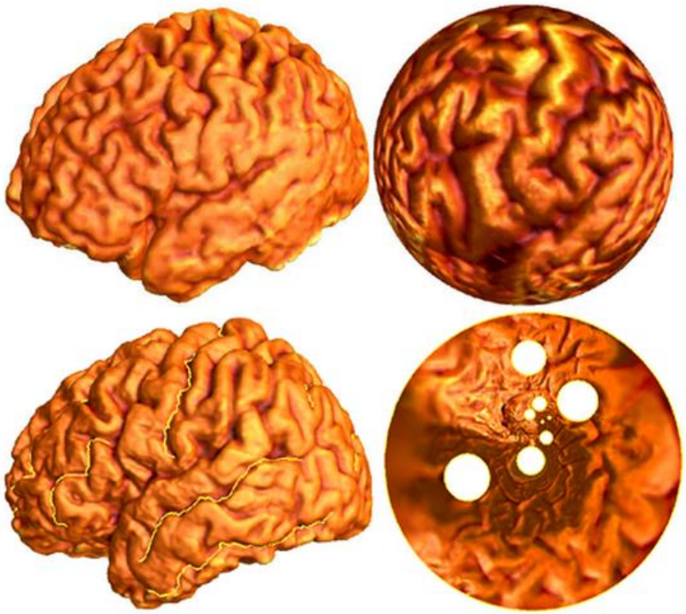
[Schreiner et al. 04]

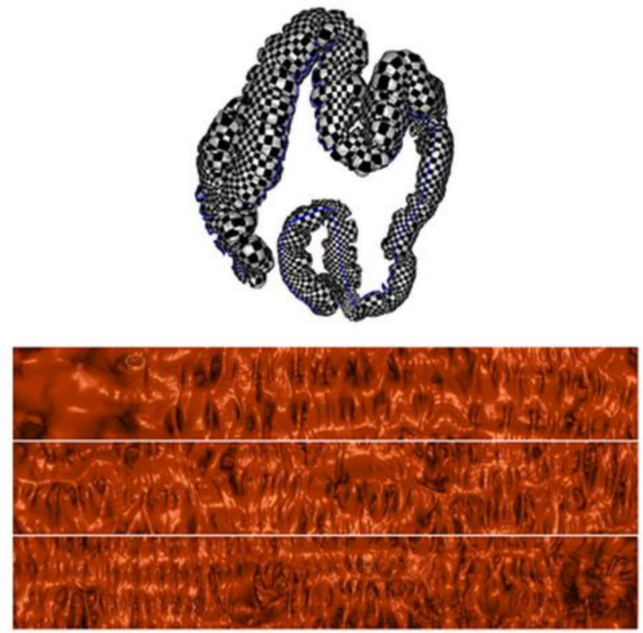


[Jin et al. 08]

“Real” Applications

Brain/colon mapping

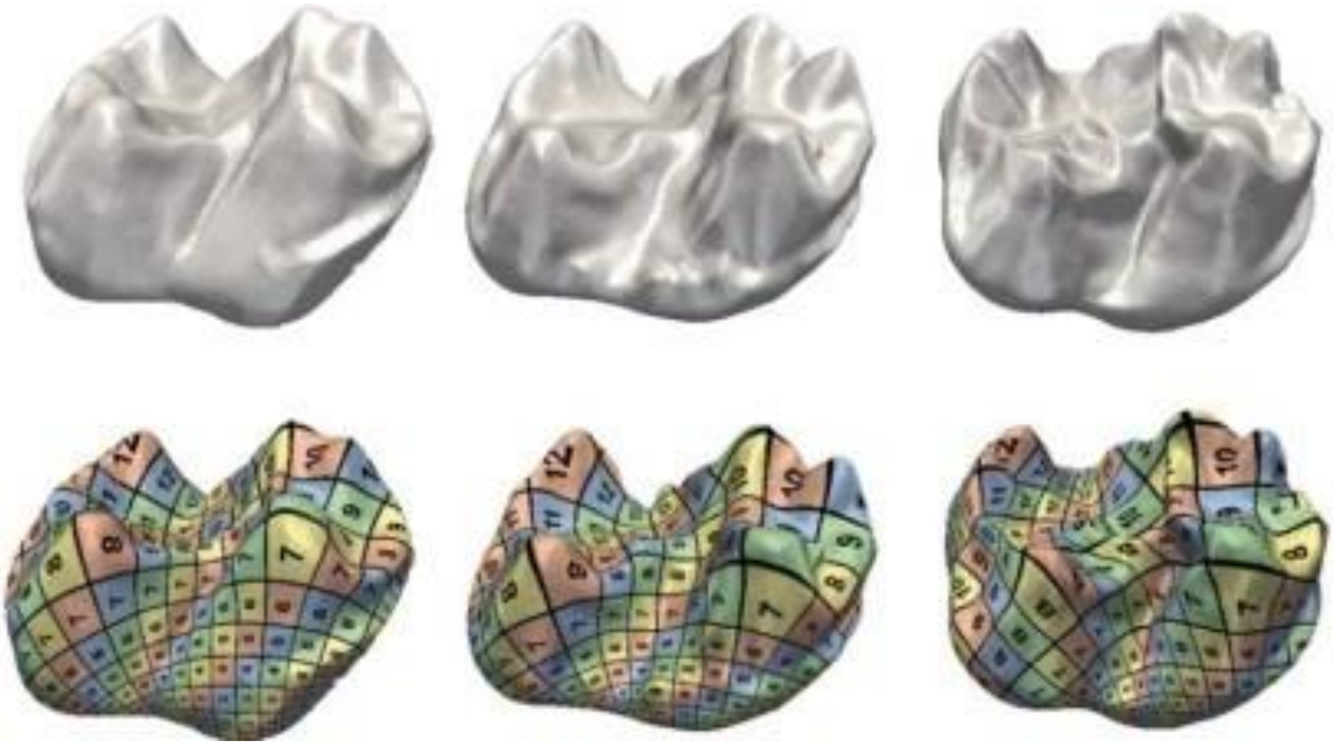




[Gu et al.]

"Real" Applications

Biological Morphology



[Boyer et al. 2012]

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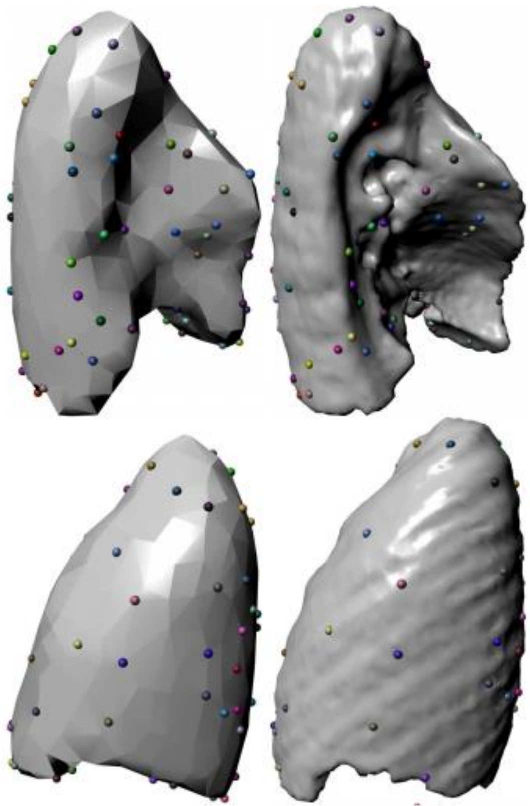
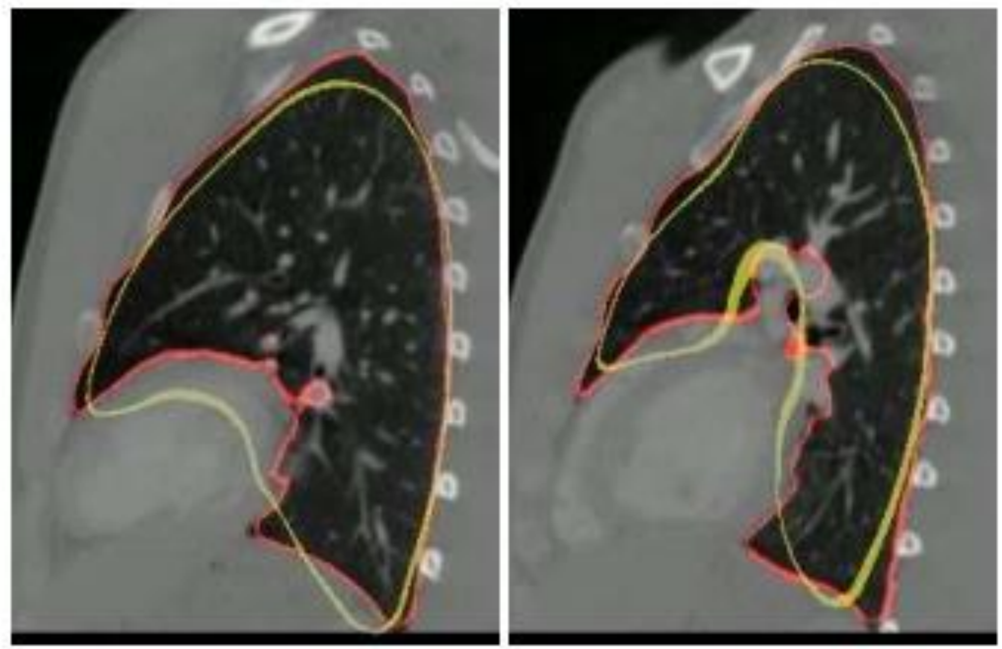
“Real” Applications

Medical segmentation/registration





[Levi and Gotsman 12]

Definition

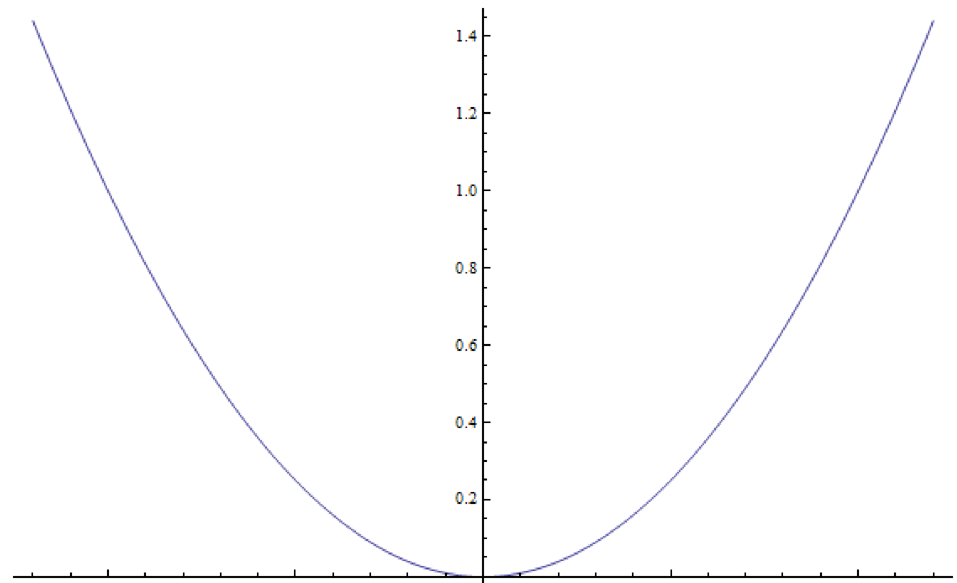
Mapping / Map :

A smooth function between shapes / spaces

Examples

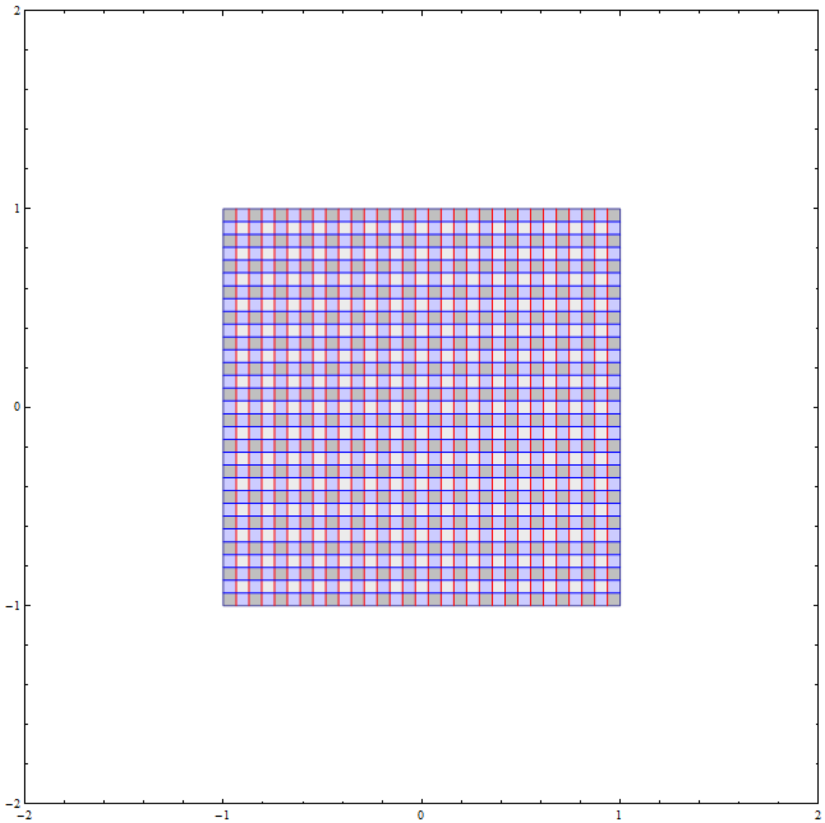
$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = x^2$$

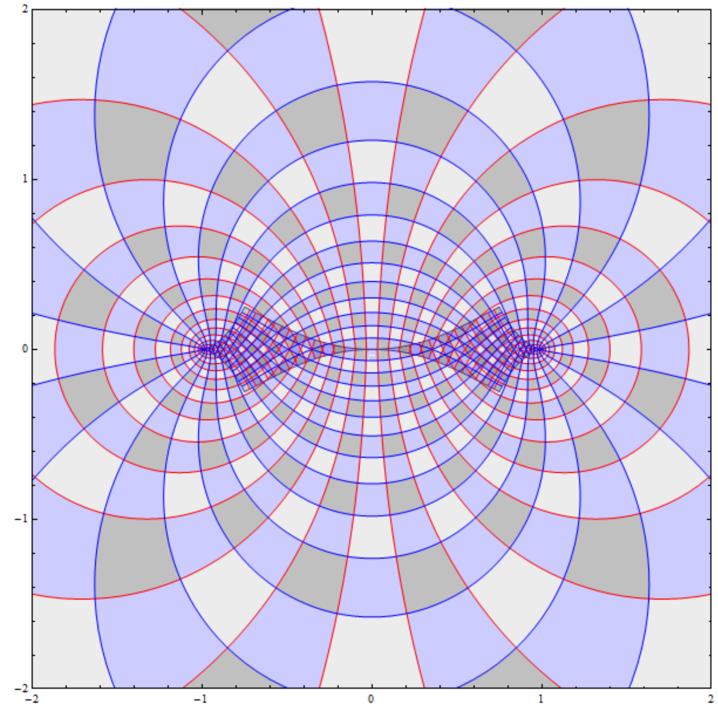


$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$f(z) = z + \frac{1}{z}$$



$\Rightarrow$



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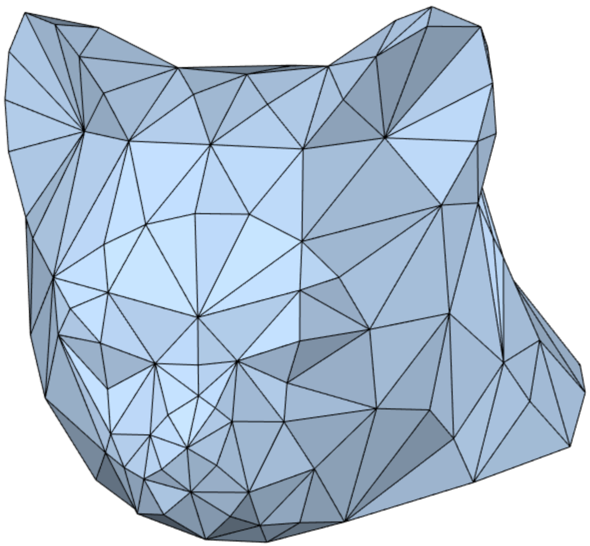
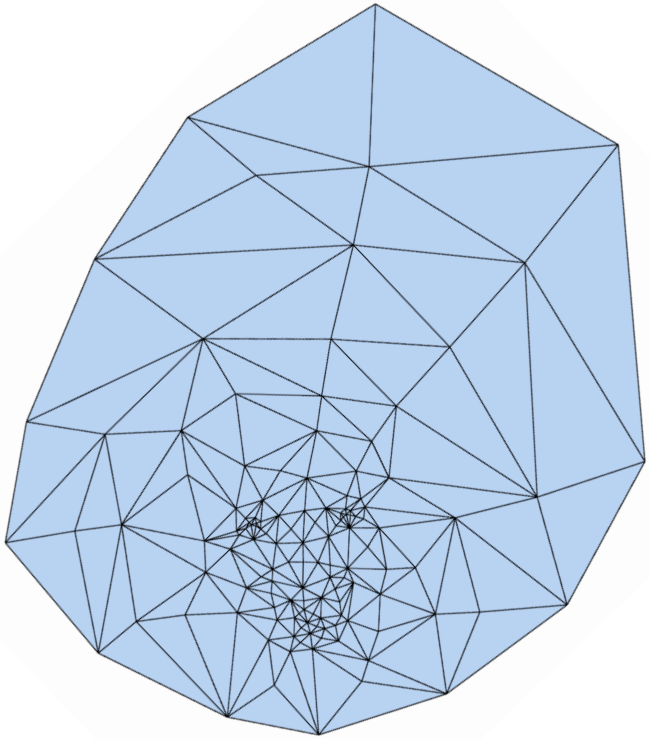
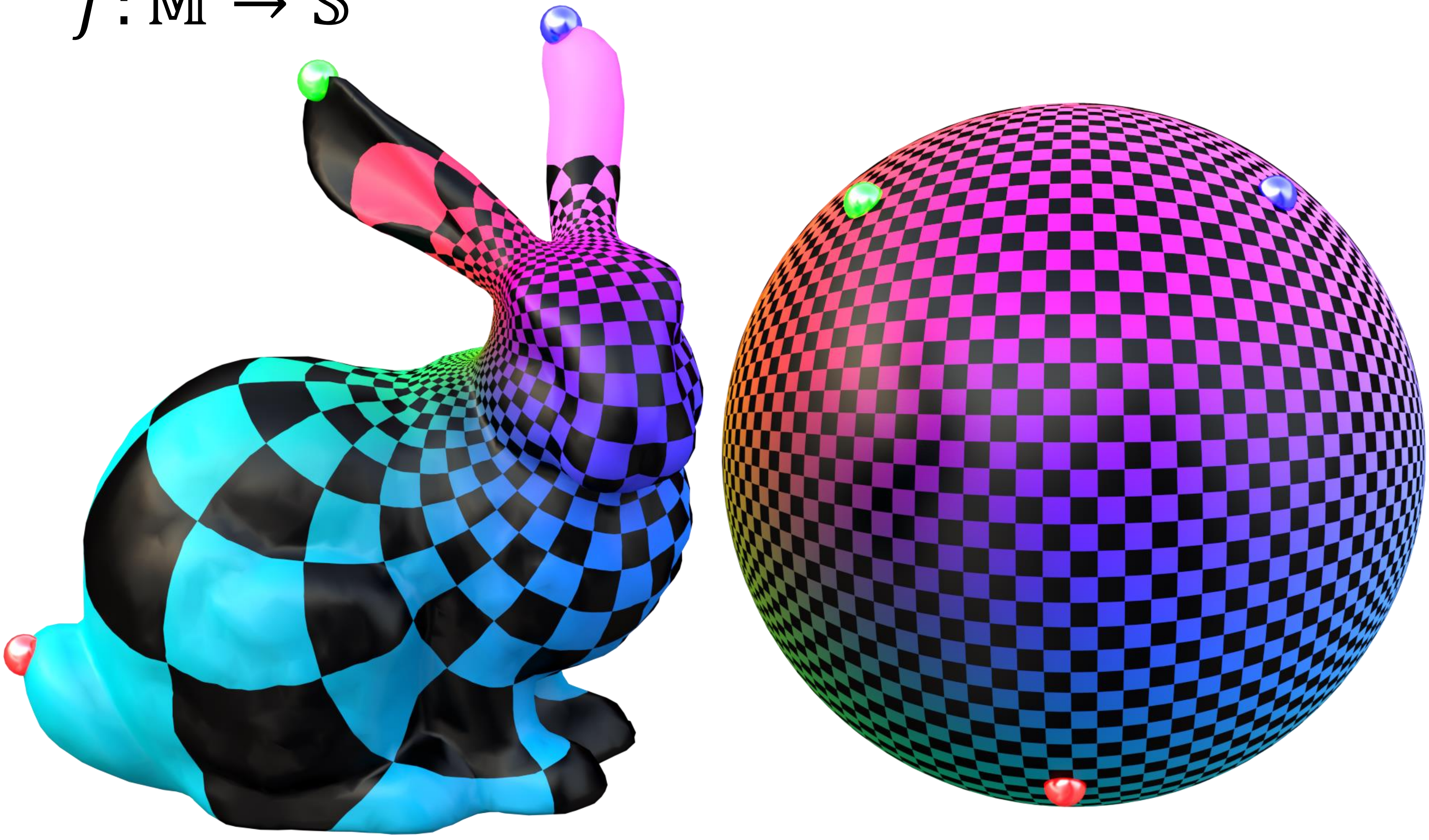
$$f : \mathbb{M} \rightarrow \mathbb{R}^2$$





$f: \mathbb{M} \to \mathbb{S}^2$



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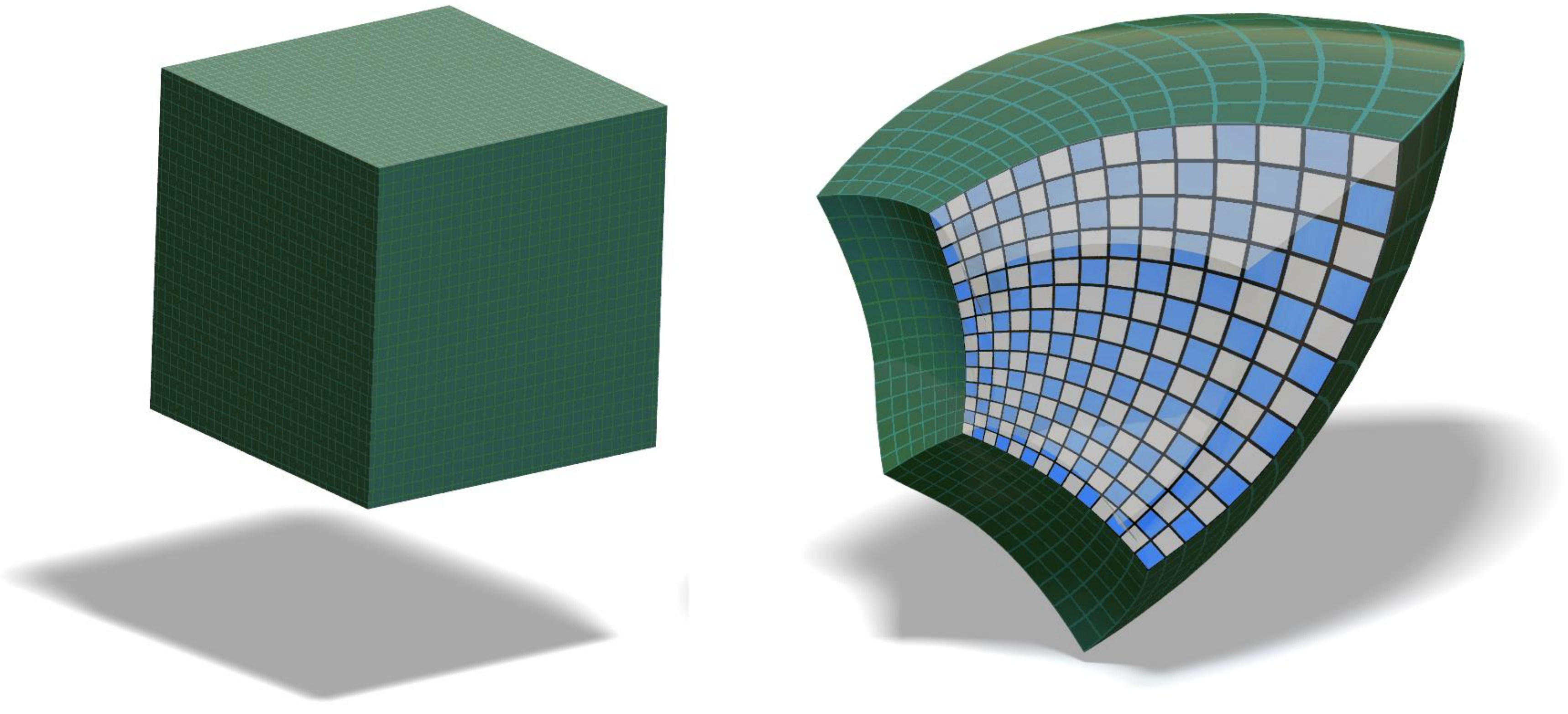
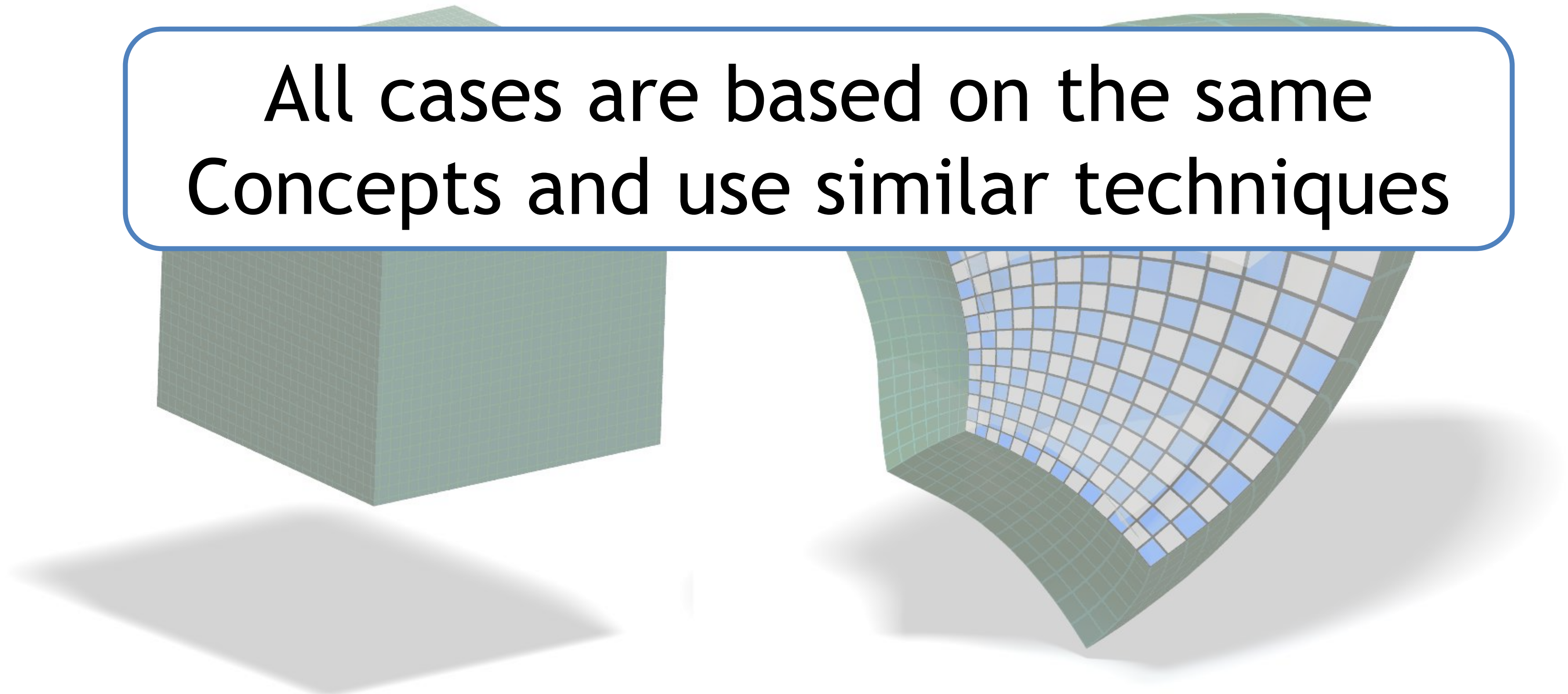
$$f: \mathbb{M} \rightarrow \mathbb{M}'$$

$$f: \mathbb{R}^3 \to \mathbb{R}^3$$



$f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$



All cases are based on the same Concepts and use similar techniques

2D Maps : Representation

$$\boxed{\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}}$$

2D Maps : Representation

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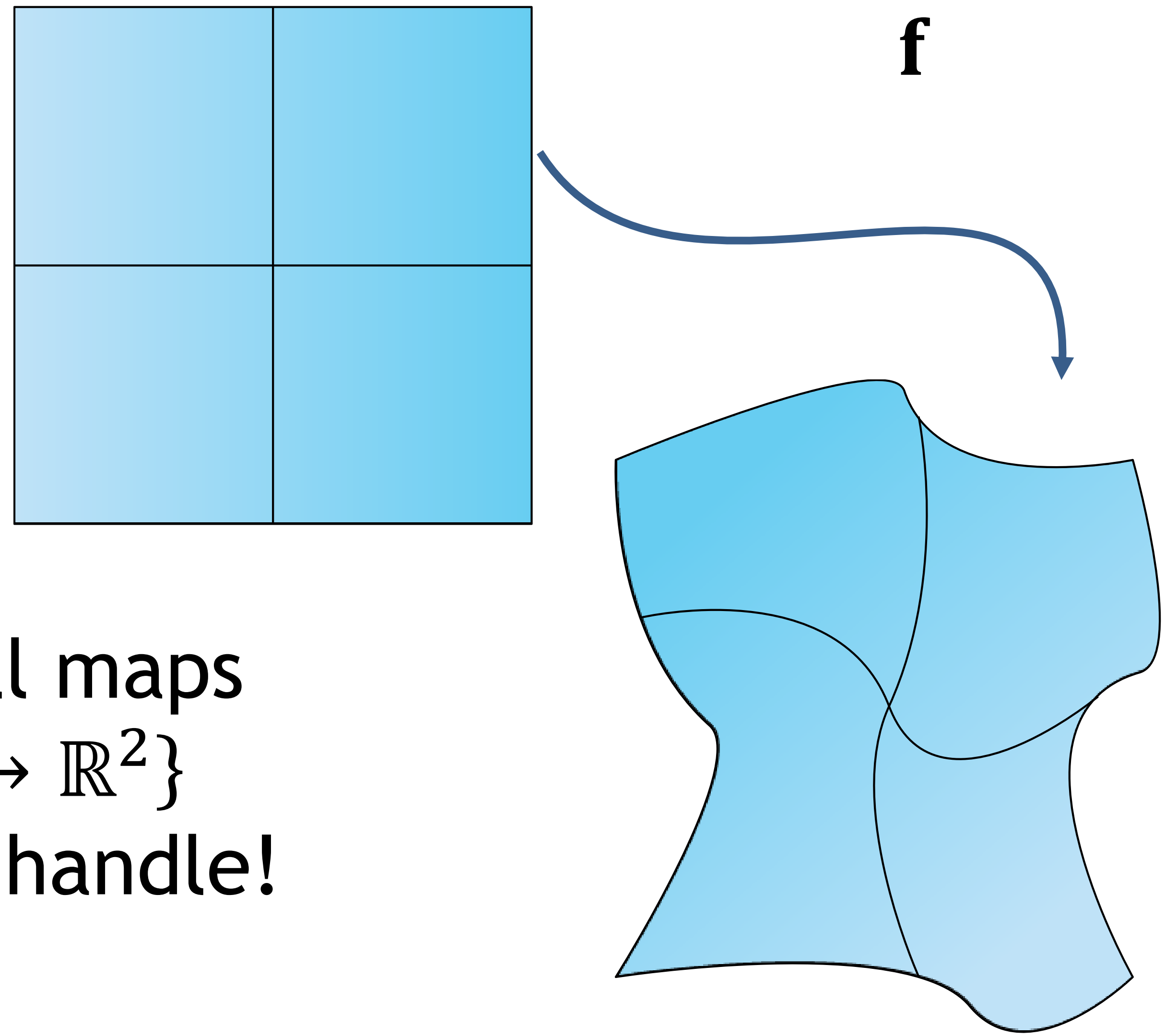


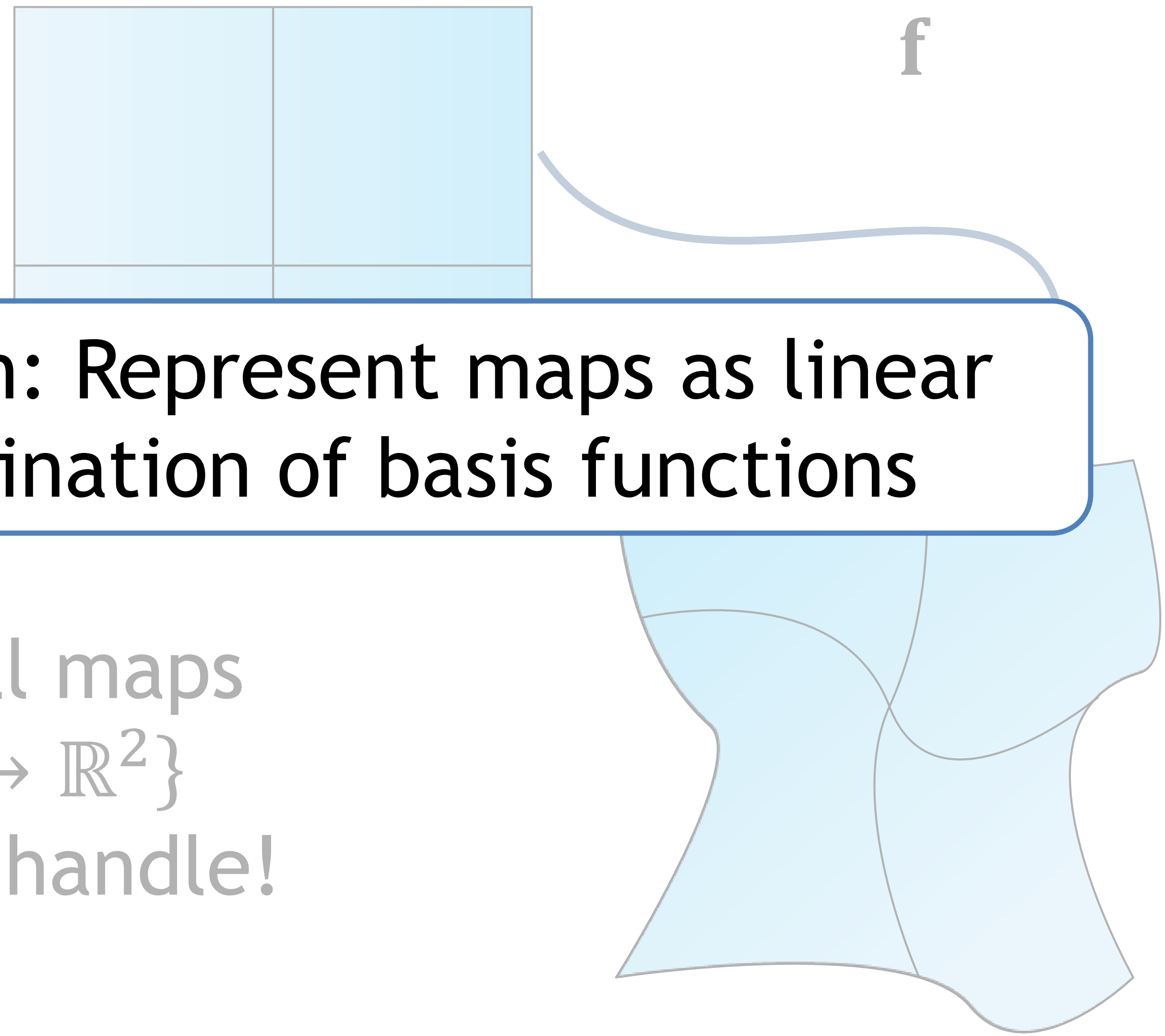
The set of all maps

$\{f(x): \mathbb{R}^2 \to \mathbb{R}^2\}$

is too big to handle!

2D Maps : Representation

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}$$



Solution: Represent maps as linear combination of basis functions

The set of all maps
$\{f(x)\colon \mathbb{R}^2 \to \mathbb{R}^2\}$
is too big to handle!

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2D Maps : Representation

$$f_1, f_2, f_3, \ldots, f_n$$

2D Maps : Representation

$$f_1, f_2, f_3, \dots, f_n$$

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}$$

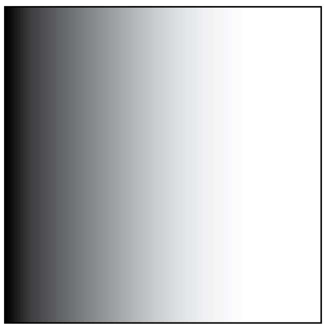
2D Maps : Representation

$$f_1, f_2, f_3, \dots, f_n$$

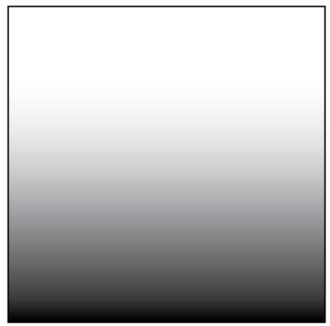
$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \sum a_i f_i(\mathbf{x}) \\ \sum b_i f_i(\mathbf{x}) \end{pmatrix}$$

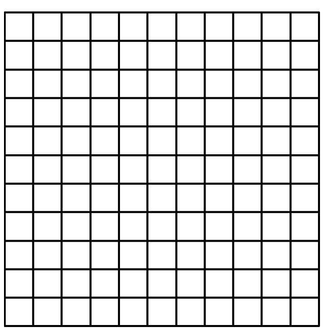
2D Maps : Representation

$f_1(x, y) = x$

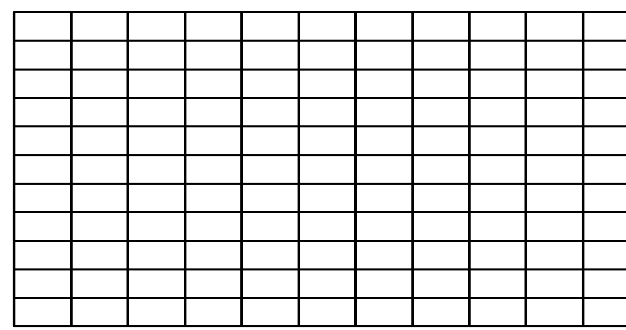


$f_2(x, y) = y$



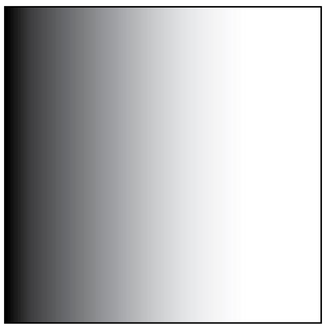


$(x, y) \Rightarrow (2f_1, f_2)$

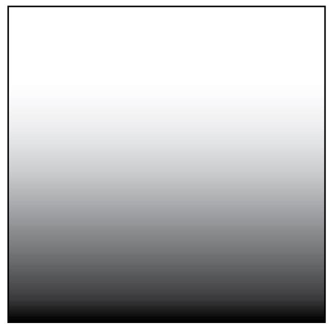


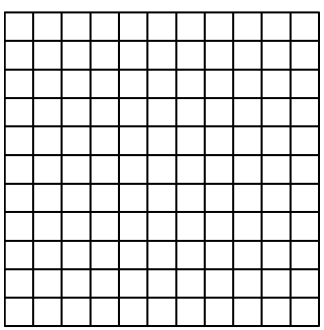
2D Maps : Representation

$f_1(x, y) = x$

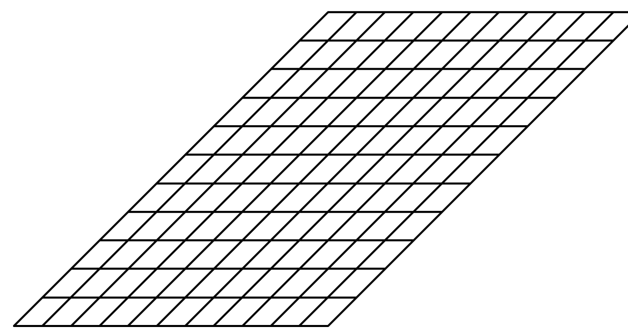


$f_2(x, y) = y$



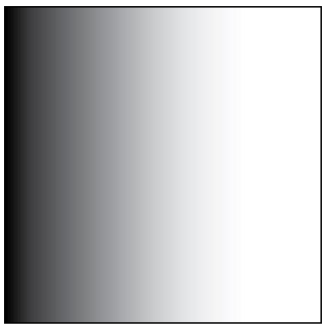
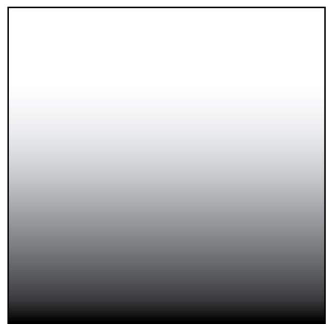
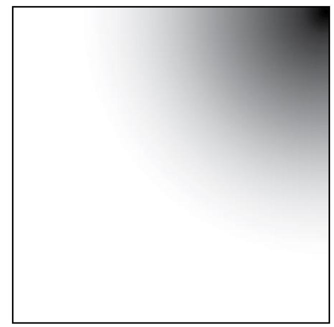


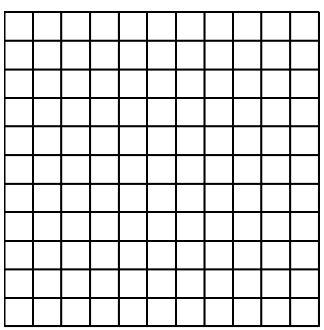
$(x, y) \Rightarrow (f_1 + f_2, f_2)$



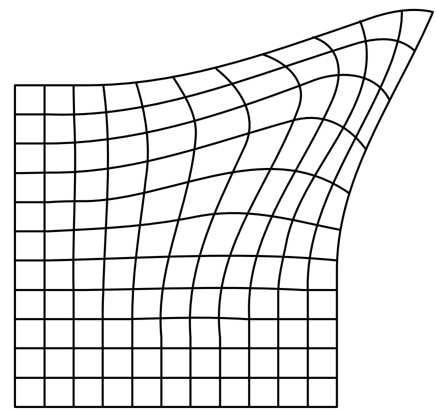
2D Maps : Representation

$f_1(x, y) = x$ $\quad$ $f_2(x, y) = y$ $\quad$ $f_3(x, y)$



$$(x, y) \Rightarrow (f_1 + f_3, f_2 + f_3)$$



Mappings for deformations

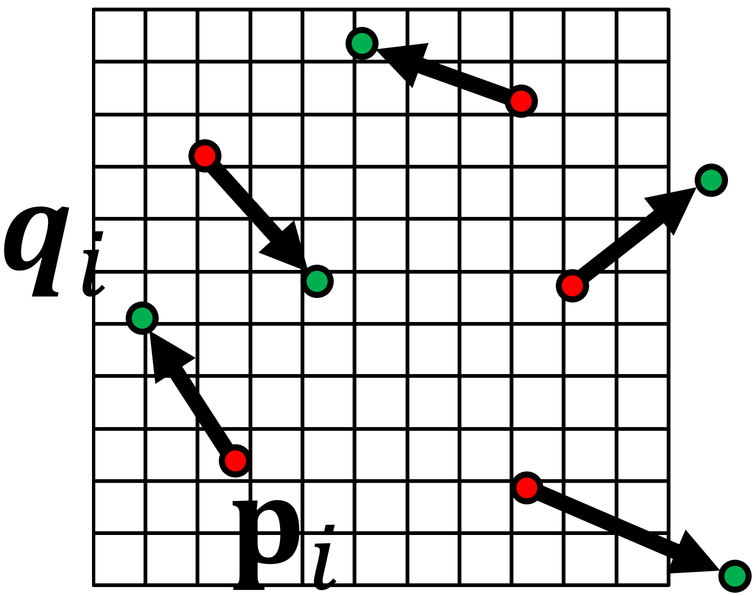
$\mathbf{q}_i$

$\mathbf{p}_i$

Mappings for deformations

$\mathbf{q}_i$

Deformation as an interpolation problem



$$\mathbf{f}(\mathbf{p}_i) = \sum \mathbf{c}_i f_i(\mathbf{x})$$

$$\mathbf{c}_i = ?$$

$$\mathbf{f}(\mathbf{p}_i) = \mathbf{q}_i, \forall i$$

$$\sum \mathbf{c}_i f_i(\mathbf{p}_i) = \mathbf{q}_i, \forall i$$

Example: Thin Plate Spline

Solve the problem

$$\min E_{\mathrm{TPS}}(\mathbf{f}) = \iint \left[\left(\frac{\partial^2 \mathbf{f}}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 \mathbf{f}}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 \mathbf{f}}{\partial y^2} \right)^2 \right]$$

Bending energy

$$\text{s.t. } \mathbf{f}(\mathbf{p}_i) = \mathbf{q}_i, \forall i$$

General solution

$$\mathbf{f}(\mathbf{p}_i) = \mathbf{c}_0 + \mathbf{c}_x \boldsymbol{x} + \mathbf{c}_y \boldsymbol{y} + \sum \mathbf{c}_i \phi(\|\mathbf{x} - \mathbf{p}_i\|)$$

$$\phi(r) = r^2 \log r$$

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Globally VS. Locally Bijective

Globally Bijective $\rightarrow$ Locally Bijective

$\mathbf{f}$ is bijective

$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective

Globally VS. Locally Bijective

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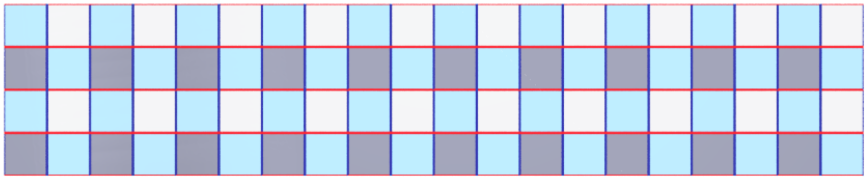

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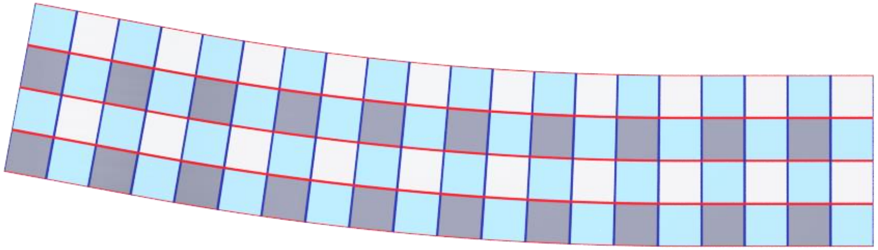

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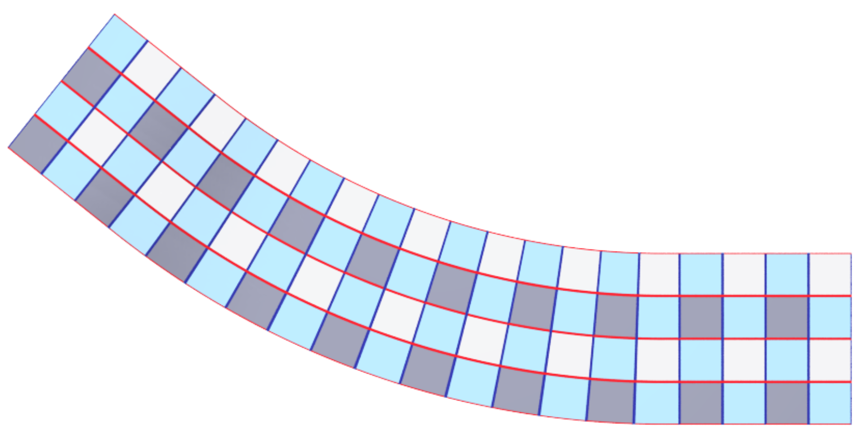

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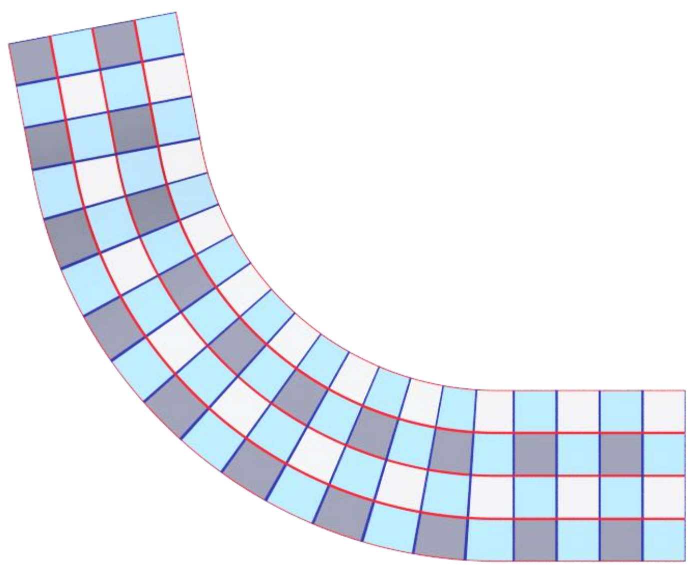

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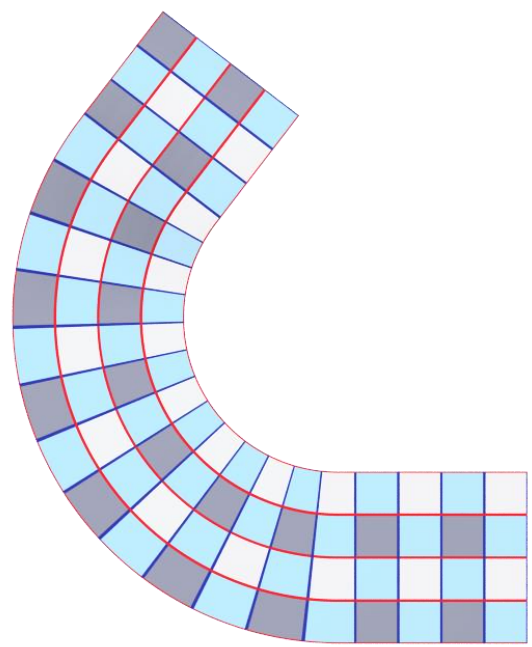

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Locally Bijective

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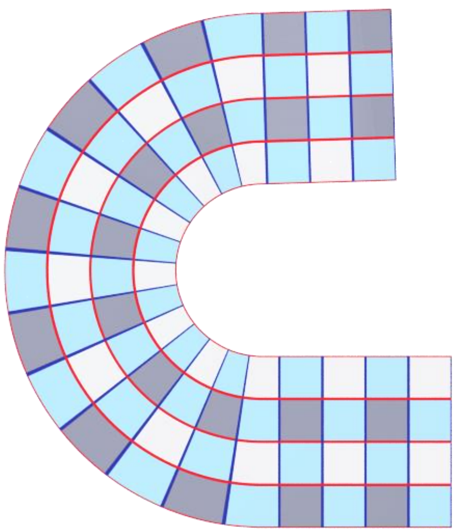
$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective

Still Bijective!



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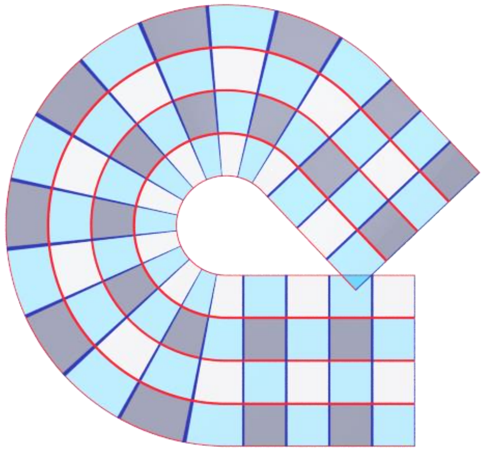

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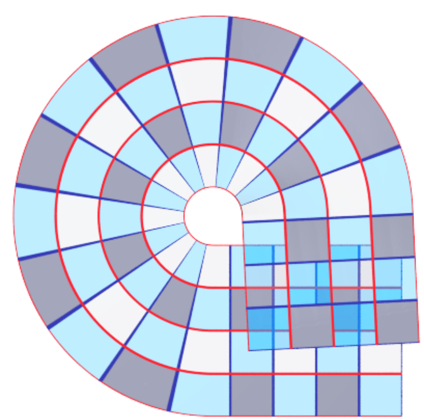

Globally VS. Locally Bijective

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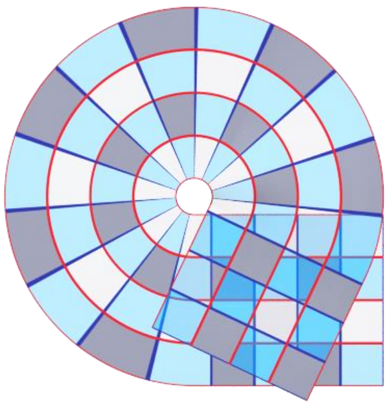

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Globally VS. Locally Bijective

Globally Bijective

$\mathbf{f}$ is bijective

Not Bijective!

Locally Bijective

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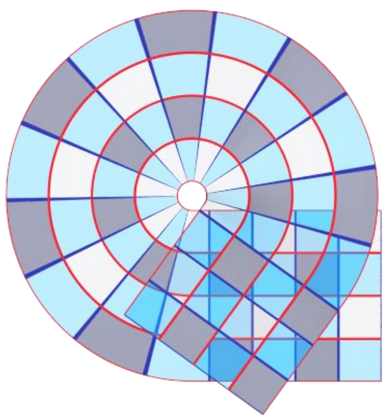
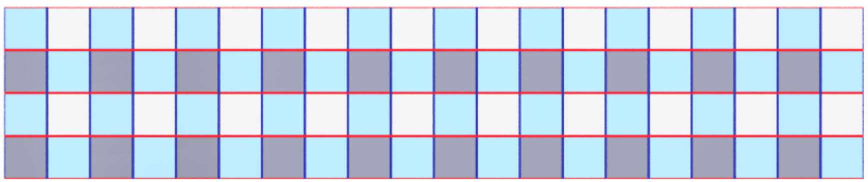
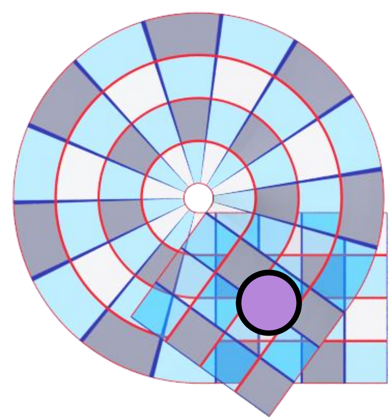


Globally VS. Locally Bijective



Two Pre-images

Not Bijective!



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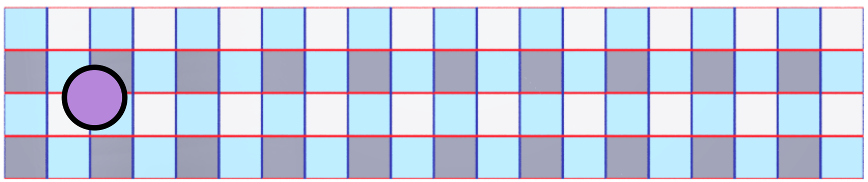
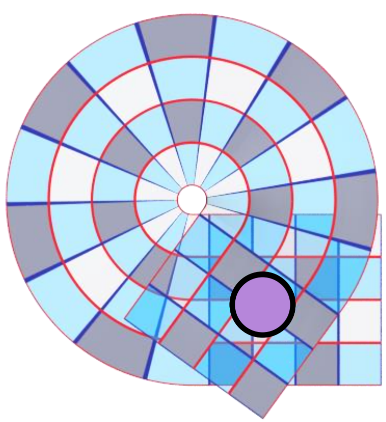
Globally Bijective VS. Locally Bijective

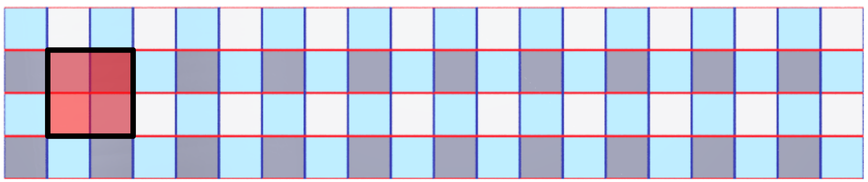


Not Bijective!

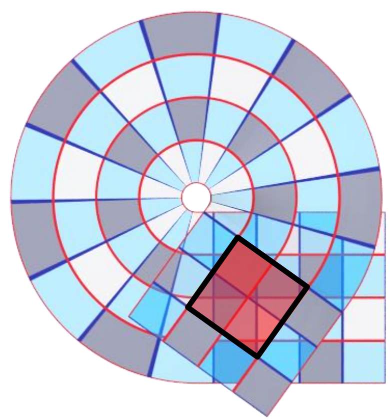


Globally Bijective VS. Locally Bijective

U

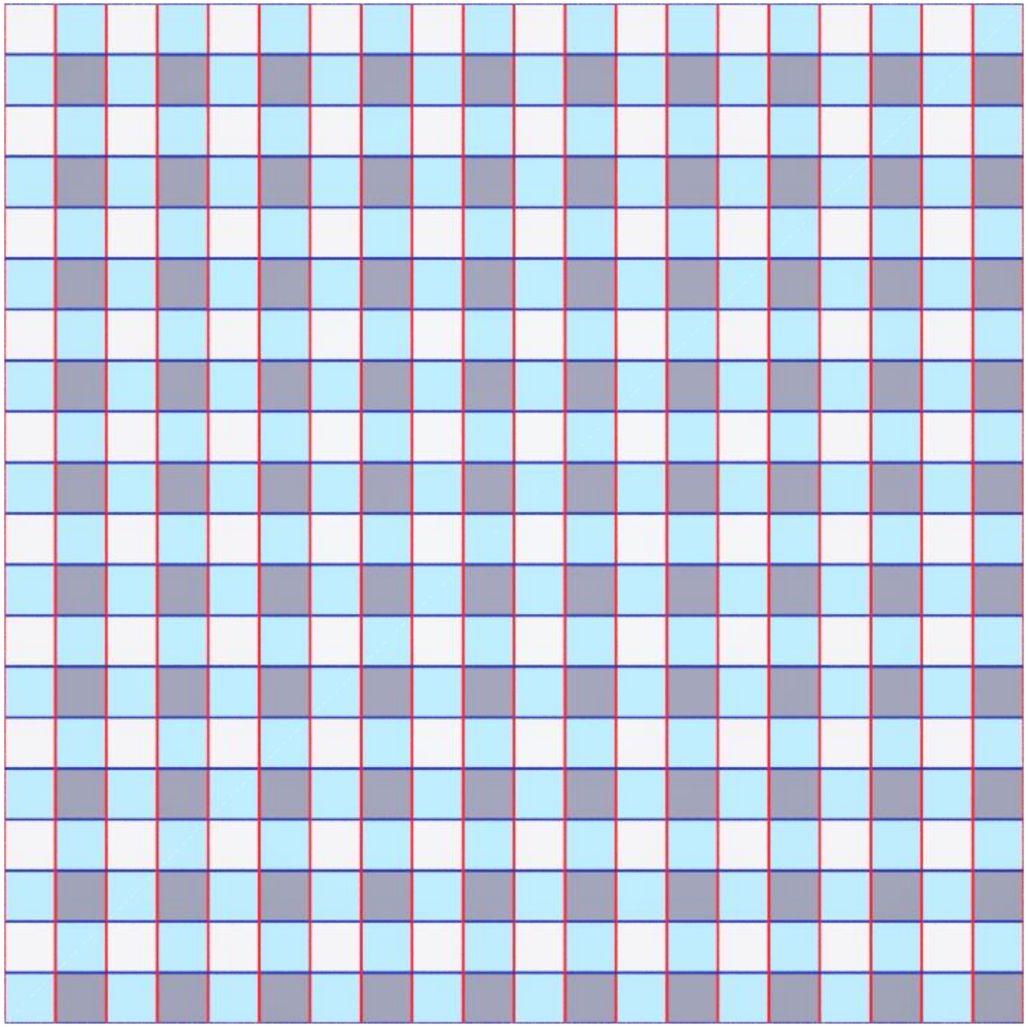


Only Locally Bijective!



f(U)

Locally Bijection – Non-example



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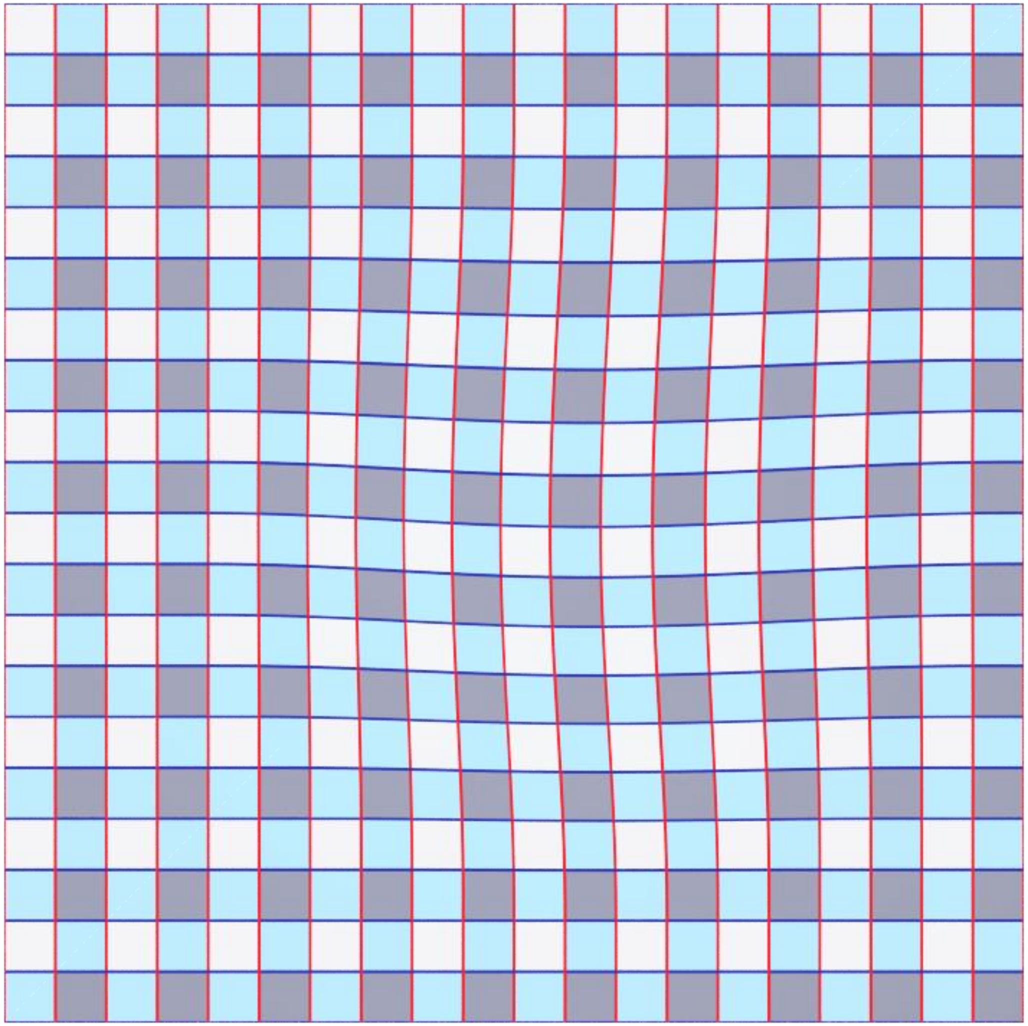
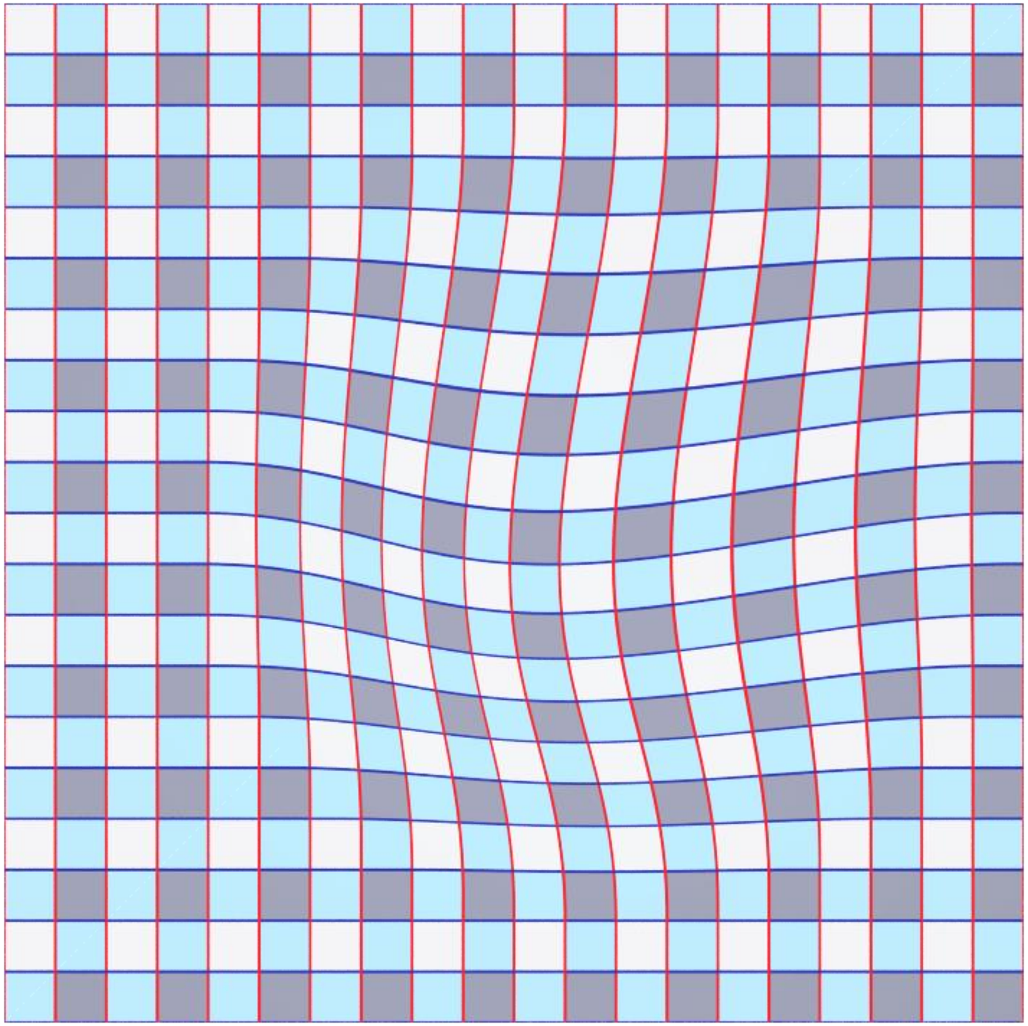
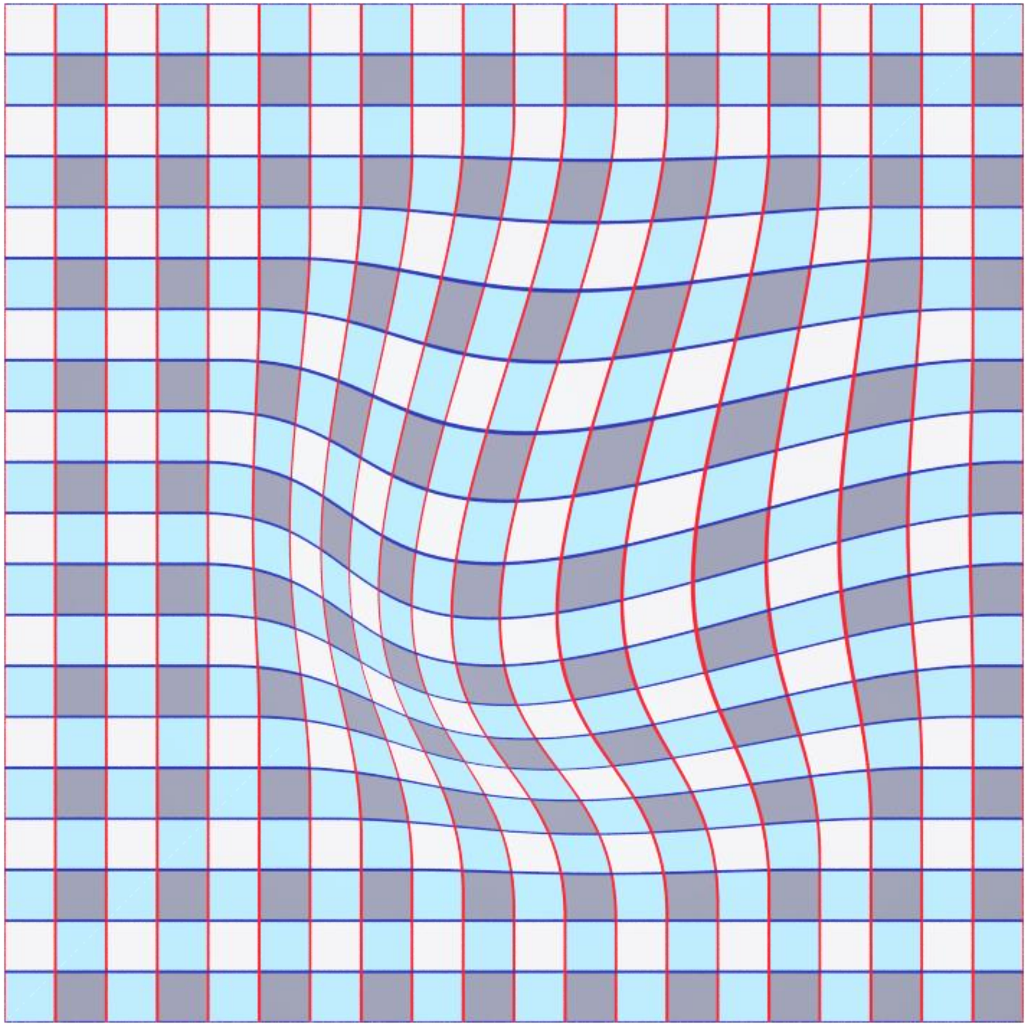
Locally Bijection – Non-example



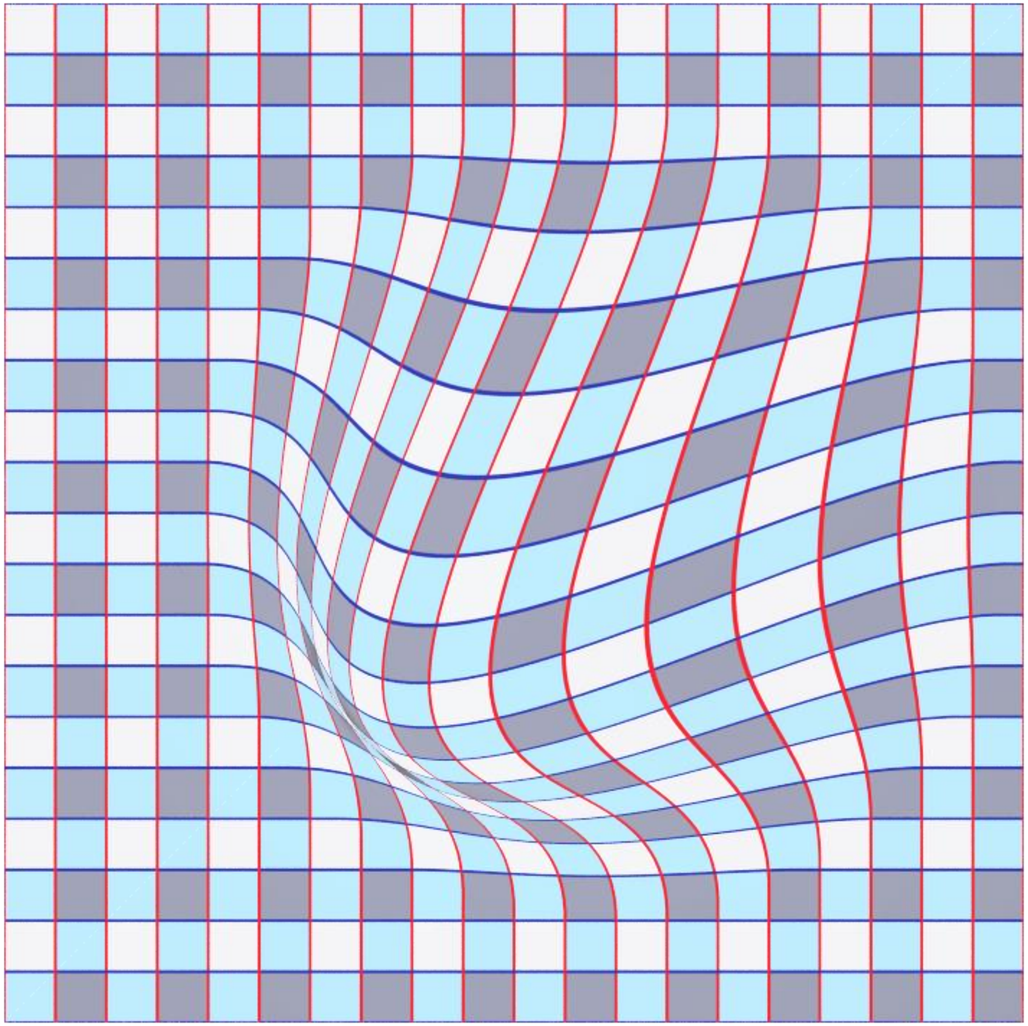
Locally Bijection – Non-example



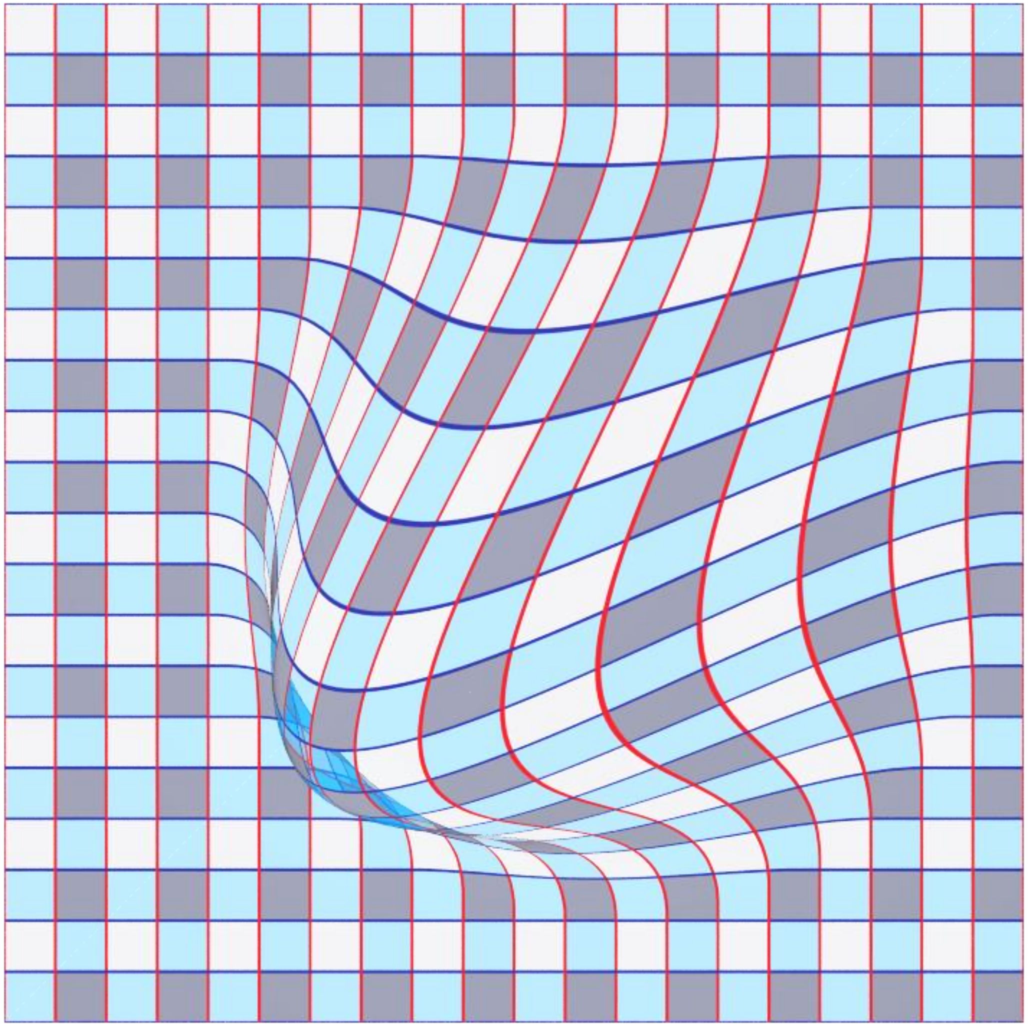
Locally Bijection – Non-example



Locally Bijection – Non-example



Locally Bijection – Non-example



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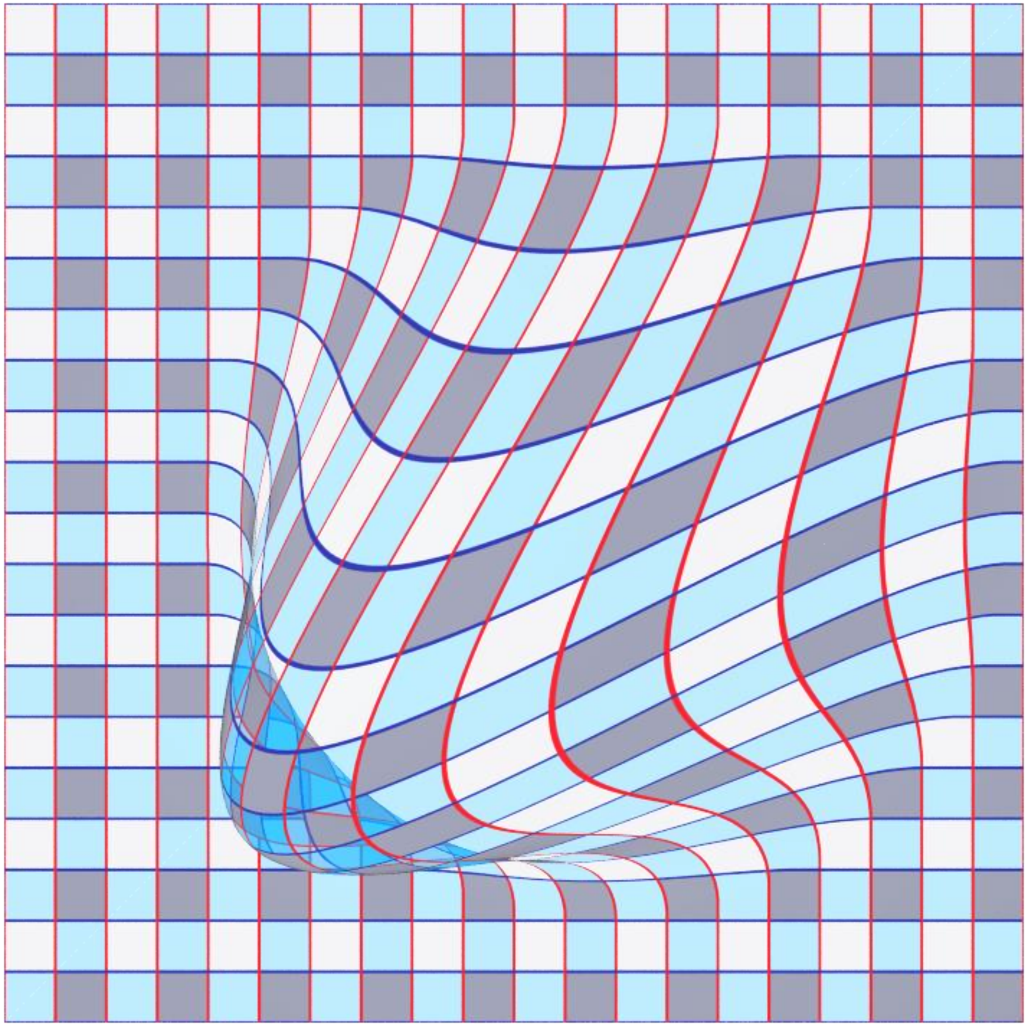
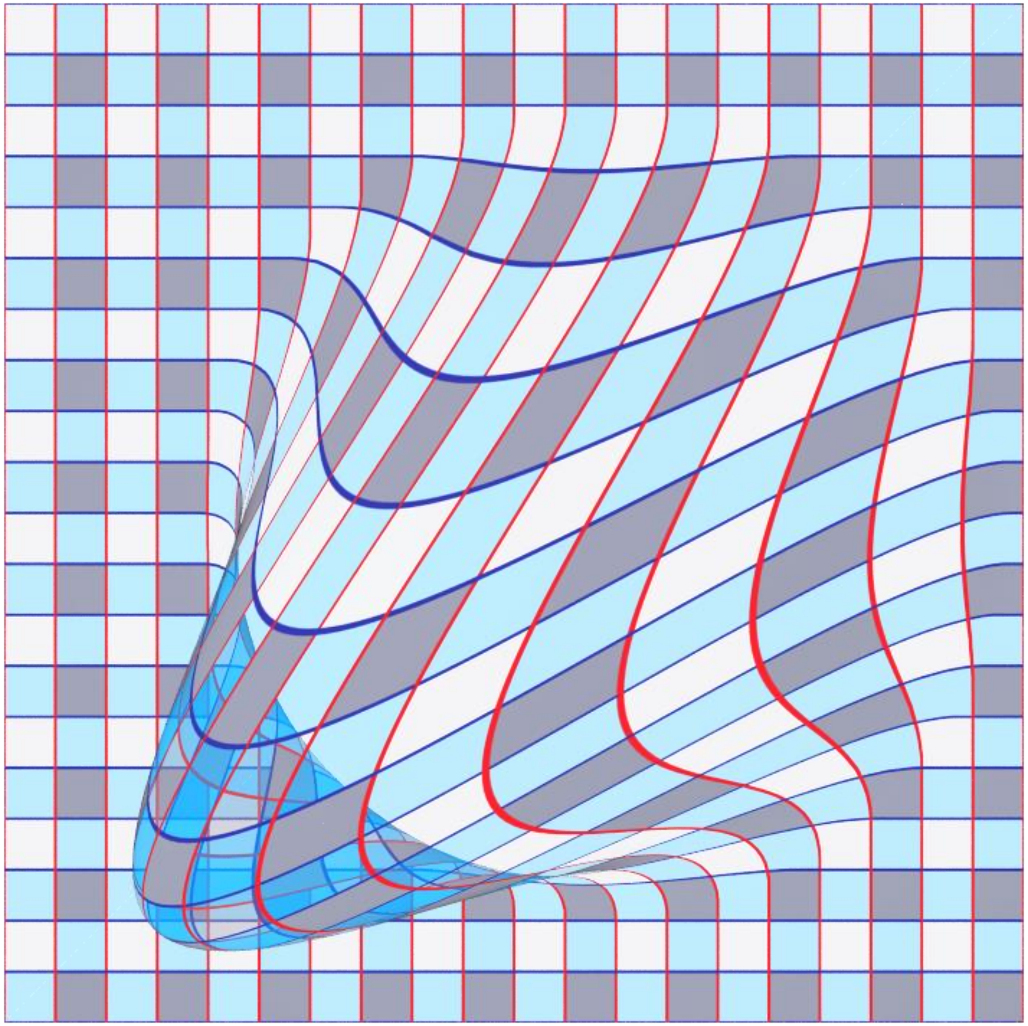
Locally Bijection – Non-example



Locally Bijection – Non-example



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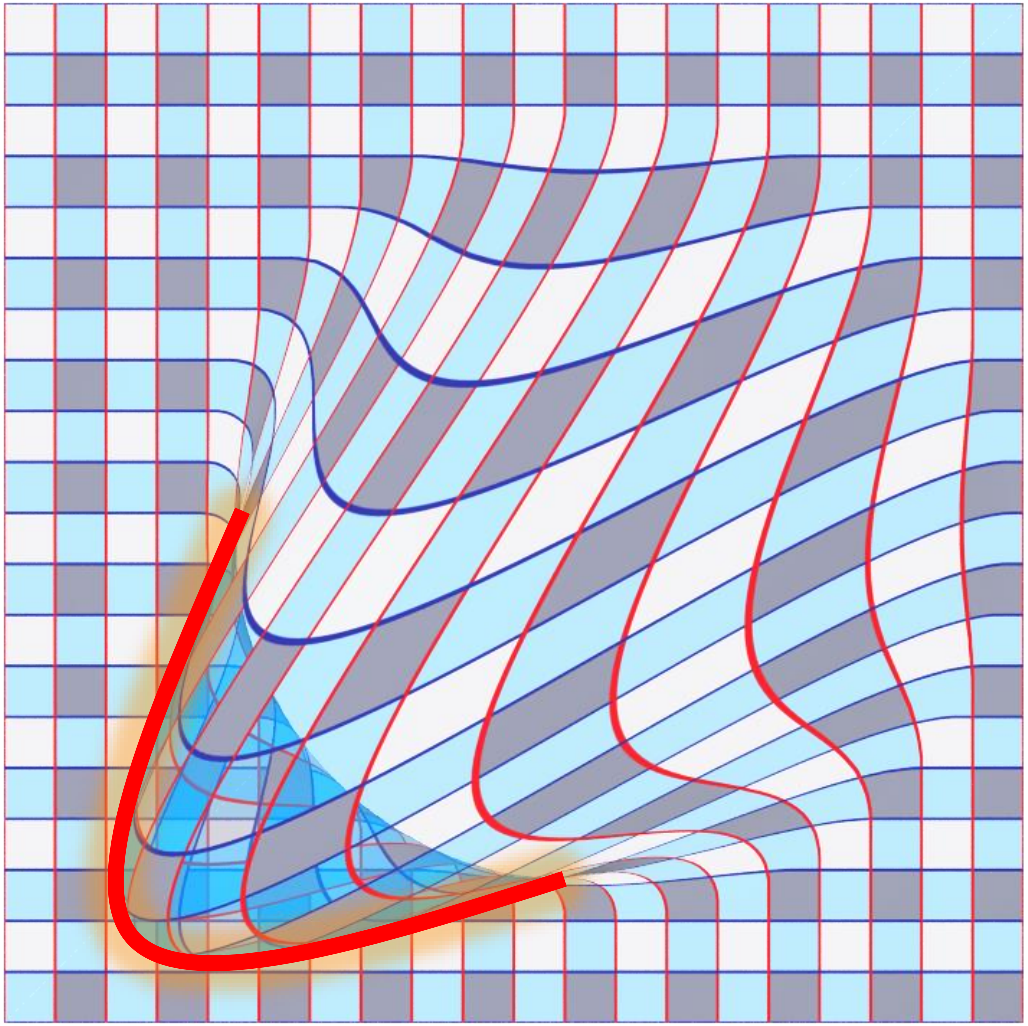
Locally Bijection – Non-example



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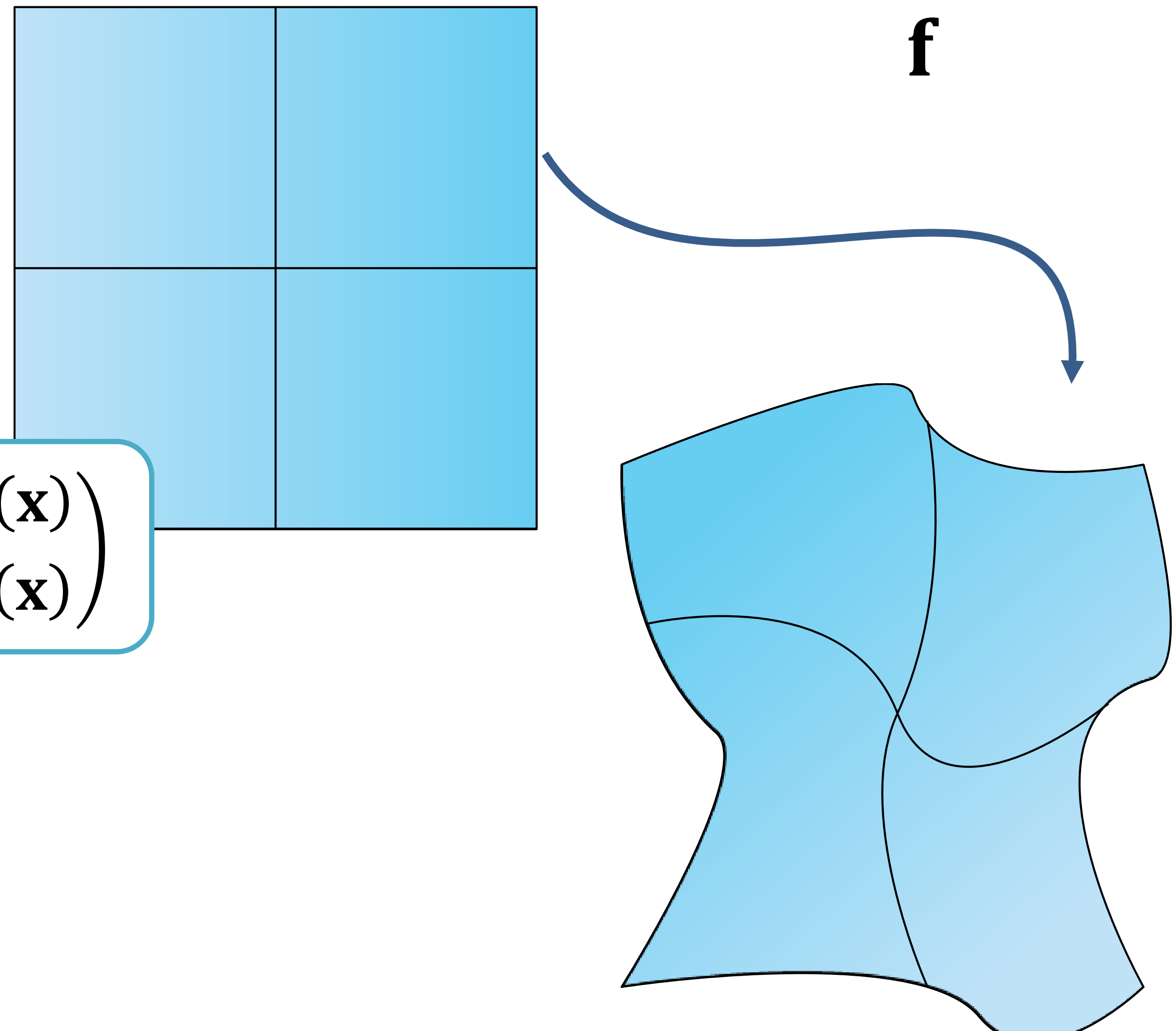
Local Bijection Sufficient condition

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}$$

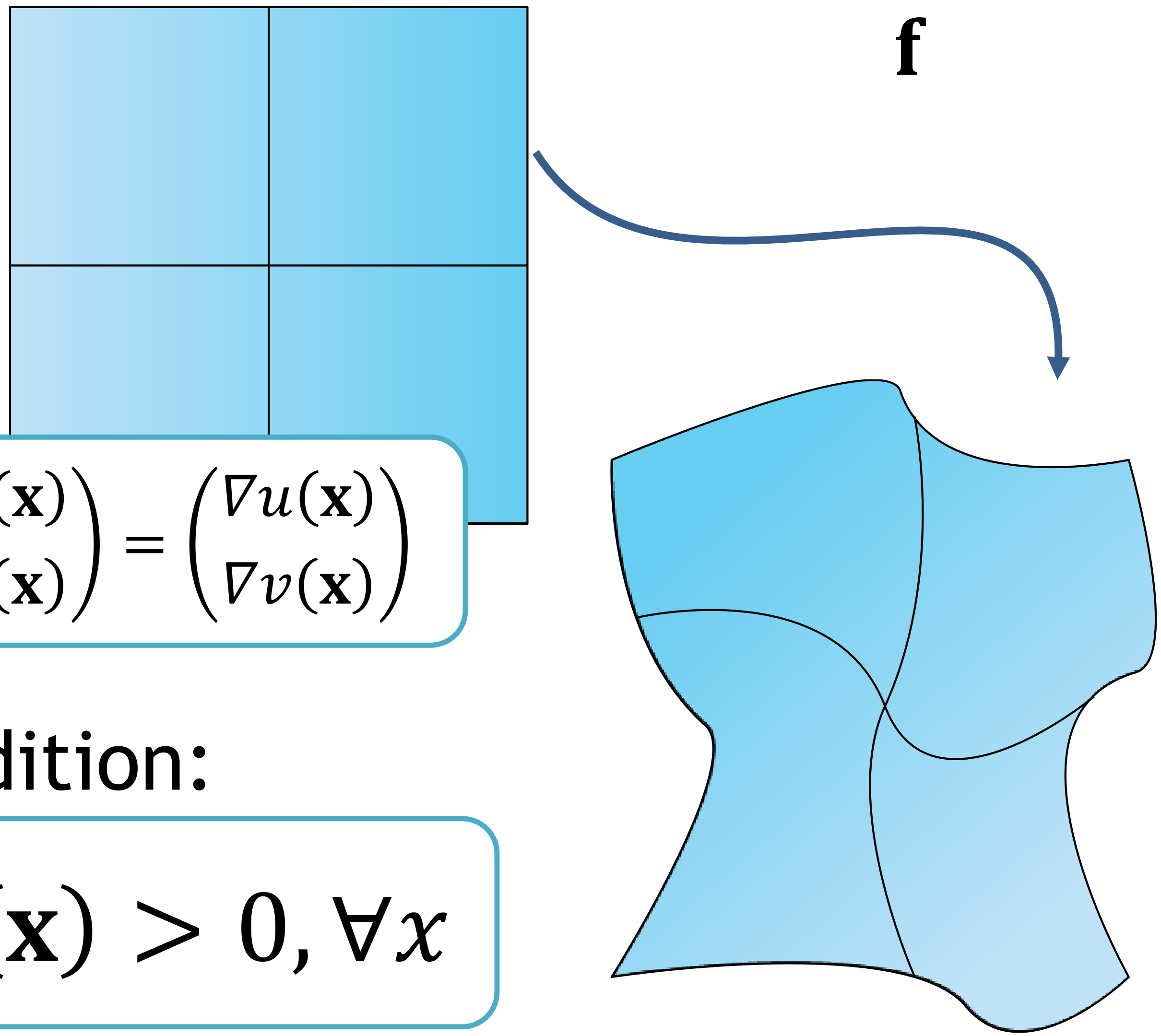


The Jacobian:

$$\mathcal{J}\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \partial_x u(\mathbf{x}) & \partial_y u(\mathbf{x}) \\ \partial_x v(\mathbf{x}) & \partial_y v(\mathbf{x}) \end{pmatrix}$$

Local Bijection Sufficient condition

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix}$$



The Jacobian:

$$\mathcal{J}\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \partial_x u(\mathbf{x}) & \partial_y u(\mathbf{x}) \\ \partial_x v(\mathbf{x}) & \partial_y v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \nabla u(\mathbf{x}) \\ \nabla v(\mathbf{x}) \end{pmatrix}$$

The Condition:

$$\det \mathcal{J}\mathbf{f}(\mathbf{x}) > 0, \forall x$$

Globally VS. Locally Bijective

Globally Bijective $\Rightarrow$ Locally Bijective

$\mathbf{f}$ is bijective

$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective

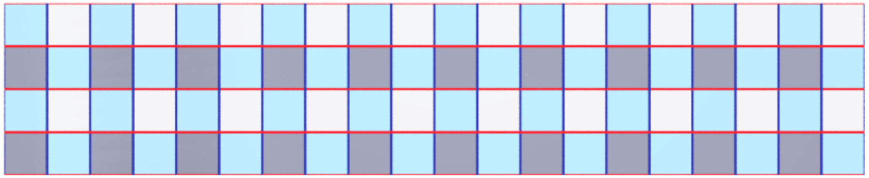
Globally VS. Locally Bijective

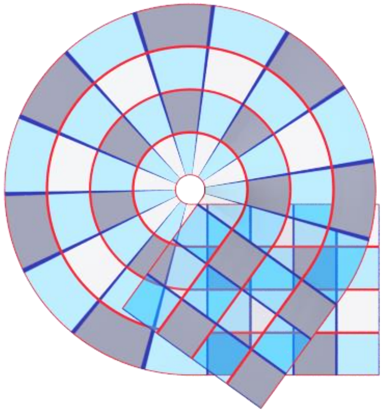
Globally Bijective

Locally Bijective

$\mathbf{f}$ is bijective

$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective





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Globally VS. Locally Bijective

Globally Bijective

Locally Bijective

$\mathbf{f}$ is bijective

$\mathbf{f}: \mathbf{U} \rightarrow \mathbf{f}(\mathbf{U})$ is bijective

“Global inversion theorems”

What are good maps?

Local

Bijectivity

Low distortion

Not Bijective

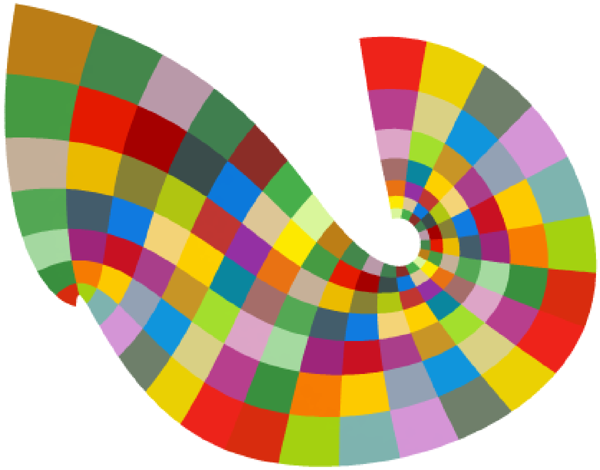
Bijective

Lower distortion

Distortion - Types

Conformal distortion

Isometric distortion

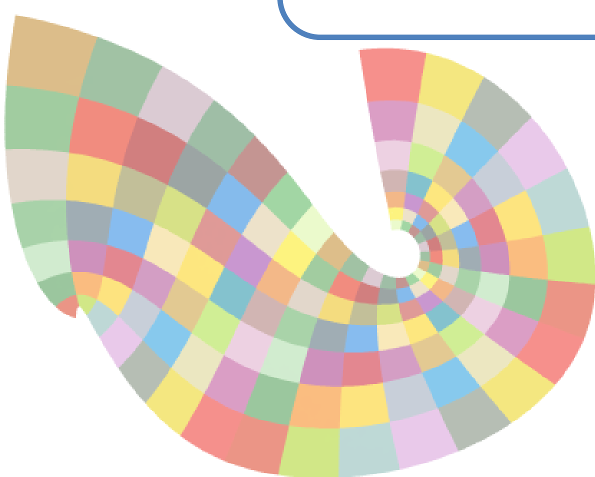


Distortion - Types

Conformal distortion

Isometric distortion

The distortion is a function of the Jacobian at a point



Distortion - LSCM [Lévy et al. 2002]

LSCM – **L**east **S**quares **C**onformal **M**ap

We want the Jacobian to be a similarity matrix

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \qquad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\begin{aligned} \partial_x u &= \partial_y v \\ \partial_y u &= -\partial_x v \end{aligned}$$

Cauchy-Riemann Equations

Distortion - LSCM

LSCM – **L**east **S**quares **C**onformal **M**ap

We want the Jacobian to be a similarity matrix

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \qquad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\mathcal{D}_{\mathrm{LSCM}} = (\partial_x u - \partial_y v)^2 + (\partial_y u + \partial_x v)^2$$

Quick Notation Change

$$\mathcal{J}\mathbf{f} \Longrightarrow \mathbf{A}$$

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \Longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{D}_{\mathrm{LSCM}} = (\partial_x u - \partial_y v)^2 + (\partial_y u + \partial_x v)^2$$

$$(a - d)^2 + (b + c)^2$$

Distortion – ASAP

[Liu et al. 2008]

ASAP– **As** Similar **As** Possible

$$\mathcal{D}_{\text{ASAP}} = \|\underbrace{\mathbf{A}}_{\text{Jacobian}} - \underbrace{\mathcal{S}_{\mathbf{A}}}_{\text{Closest Similarity}}\|_F^2$$

How to compute closest similarity?

ETH zürich

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

In 2D:

$$\min_{\mathcal{S}} \|\mathbf{A} - \mathcal{S}\|_F^2$$
$$\text{s.t.} \quad \mathcal{S} \text{ is similarity}$$

$$\min_{\alpha,\beta} \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \right\|_F^2$$

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

In 2D:

$$\min_{\mathcal{S}} \|\mathbf{A} - \mathcal{S}\|_F^2$$
$$\text{s.t.} \quad \mathcal{S} \text{ is similarity}$$

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} a+d & c-b \\ b-c & a+d \end{pmatrix}$$

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

In 2D:

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} a+d & c-b \\ b-c & a+d \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a-d & c+b \\ b+c & d-a \end{pmatrix}$$

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

How to compute closest similarity?

In 2D:

$$\mathbf{A} = \underbrace{\overbrace{\frac{1}{2}\begin{pmatrix} a+d & c-b \\ b-c & a+d \end{pmatrix}}^{\mathcal{S}_A}}_{\text{Similarity}} + \underbrace{\overbrace{\frac{1}{2}\begin{pmatrix} a-d & c+b \\ b+c & d-a \end{pmatrix}}^{\mathcal{S}_A^{\perp}}}_{\text{Anti-Similarity}}$$

Distortion – ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

$$\| \quad \mathcal{S}_A^{\perp} \quad \|_F^2$$

Measure of antisimilarity

Jacobian

Closest Similarity

$$\left\| \begin{pmatrix} a-d & c+b \\ b+c & d-a \end{pmatrix} \right\|_F^2$$

$$(a-d)^2 + (b+c)^2$$

Distortion – ASAP

ASAP- As Similar As Possible

$$\| \quad \mathcal{S}_A^{\perp} \quad \|_F^2$$

y

LSCM = ASAP

$$\left\| \begin{pmatrix} a-d & c+b \\ b+c & d-a \end{pmatrix} \right\|_F^2$$

$$(a-d)^2+(b+c)^2$$

Distortion - ARAP

ARAP– **As Rigid As Possible**

$$\mathcal{D}_{\text{ARAP}} = \|\mathbf{A} - \mathcal{R}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Rotation

How to compute closest rotation?

Singular Value Decomposition

Every Matrix M has a factorization of the form

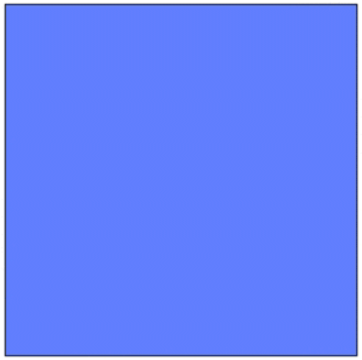
$$M \quad = \quad U \qquad\qquad S \qquad\qquad V^T$$

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad \sigma_1 > \sigma_2$$

Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad S \qquad\qquad V^T$$



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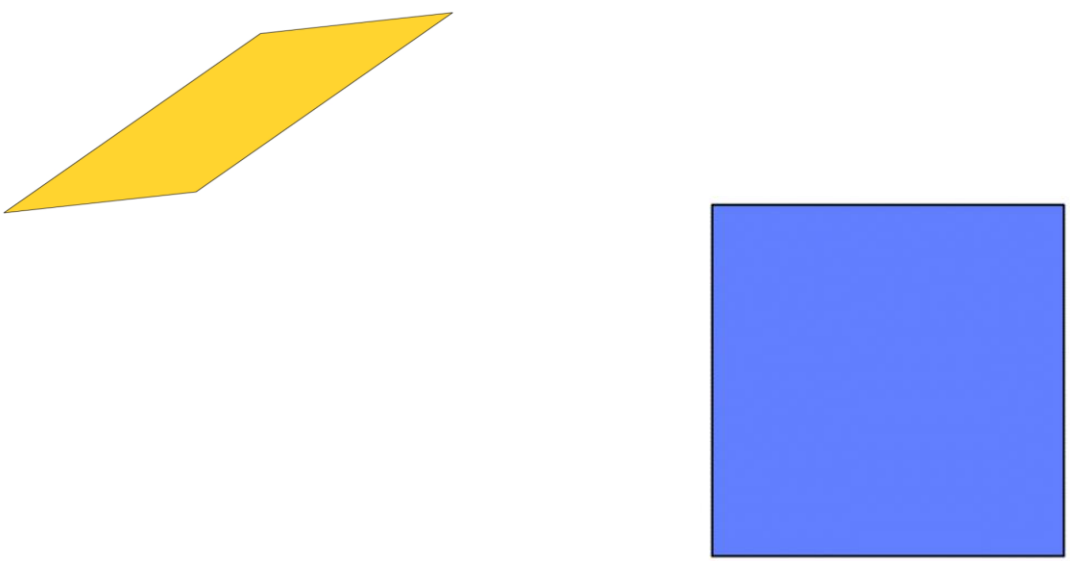

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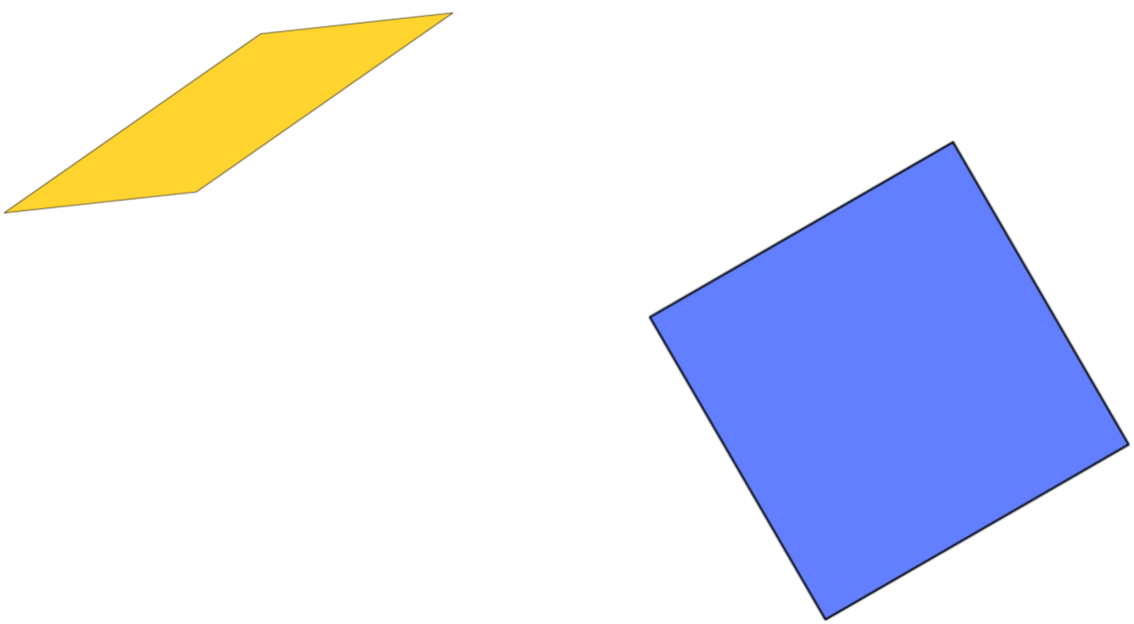

Singular Value Decomposition

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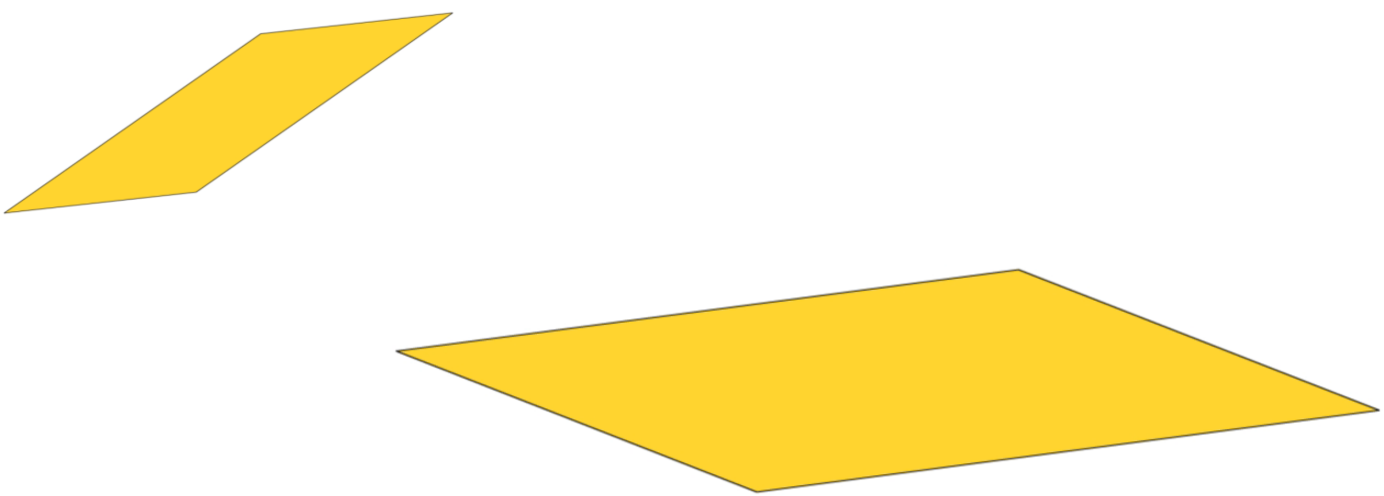


Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad S \qquad\qquad V^T$$

U and V are not rotations!

Singular Value Decomposition

Every Matrix M has a factorization of the form

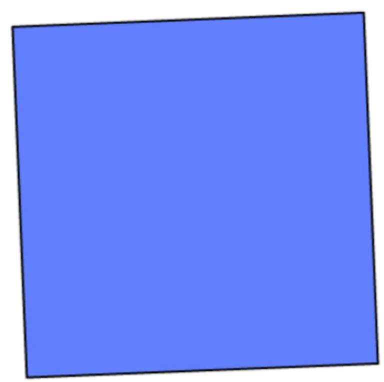
$$M \quad = \quad U \qquad\qquad S \qquad\qquad V^T$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Singular Value Decomposition

Every Matrix M has a factorization of the form

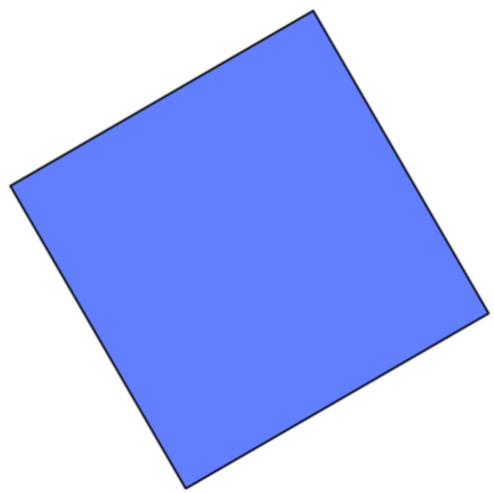
$$M \quad = \quad U \qquad\qquad S \qquad\qquad RV^T$$



Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad S \qquad\qquad RV^T$$



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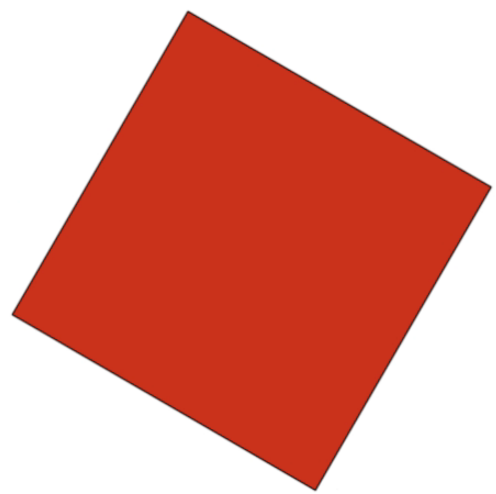
Singular Value Decomposition

Every Matrix M has a factorization of the form

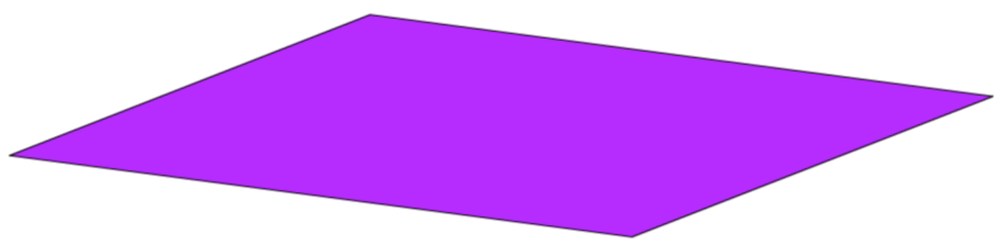
$$M \quad = \quad U \qquad\qquad S \qquad\quad RV^T$$



Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad S \qquad\qquad RV^T$$



Signed

Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M = U \quad S \quad RV^T$$

Signed Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M = U \quad SR \quad V^T$$

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \sigma_1 > \sigma_2$$

Now U and V are rotations!

Signed

Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M = U \quad SR \quad V^T$$

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \sigma_1 > \sigma_2$$

Now U and V are rotations!

What if U and V both had reflections?

Signed Singular Value Decomposition

Every Matrix M has a factorization of the form

$$M \quad = \quad U \qquad\qquad SR \qquad\qquad V^T$$

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \sigma_1 > \sigma_2$$

Now U and V are rotations!

What if U and V both had reflections?

$$\operatorname{sign}\det M = \operatorname{sign}(\sigma_2)$$

Distortion - ARAP

ARAP- **As Rigid As Possible**

$$\mathcal{D}_{\text{ARAP}} = \|\mathbf{A} - \mathcal{R}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Rotation

How to compute closest Rotation?

Distortion - ARAP

ARAP– **As Rigid As Possible**

$$\mathcal{D}_{\text{ARAP}} = \|\mathbf{A} - \mathcal{R}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Rotation

$$\mathbf{A} = USV^T \qquad \mathcal{R}_{\mathbf{A}} = UV^T$$

Proof: Using Lagrange multipliers

Distortion – ASAP

ASAP– **As** Similar **As** Possible

$$\mathcal{D}_{\text{ASAP}} = \|\mathbf{A} - \mathcal{S}_{\mathbf{A}}\|_F^2$$

Jacobian

Closest Similarity

$$\mathbf{A} = USV^T$$

$$\mathcal{S}_{\mathbf{A}} = \bar{\sigma}UV^T$$

$$\bar{\sigma} = \frac{\sigma_1 + \sigma_2}{2}$$

Distortion - ARAP

ARAP– **As Rigid As Possible**

$$\mathcal{D}_{\text{ARAP}} = \|\underbrace{\mathbf{A}}_{\text{Jacobian}} - \underbrace{\mathcal{R}_{\mathbf{A}}}_{\text{Closest Rotation}}\|_F^2$$

$$\mathbf{A} = USV^T \qquad \mathcal{R}_{\mathbf{A}} = UV^T$$

Proof: Using Lagrange multipliers

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Distortion - ARAP

ARAP– **A**s **R**igid **A**s **P**ossible

$$\mathcal{D}_{\text{ARAP}} = \|\mathbf{A} - \mathcal{R}_{\mathbf{A}}\|_F^2 = \|\mathbf{A} - UV^T\|_F^2$$

$$= \|USV^T - UV^T\|_F^2$$

$$= \|U(S - I)V^T\|_F^2$$

$$= \|(S - I)\|_F^2$$

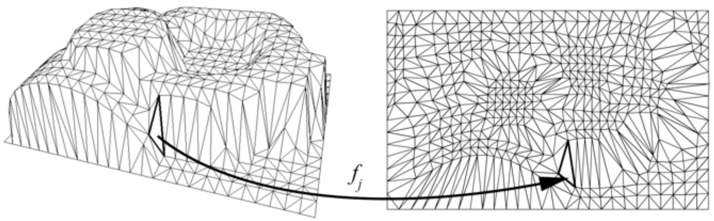
$$= (\sigma_1 - 1)^2 + (\sigma_2 - 1)^2$$

Distortion - ASAP

ASAP– **A**s **S**imilar **A**s **P**ossible

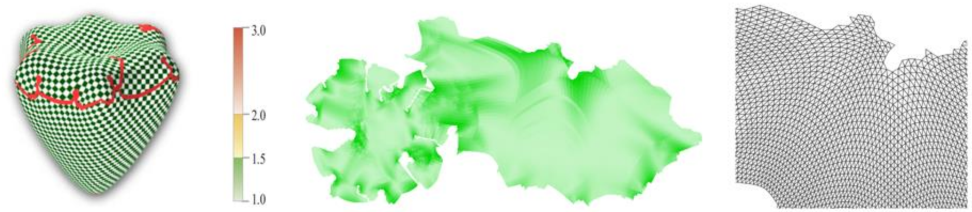
$$
\begin{aligned}
\mathcal{D}_{\mathrm{ASAP}} &= \|\mathbf{A} - \mathcal{S}_{\mathbf{A}}\|_F^2 \\
&= \|USV^T - \bar{\sigma}UV^T\|_F^2 \\
&= \|U(S - \bar{\sigma}I)V^T\|_F^2 \\
&= (\sigma_1 - \bar{\sigma})^2 + (\sigma_2 - \bar{\sigma})^2 \\
&= (\sigma_1 - \sigma_2)^2
\end{aligned}
$$

$$\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$



MIPS [Hormann & Greiner 2000]

$$\max\left\{\sigma_1, \frac{1}{\sigma_2}\right\}$$

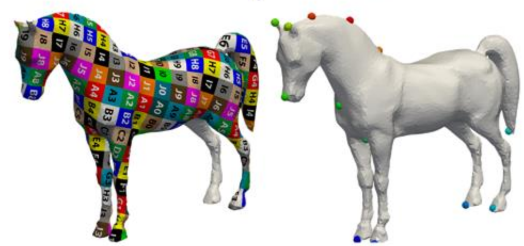


[Sorkine et al. 2000]

$$\sigma_1^2 + \sigma_2^2 + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

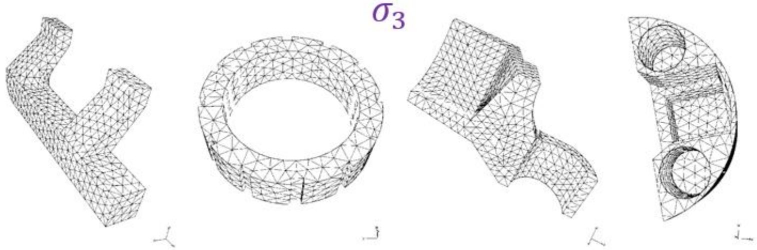
[Schreiner et al. 2014]

$$\sqrt{\sigma_1^2 + \frac{1}{\sigma_2^2}}$$



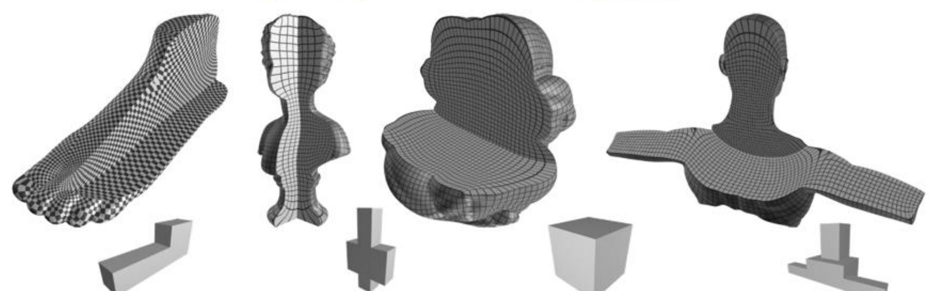
[Aigerman et al. 2014]

$$\frac{\sigma_1}{\sigma_3}$$



[Freitag & Knupp 2002]

$$\frac{\sigma_1}{\sigma_3} + \frac{\sigma_3}{\sigma_1}; \quad \sigma_1\sigma_2\sigma_3 + \frac{1}{\sigma_1\sigma_2\sigma_3}$$



[Paillé & Poulin 2012]

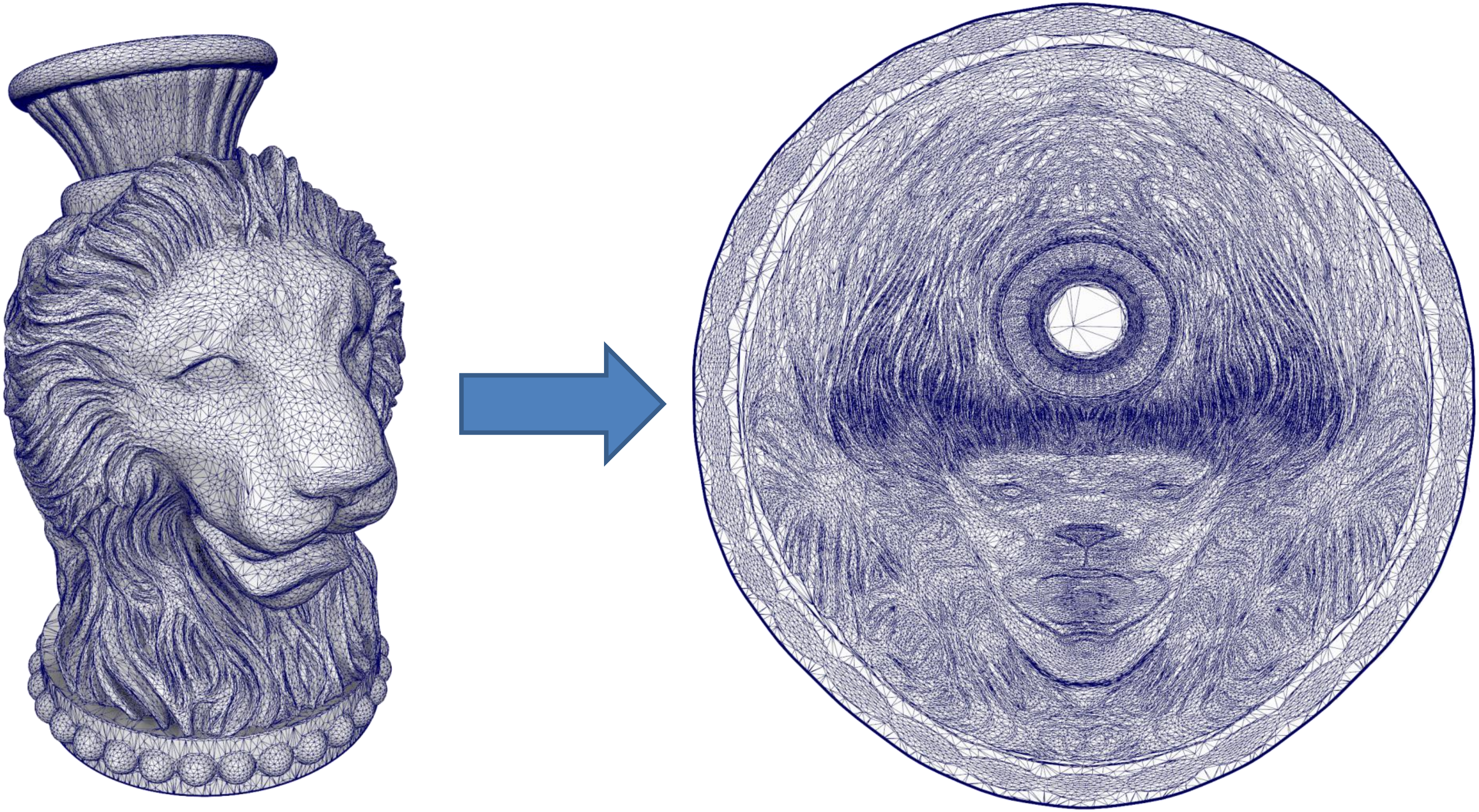
252-0538-00L, Spring 2018

Shape Modeling and Geometry Processing

Mesh Parameterization

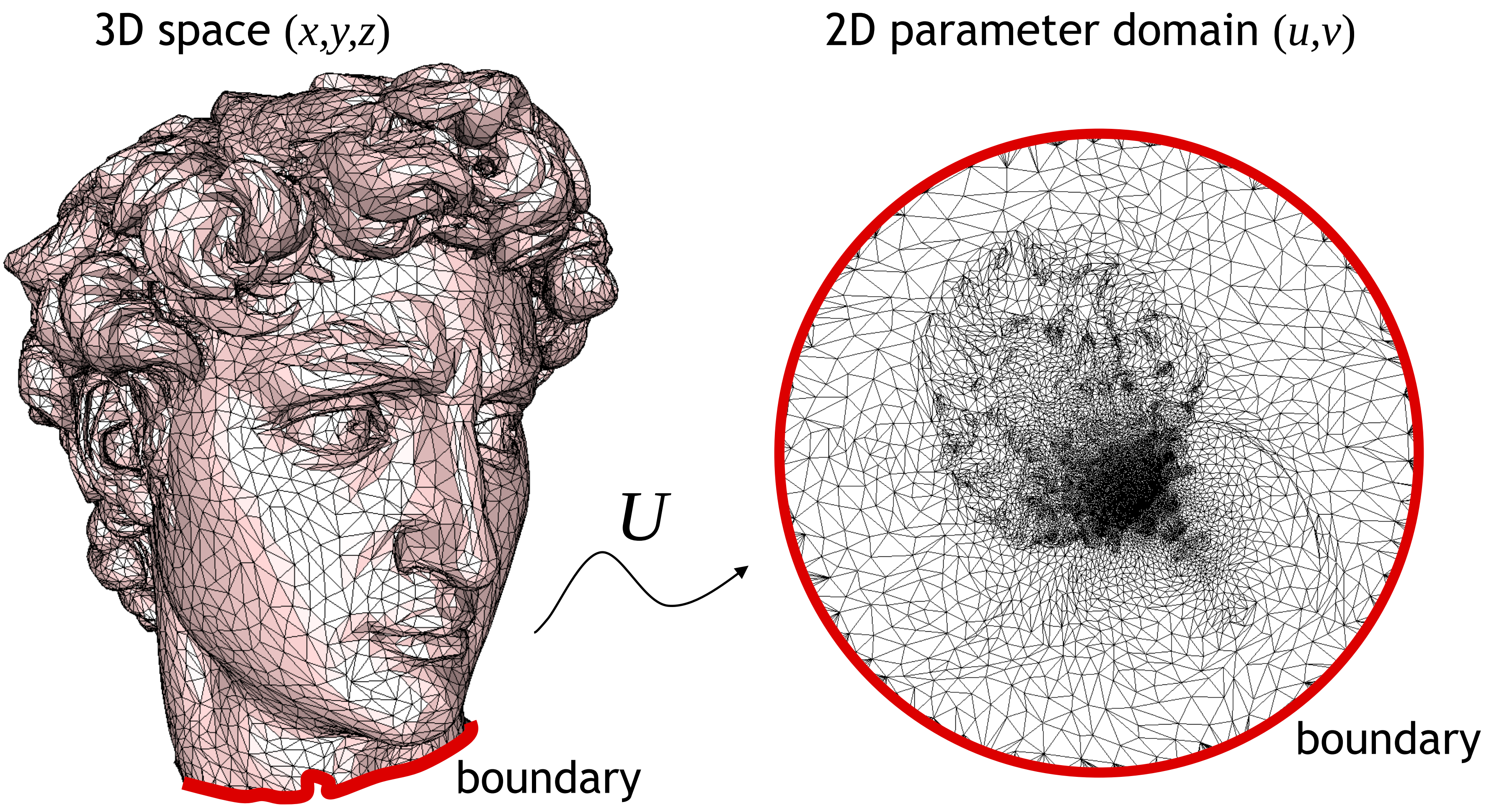
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

What Is a Parameterization?



What Is a Parameterization?

What Is a Parameterization?



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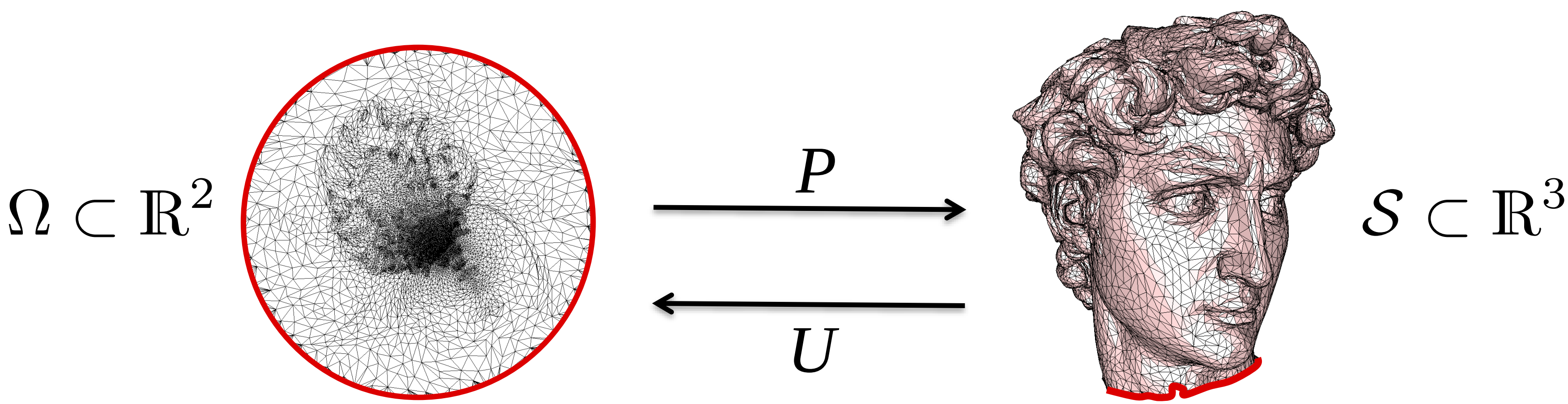
Parameterization – Definition

- Mapping P between a 2D domain Ω and the mesh S embedded in 3D (the inverse = flattening)
- Each mesh vertex has a corresponding 2D position:
$$U(\mathbf{v}_i) = (u_i, v_i)$$
- Inside each triangle, the mapping is affine (barycentric coordinates)

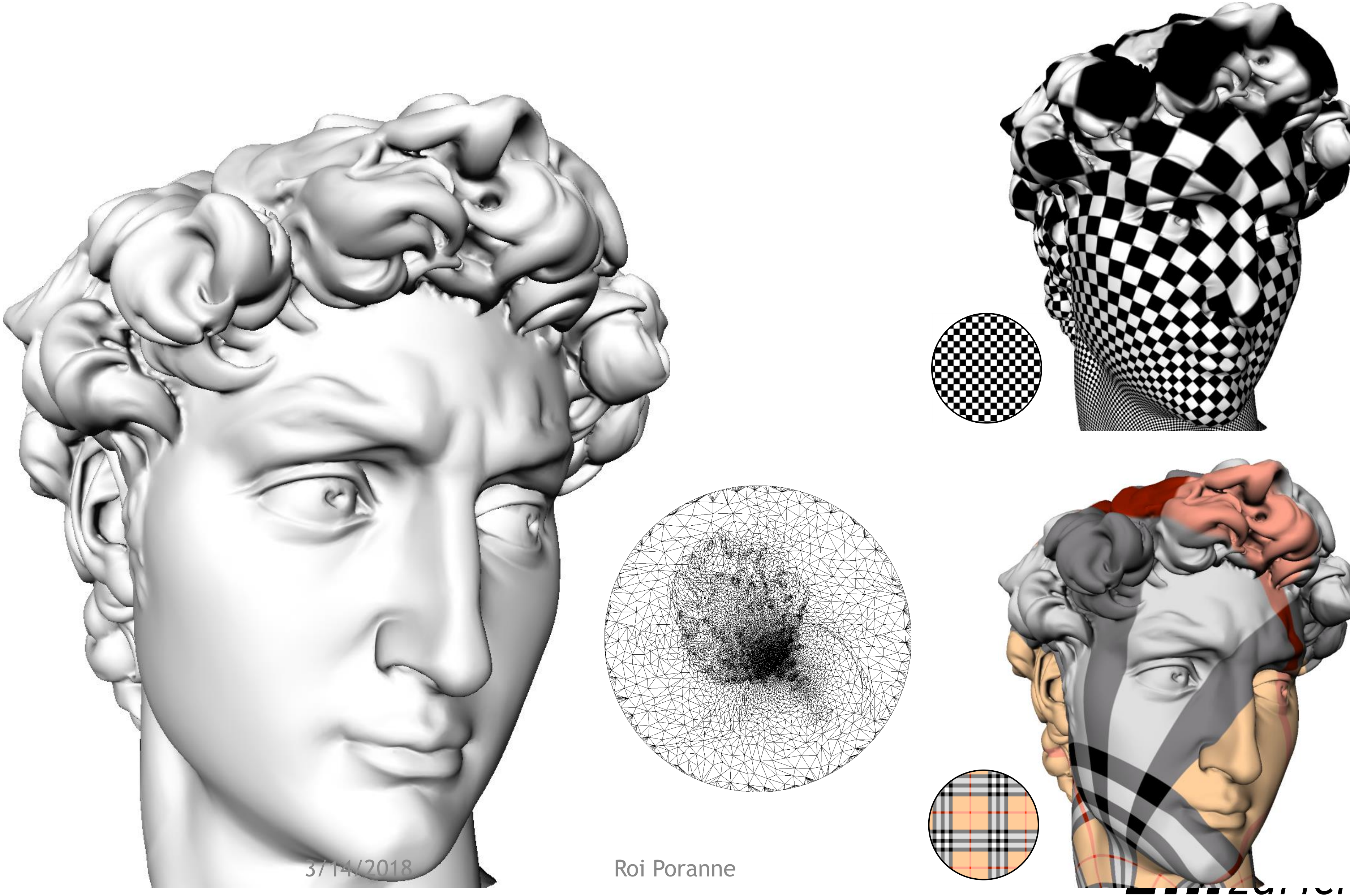


Why Parameterization?

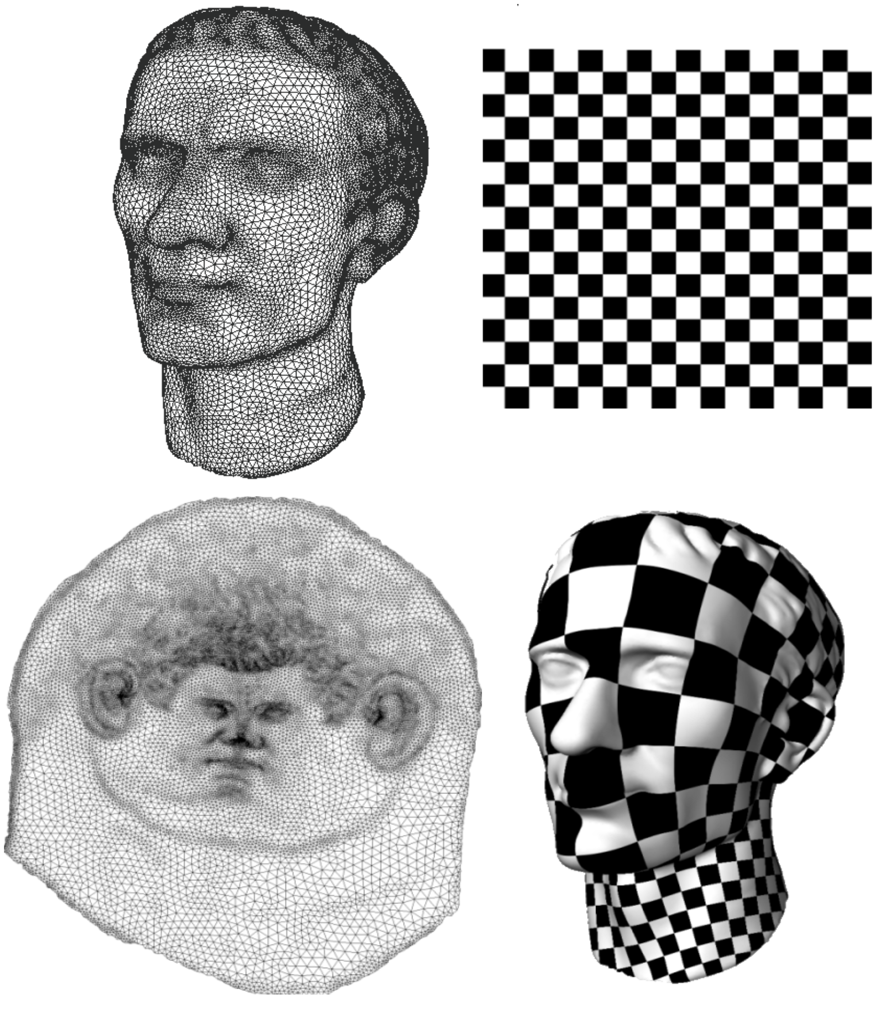
- Allows us to do many things in 2D and then map those actions onto the 3D surface
- It is often easier to operate in the 2D domain

- Mesh parameterization allows to use some notions from continuous surface theory

Main Application: Texture Mapping



Main Application: Texture Mapping



Texture Mapping

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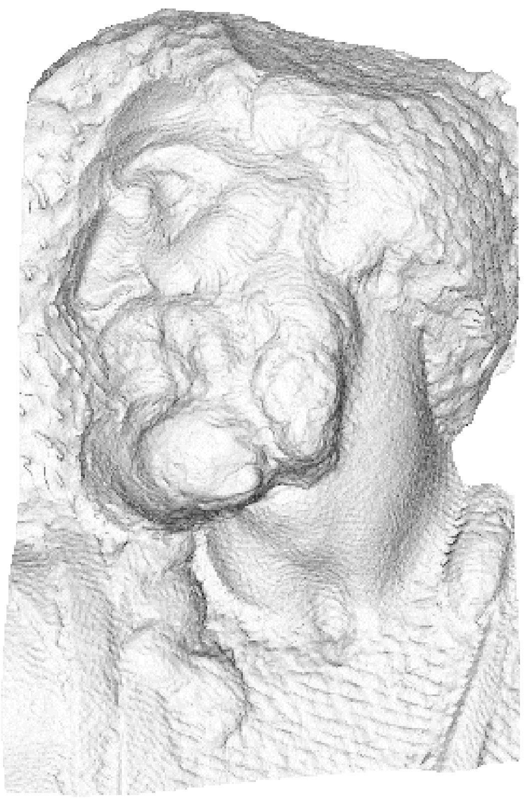
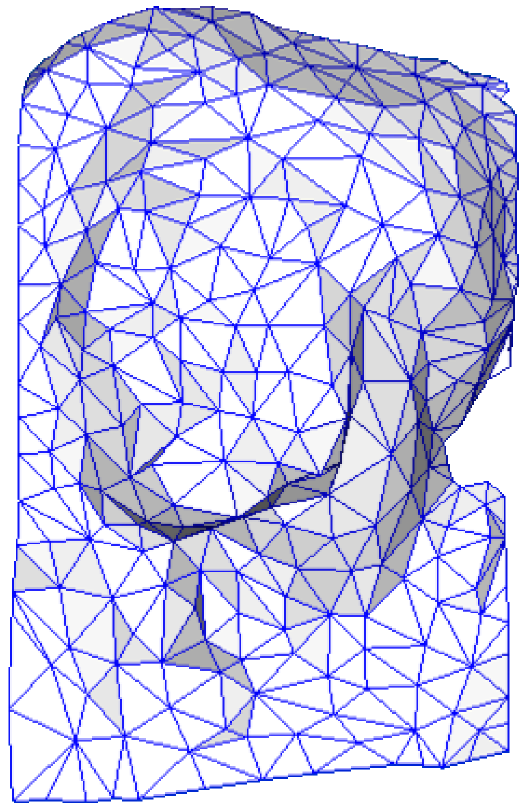
Texture Mapping

Image from Vallet and Levy, techreport INRIA

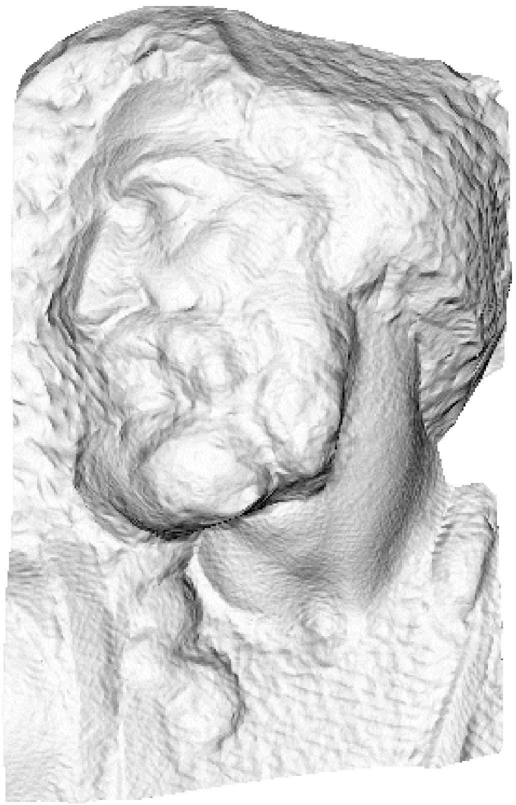
Normal/Bump Mapping



original mesh
4M triangles

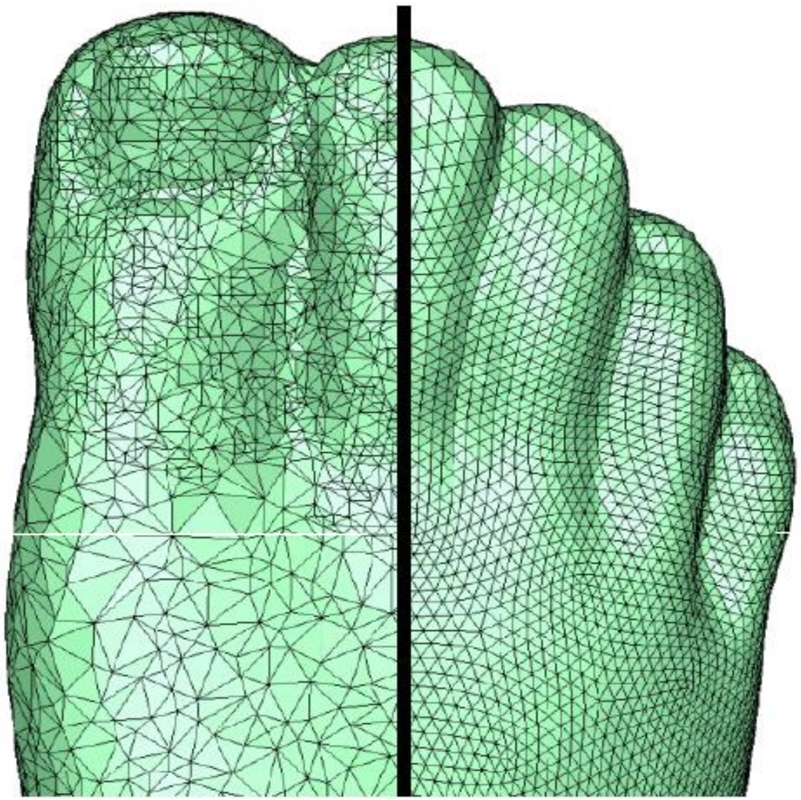


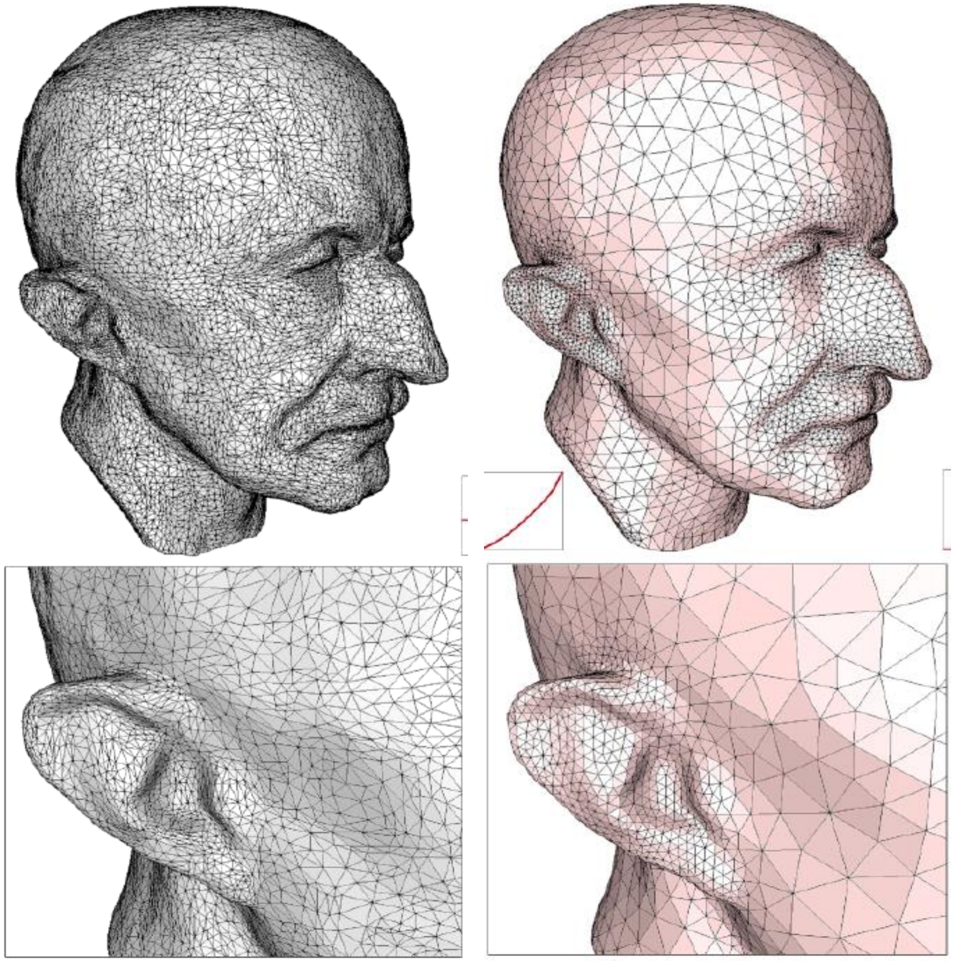
simplified mesh
500 triangles



simplified mesh
and normal mapping
500 triangles

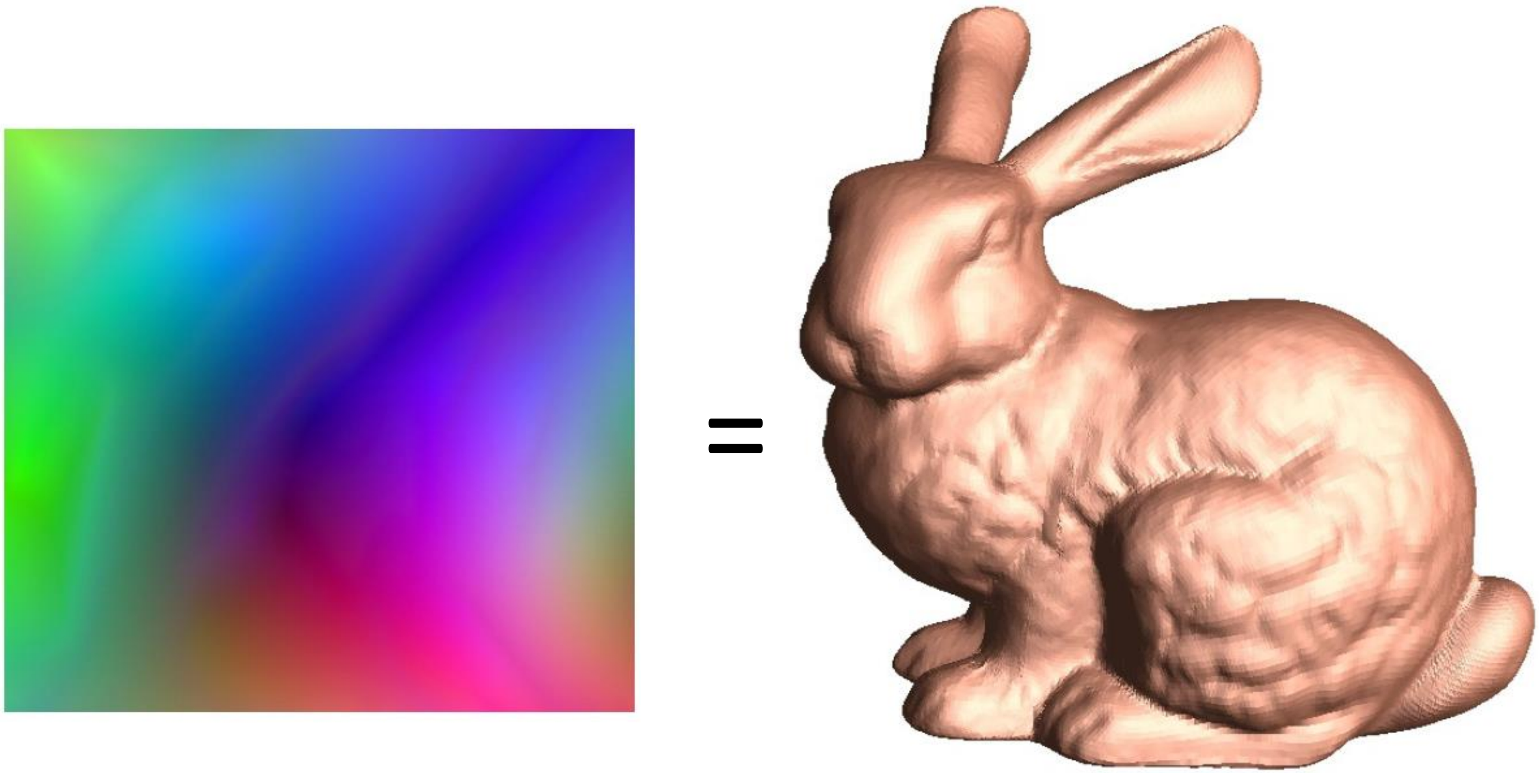
Remeshing





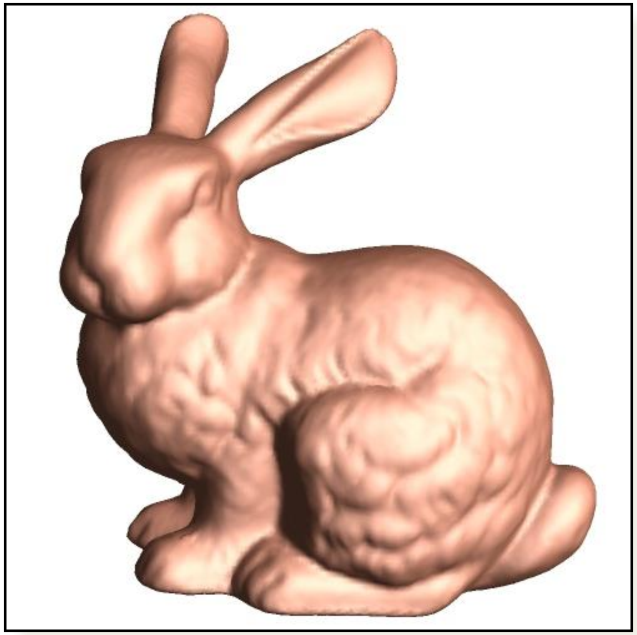
“Interactive Geometry Remeshing”, Alliez et al., SIGGRAPH 2002

Compression

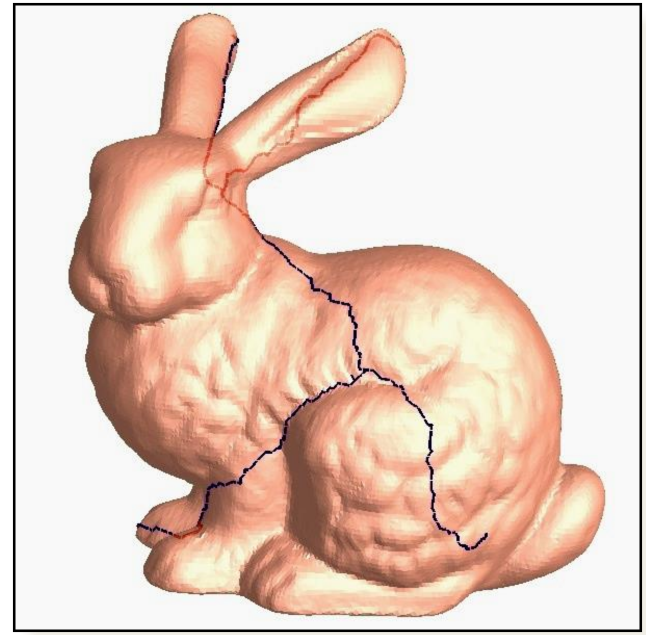


“Geometry images”, Gu et al., SIGGRAPH 2002
http://research.microsoft.com/en-us/um/people/hoppe/proj/gim/

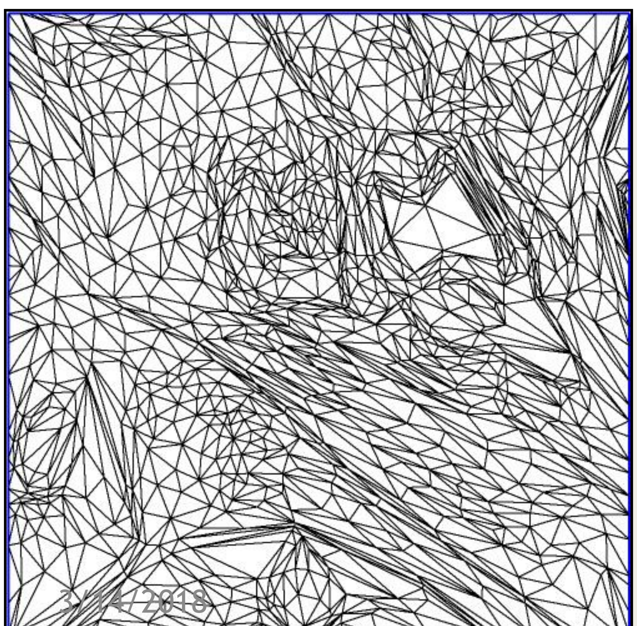
Geometry Images



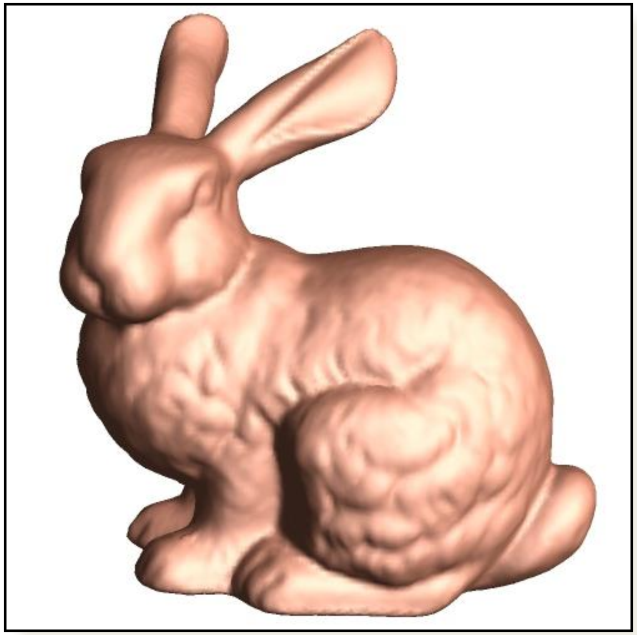
cut

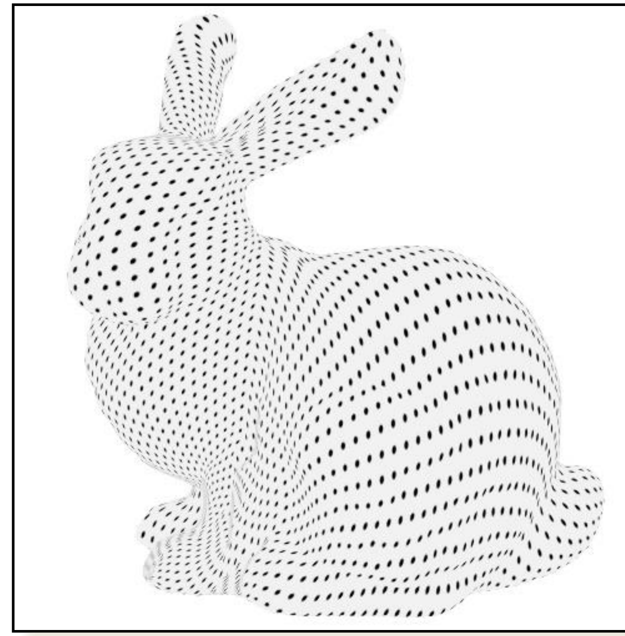


parametrize

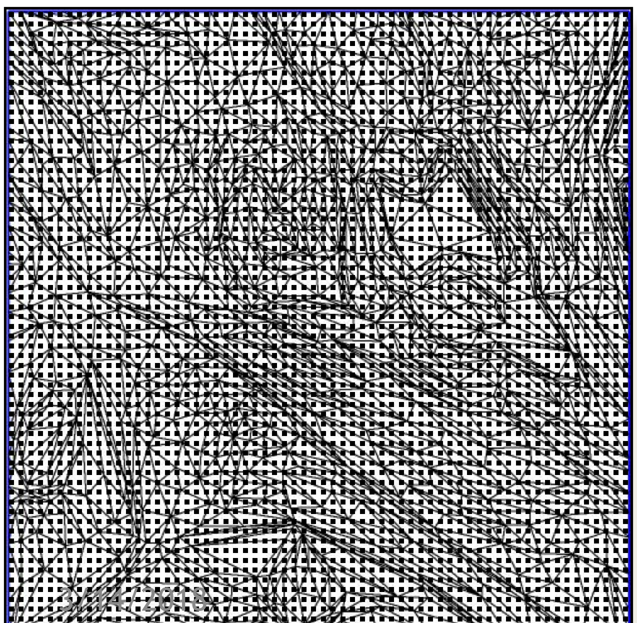


Geometry Images



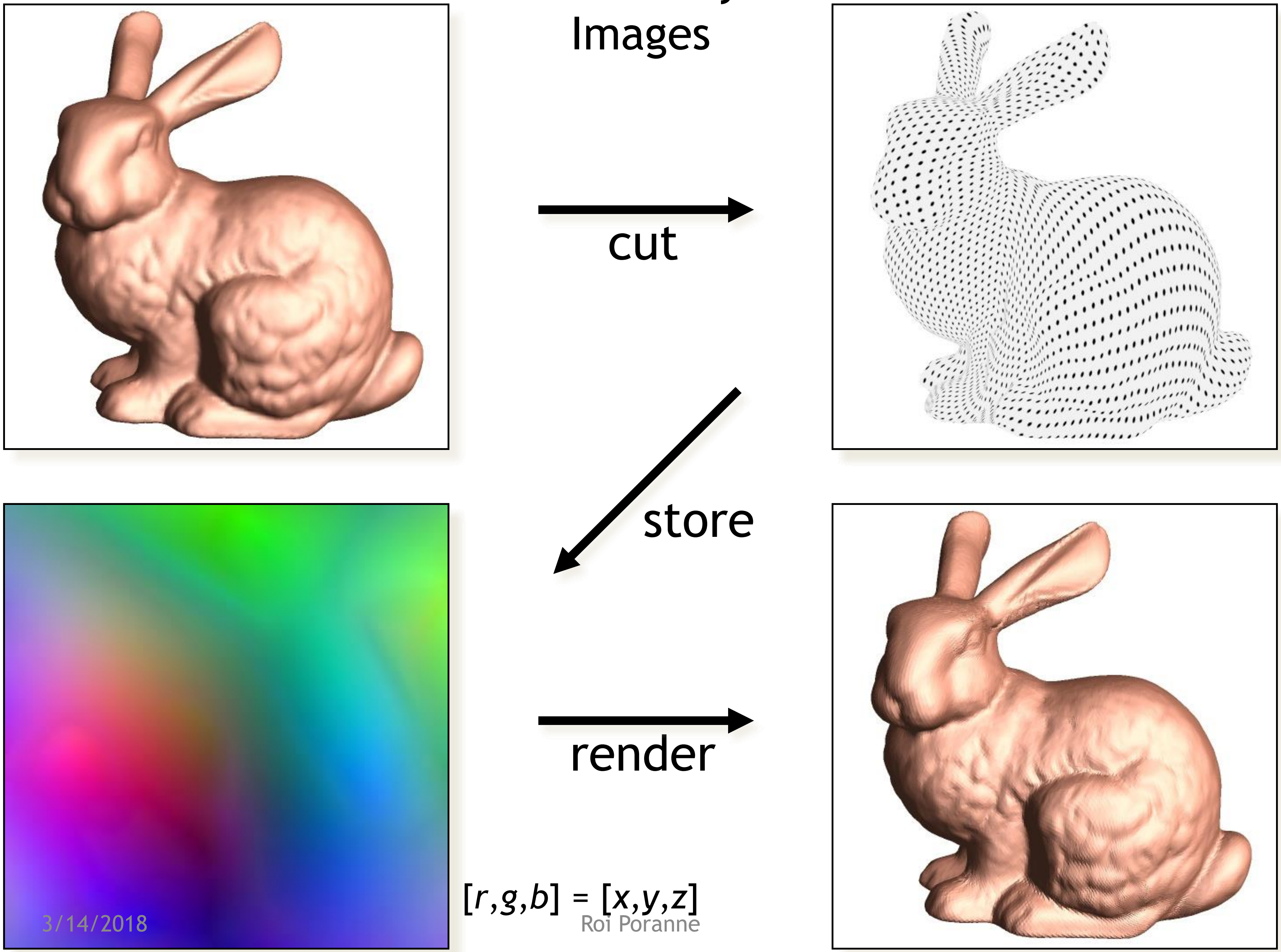


cut

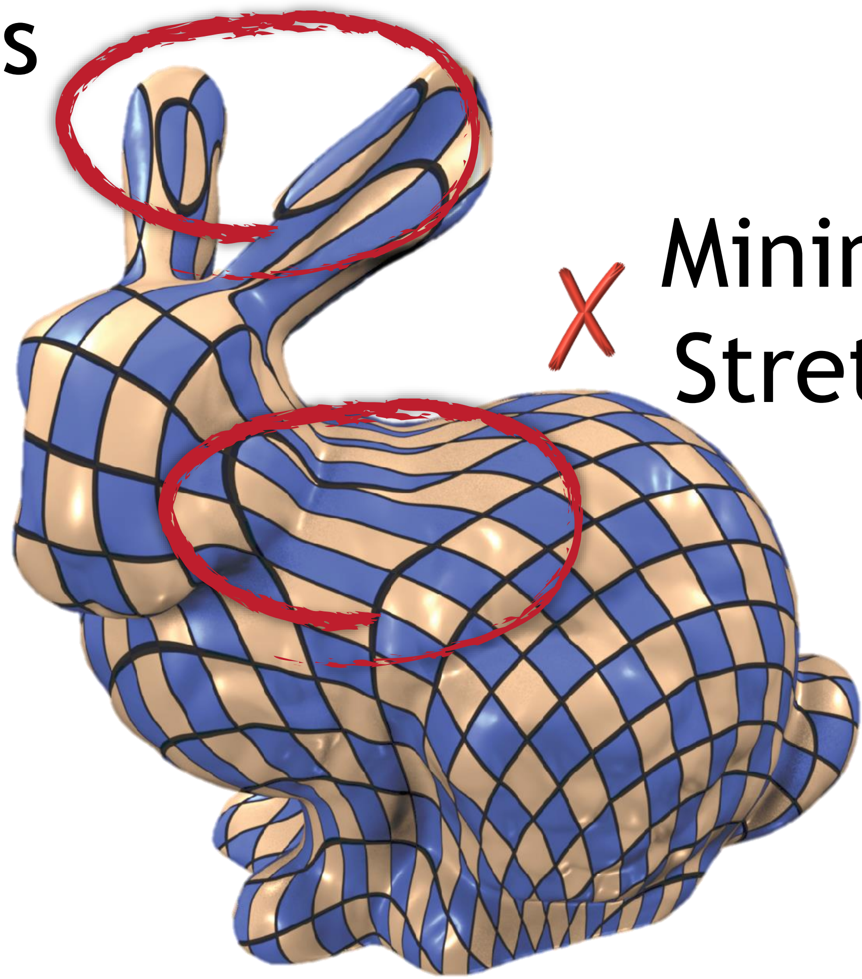


sample

Geometry Images



Good Parameterization



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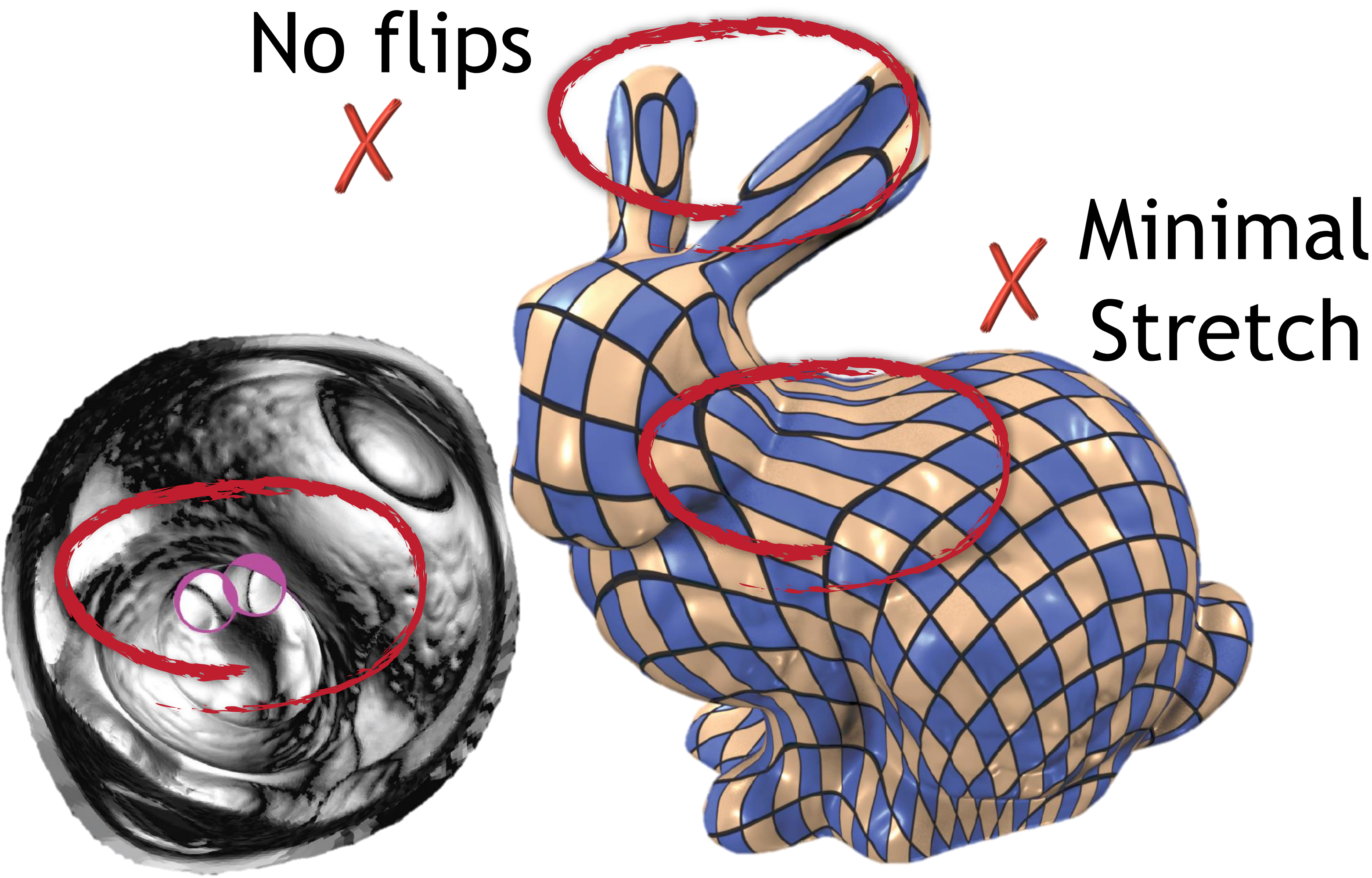
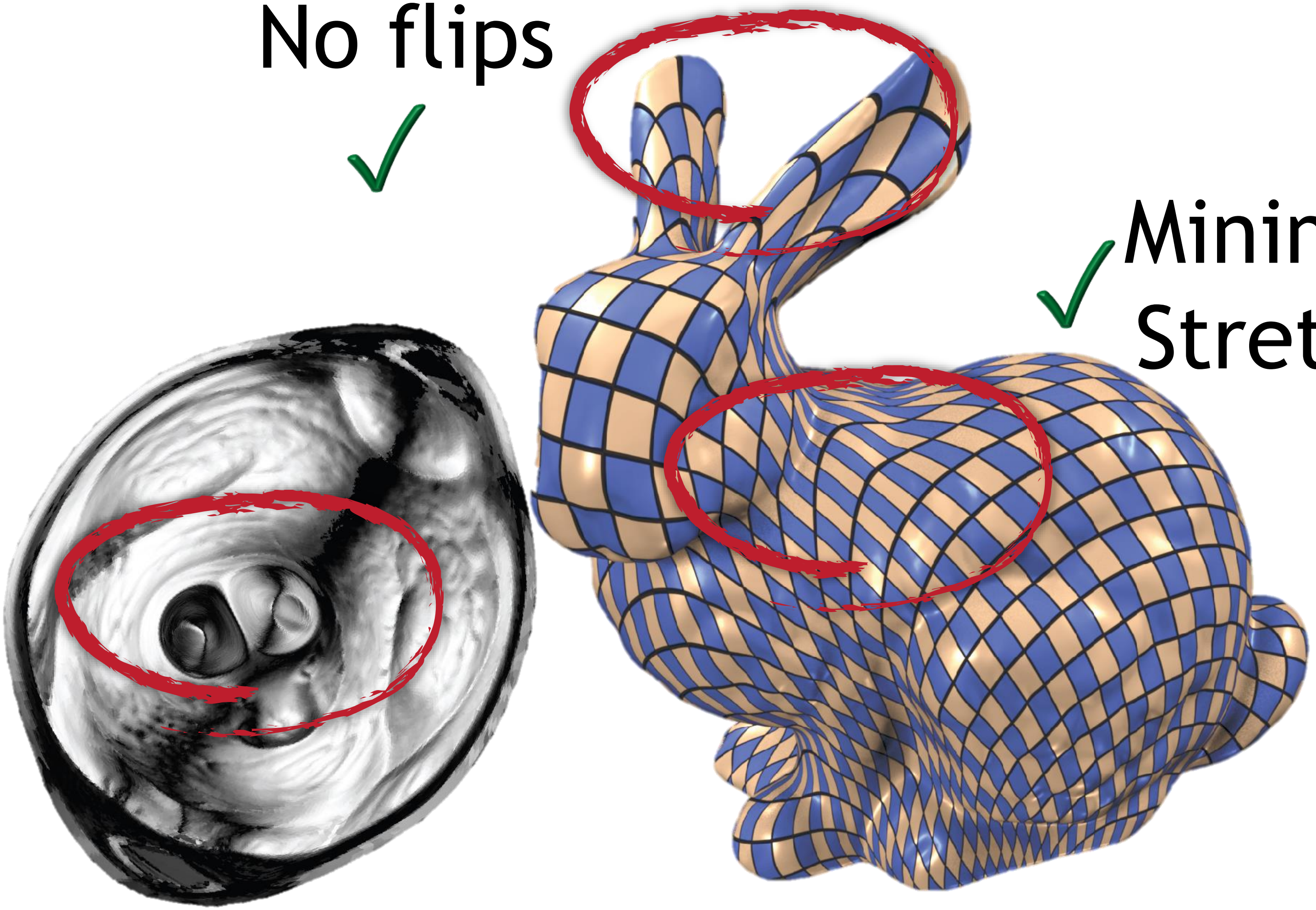
Good Parameterization



Good Parameterization

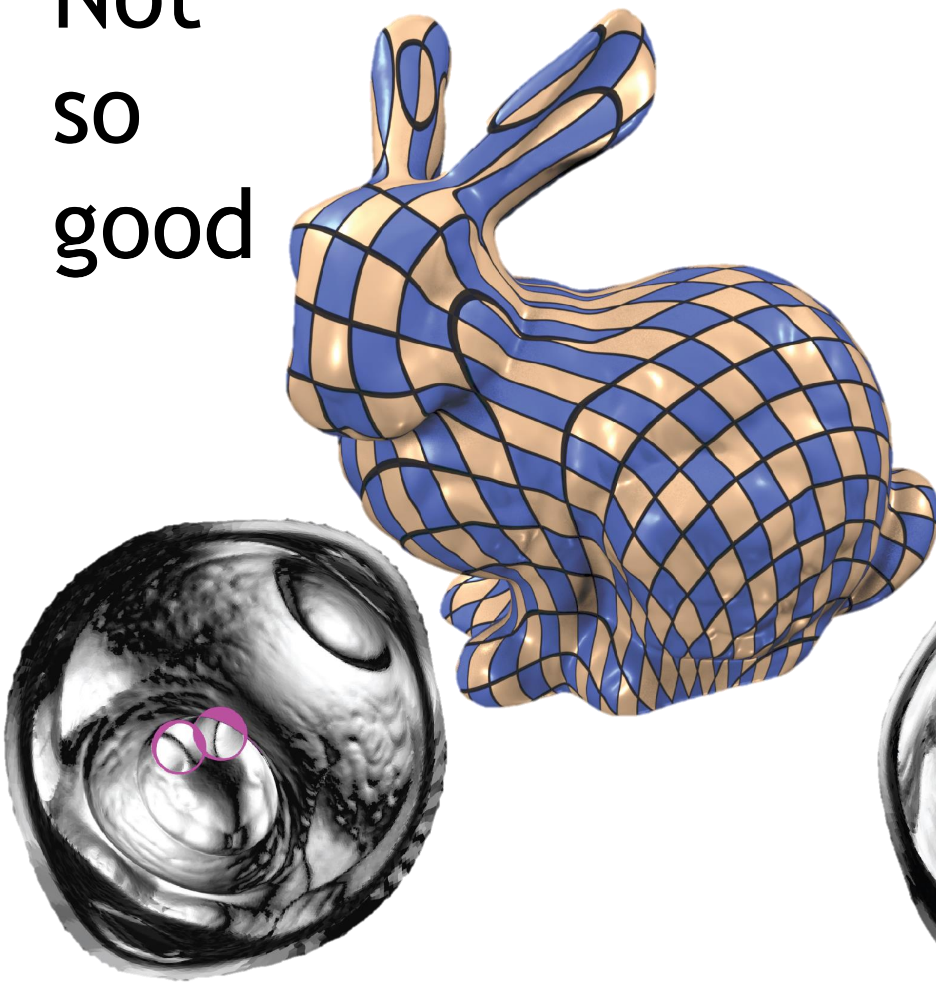
No flips ✓

✓ Minimal Stretch

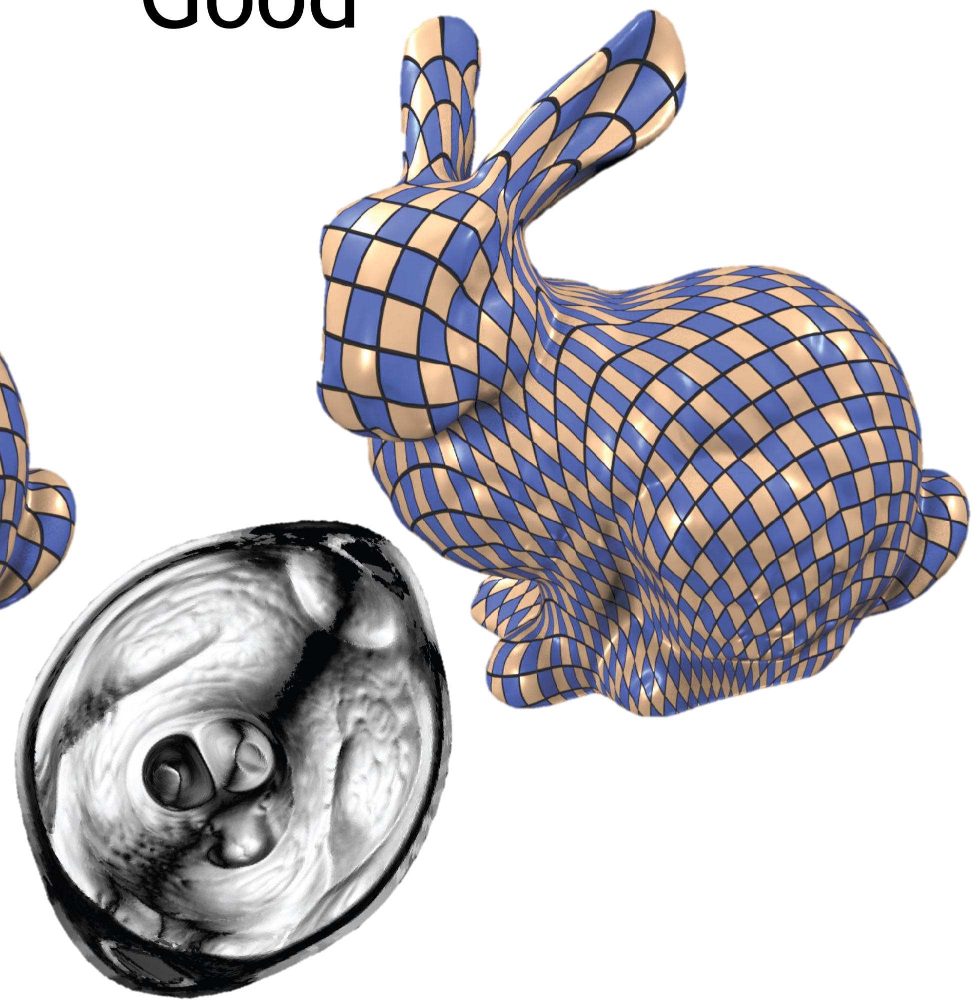


Good Parameterization

Not so good

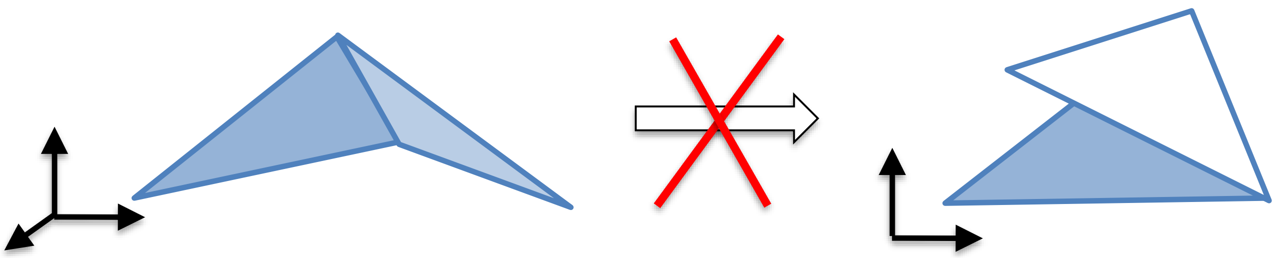


Good



Bijectivity

- Locally bijective (1-1 and onto): No triangles fold over.



- Globally bijective: locally bijective + no “distant” areas overlap

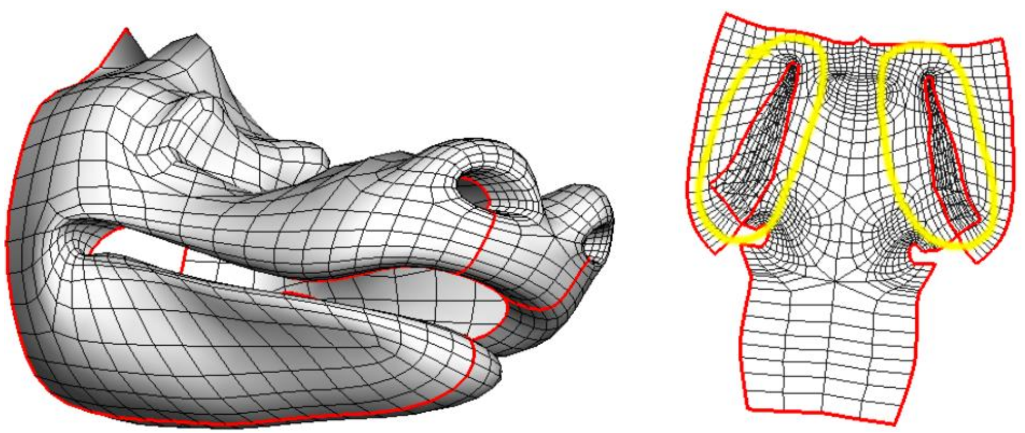
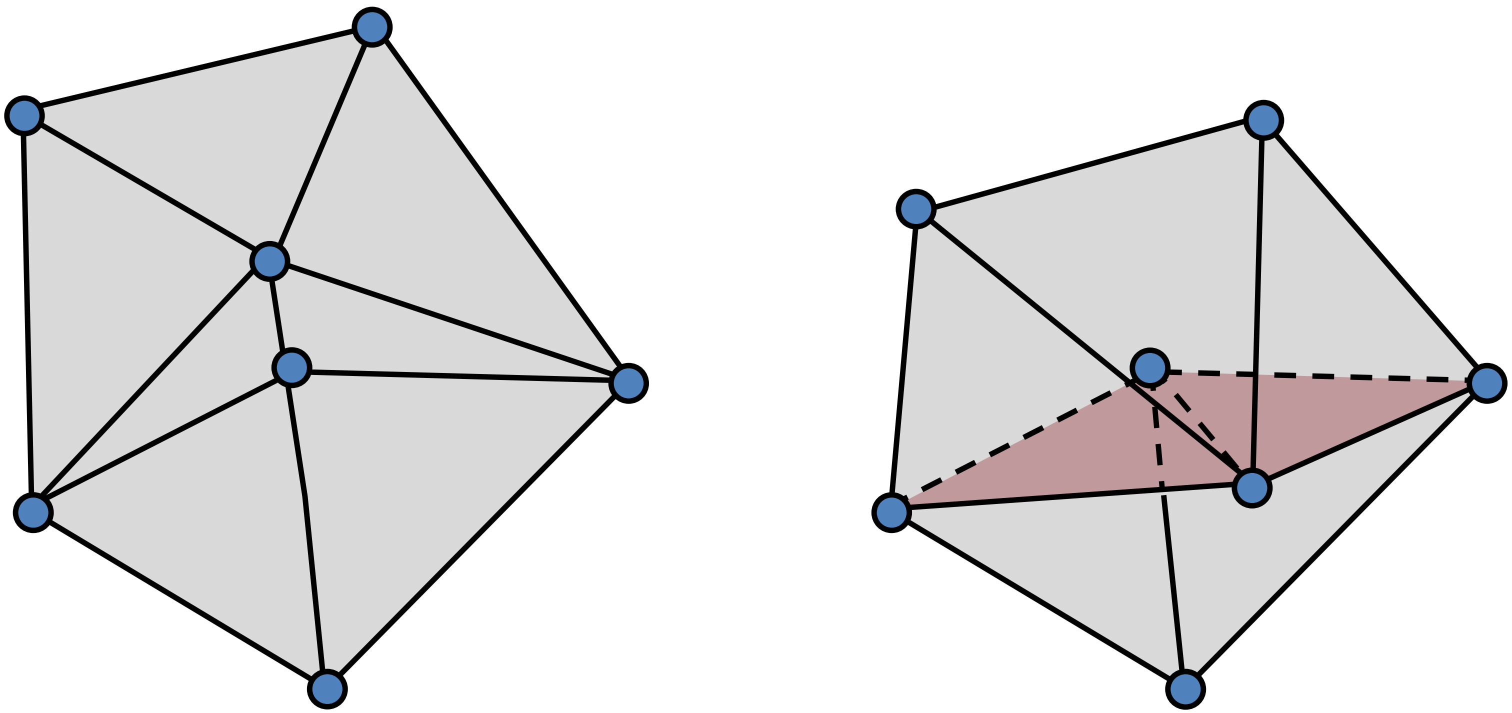
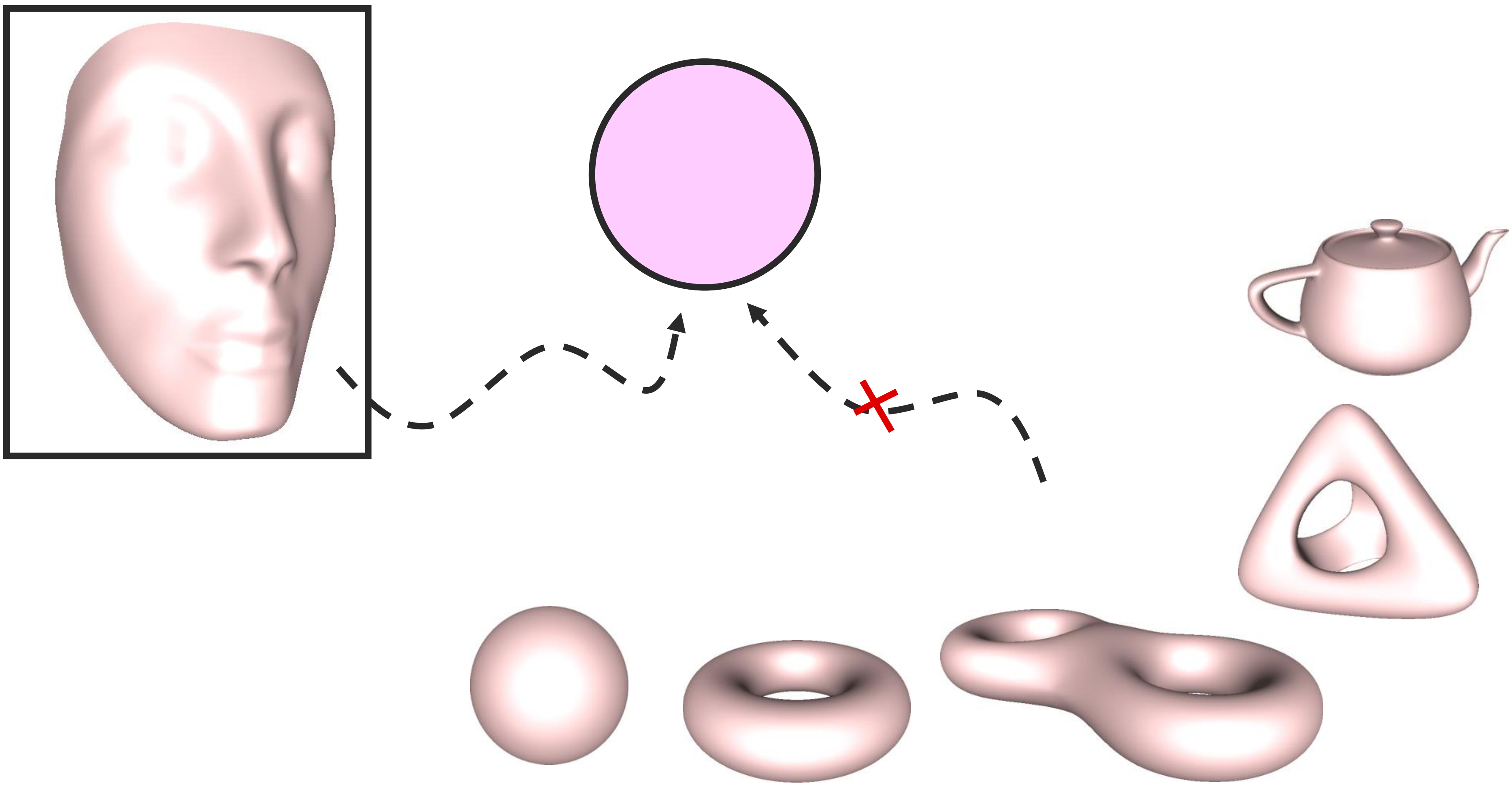


image from “Least Squares Conformal Maps”, Lévy et al., SIGGRAPH 2002

Local Foldovers



Bijectivity: Non-Disk Domains



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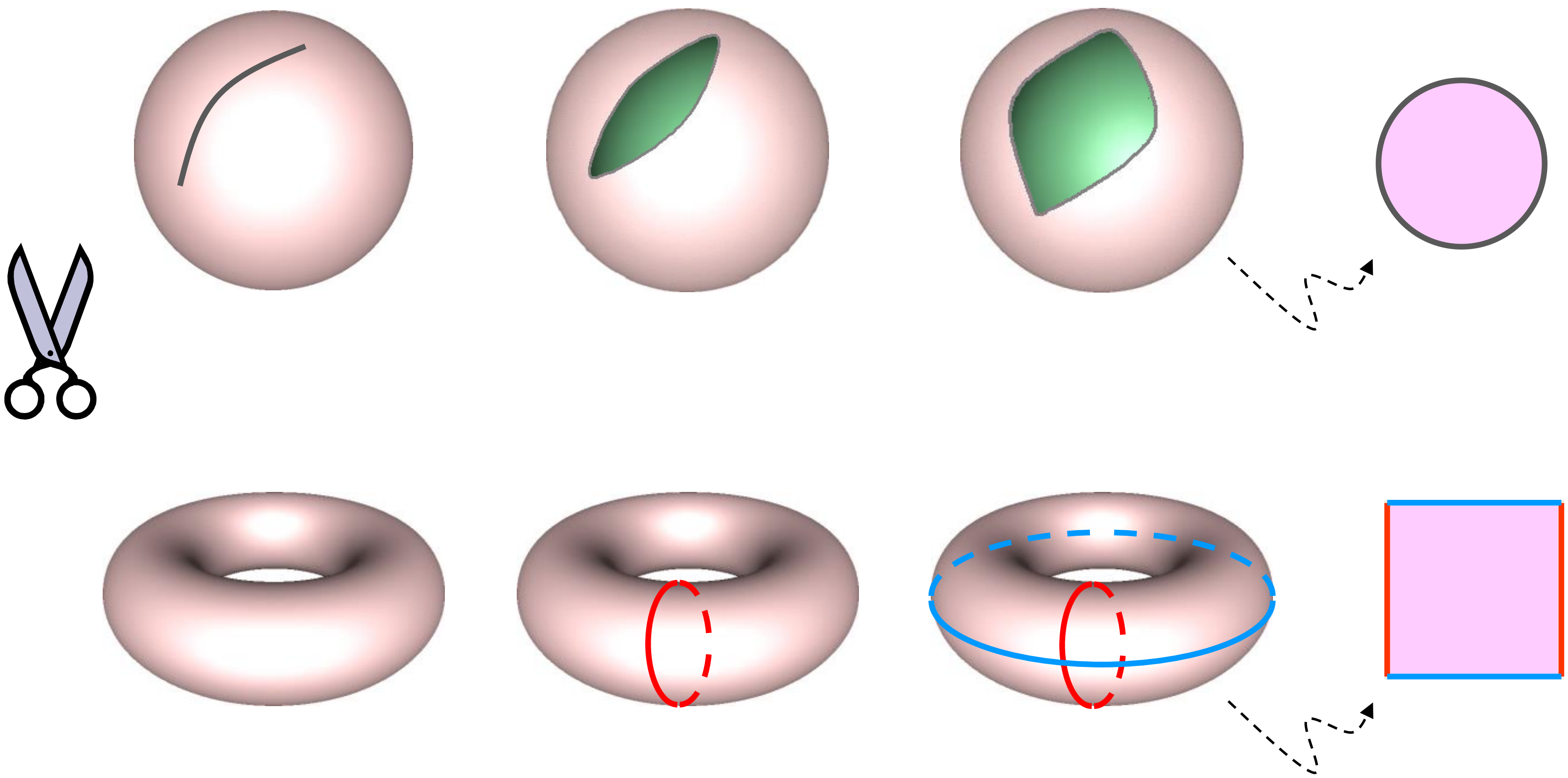
Mesh Cutting



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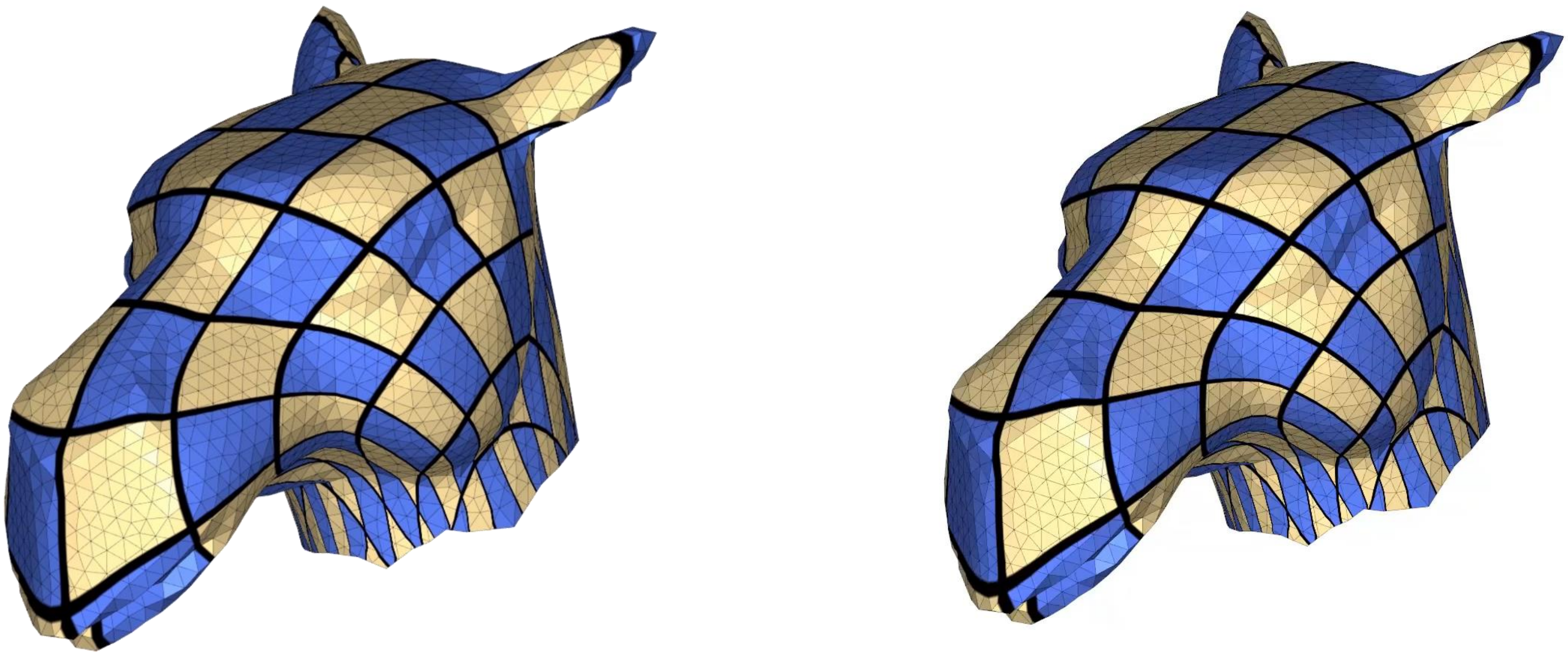
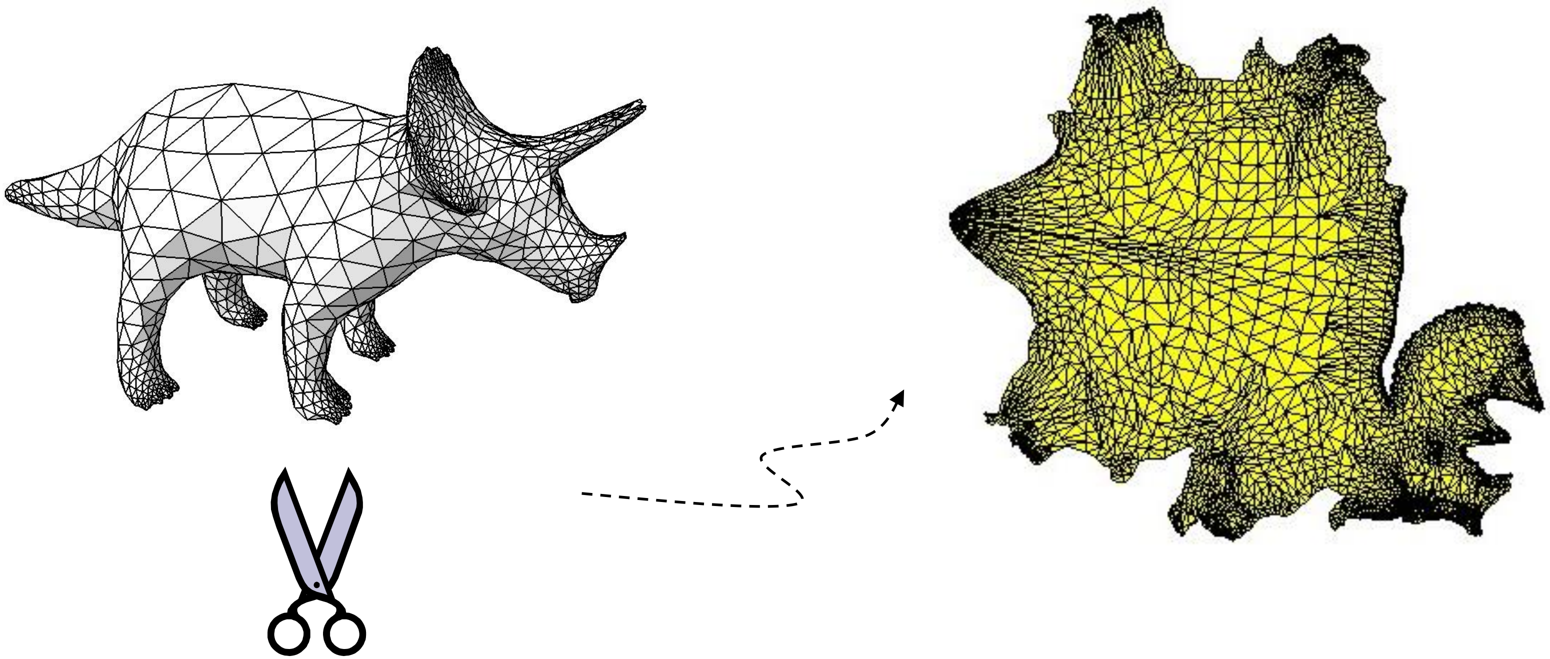
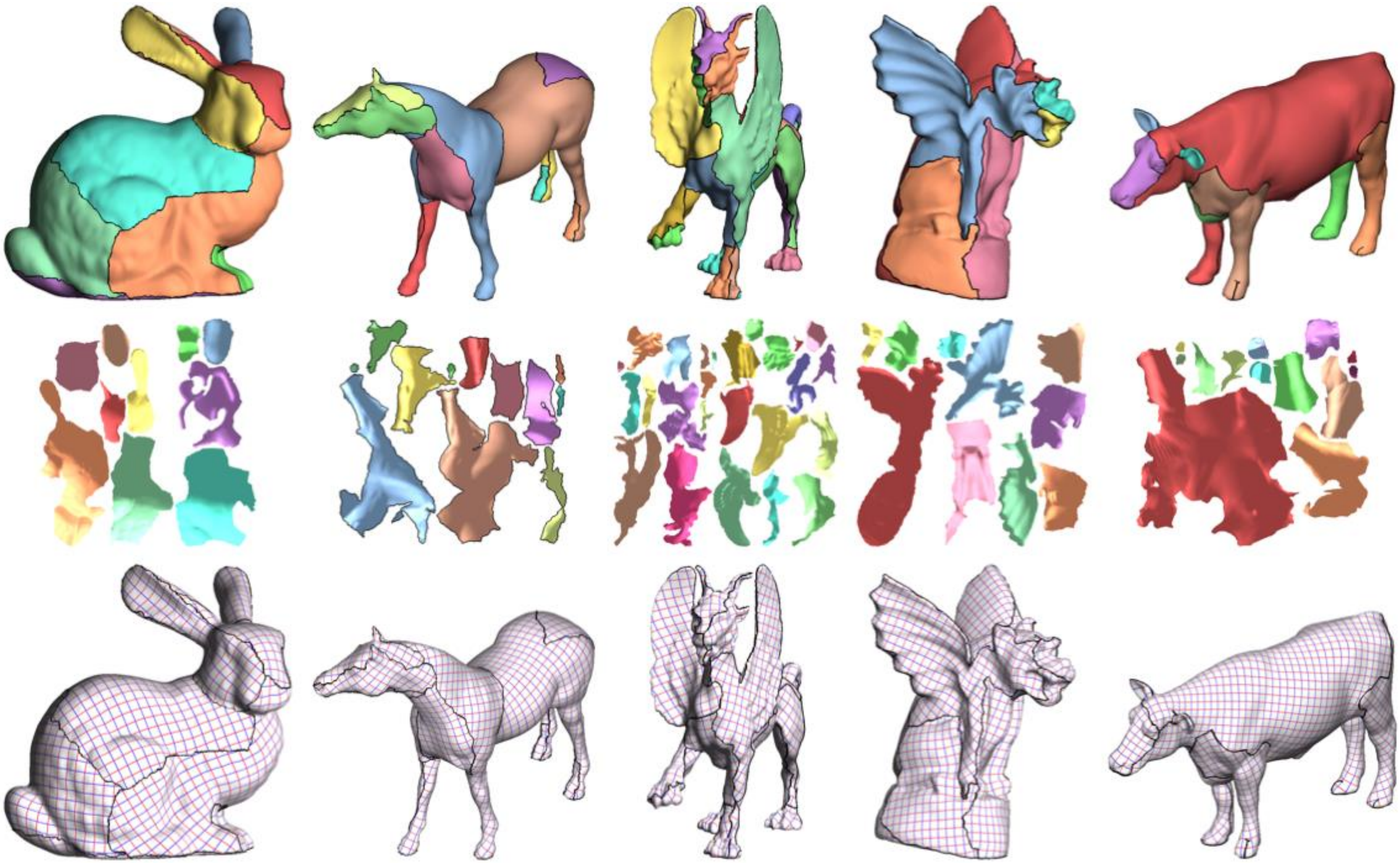


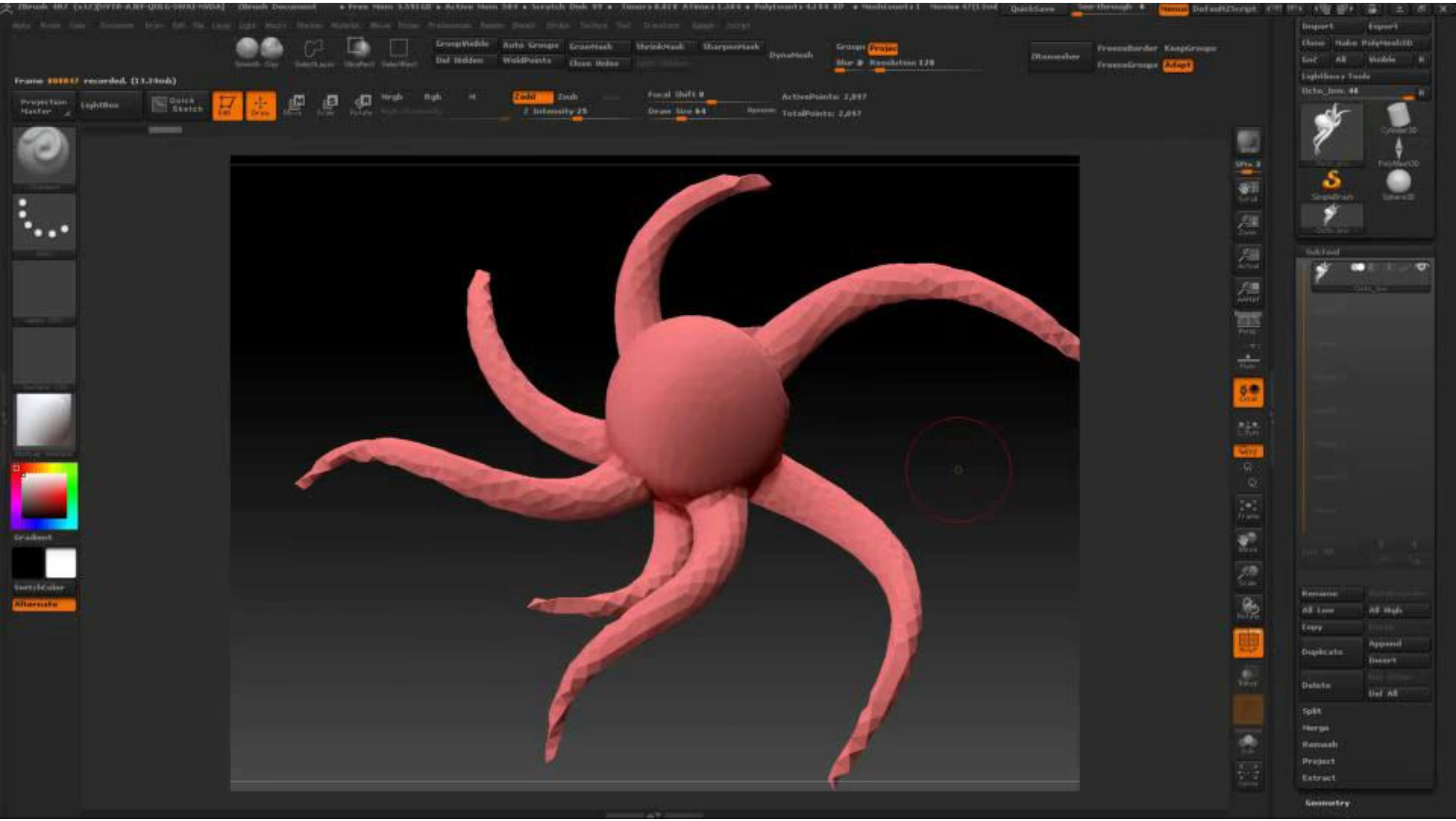
Mesh Cutting



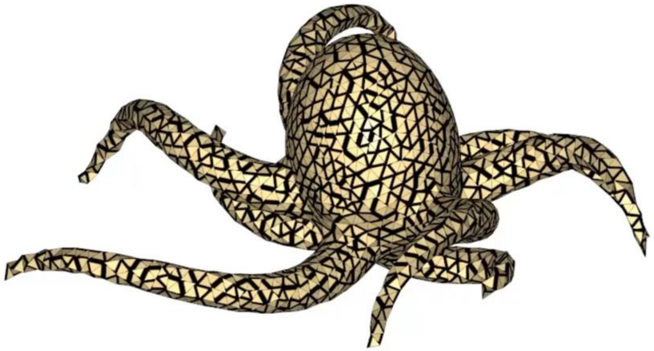
A. Sheffer, J. Hart:
Seamster: Inconspicuous Low-Distortion Texture Seam Layout, IEEE Vis 2002
http://www.cs.ubc.ca/~sheffa/papers/VIS02.pdf

Segmentation

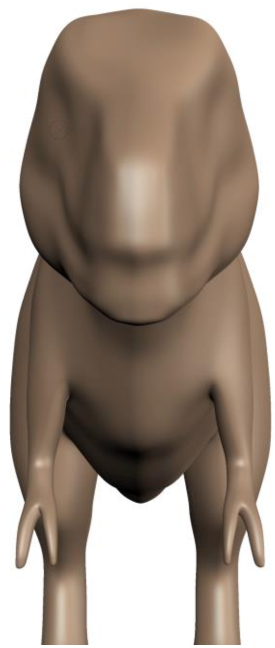



Projection Master
LightBox
Quick Sketch
Gradient
SwitchColor
Alternate
DynaMesh
Rename
All Low
All High
Copy
Duplicate
Append
Insert
Delete
Del All
Split
Merge
Remesh
Project
Extract
Geometry

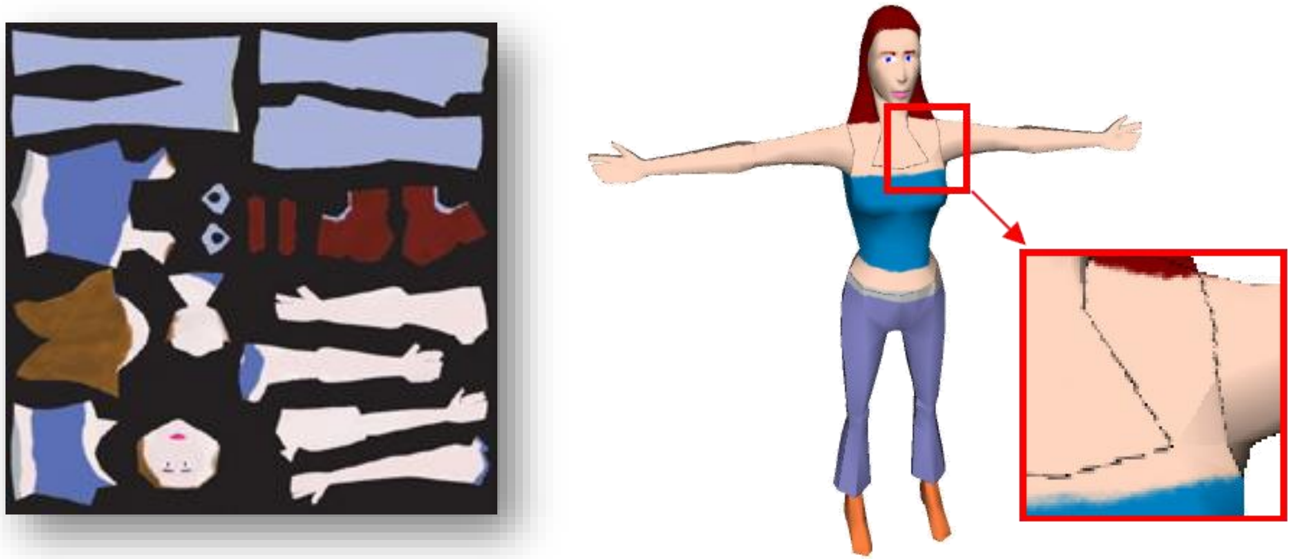

Lambda
Delta



3D painting



Texture artifacts



Texture artifacts

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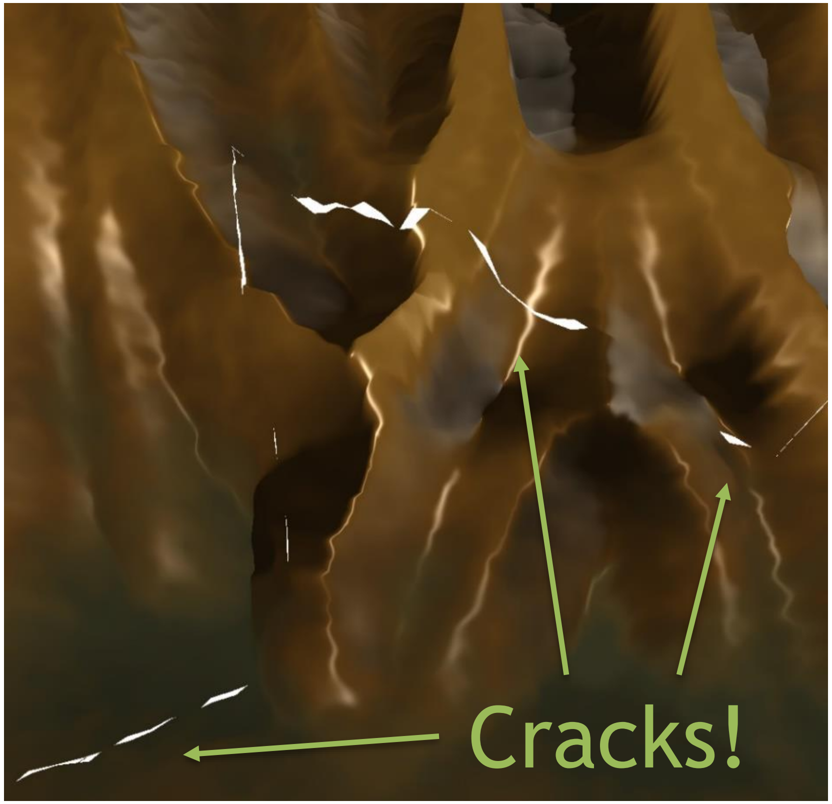
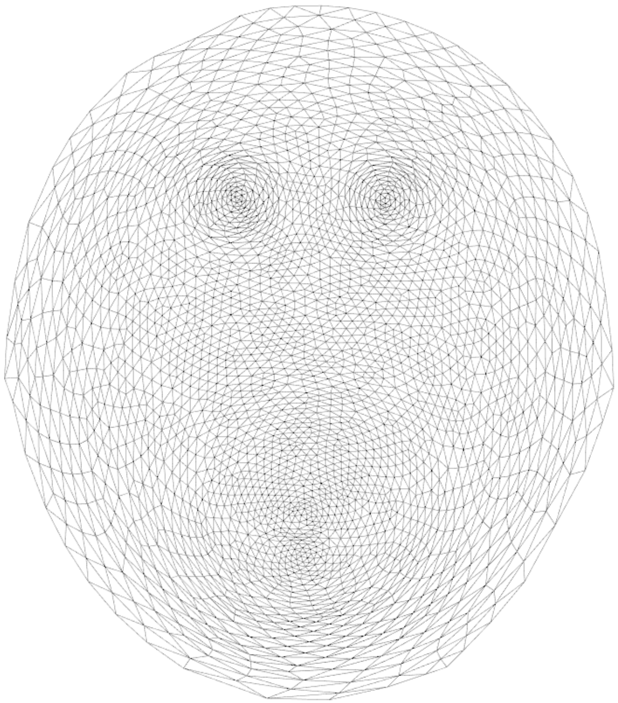
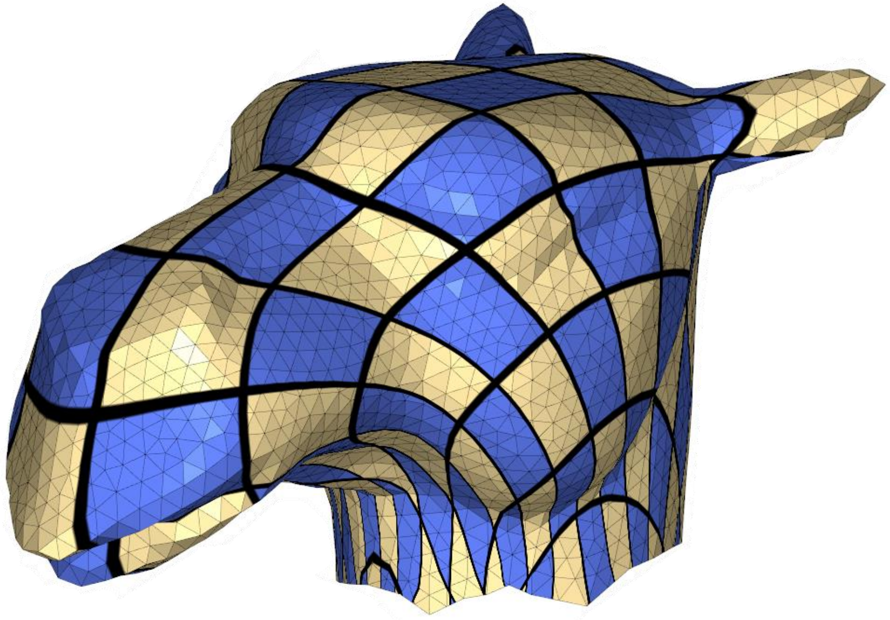
Texture artifacts

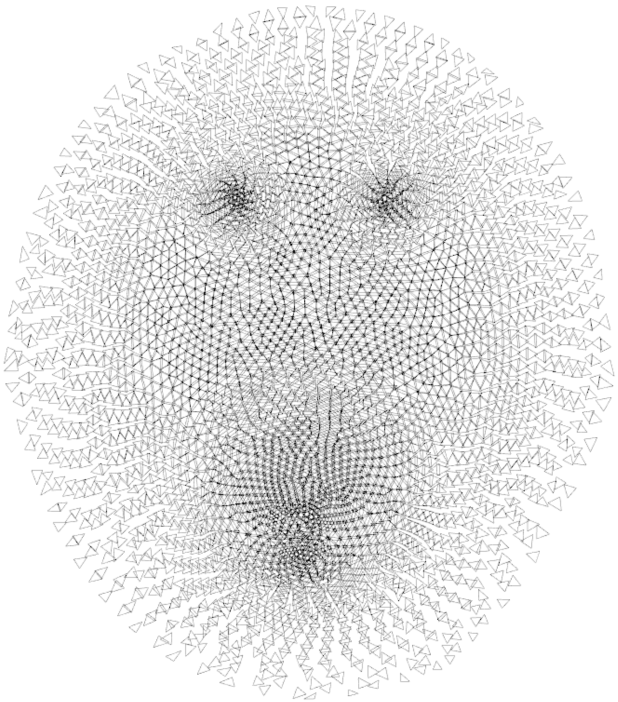
Texture artifacts

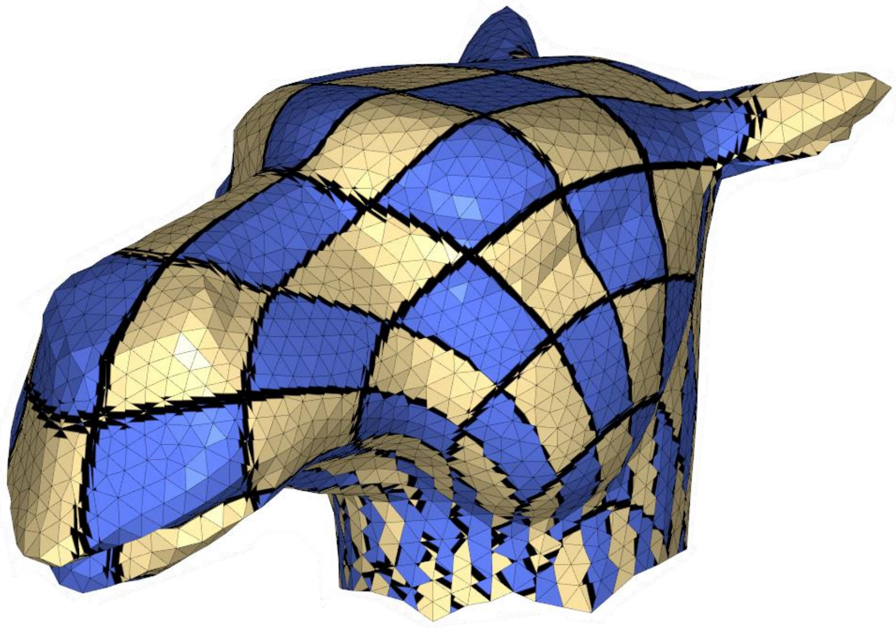


The balance

The balance





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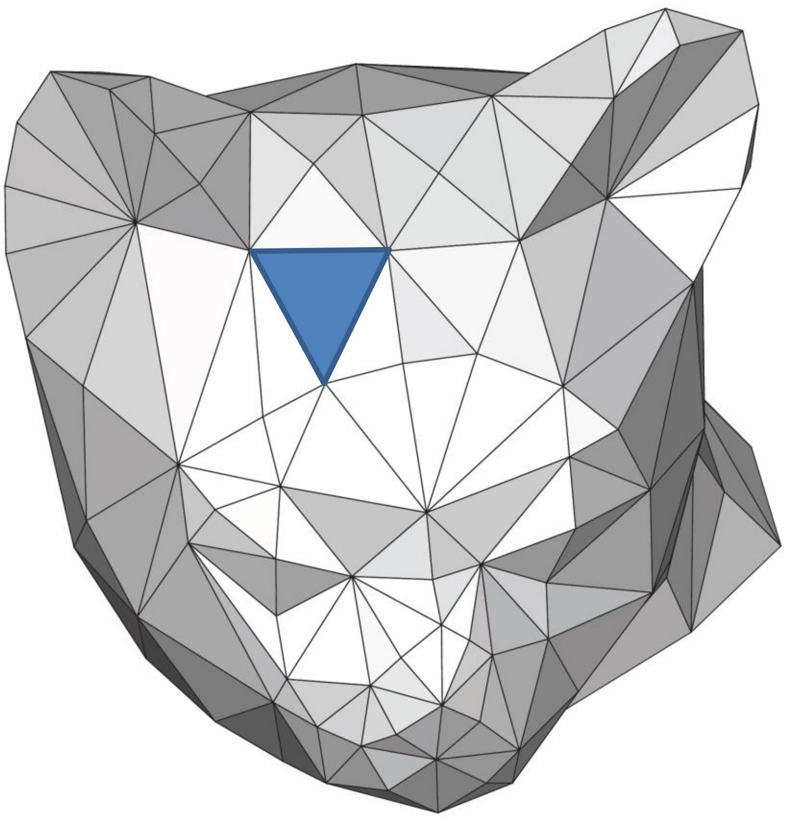
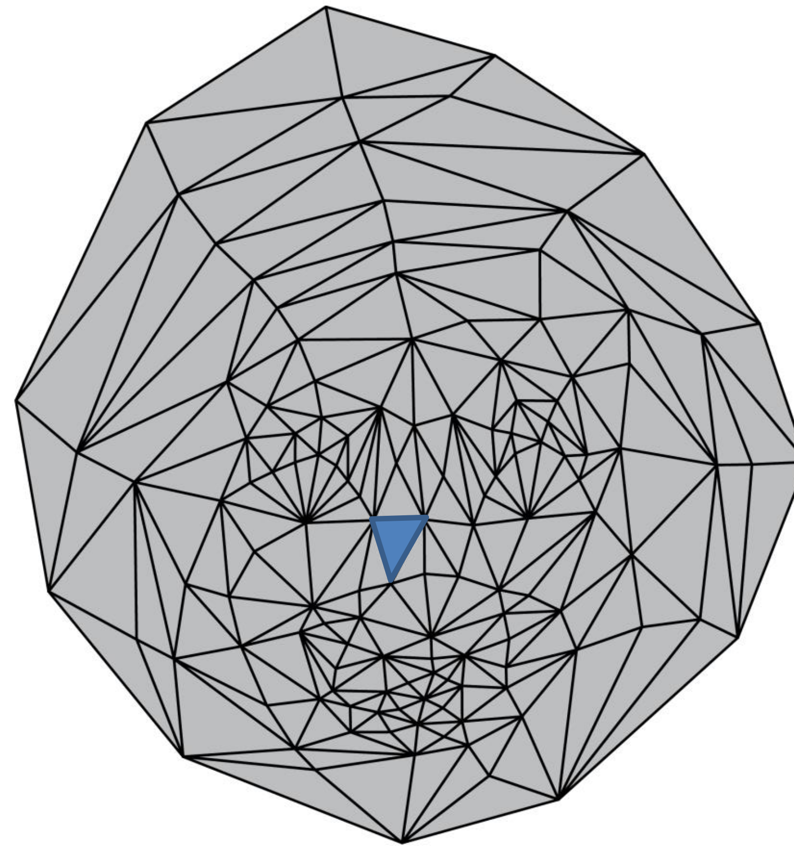
Distortion on Triangle Meshes





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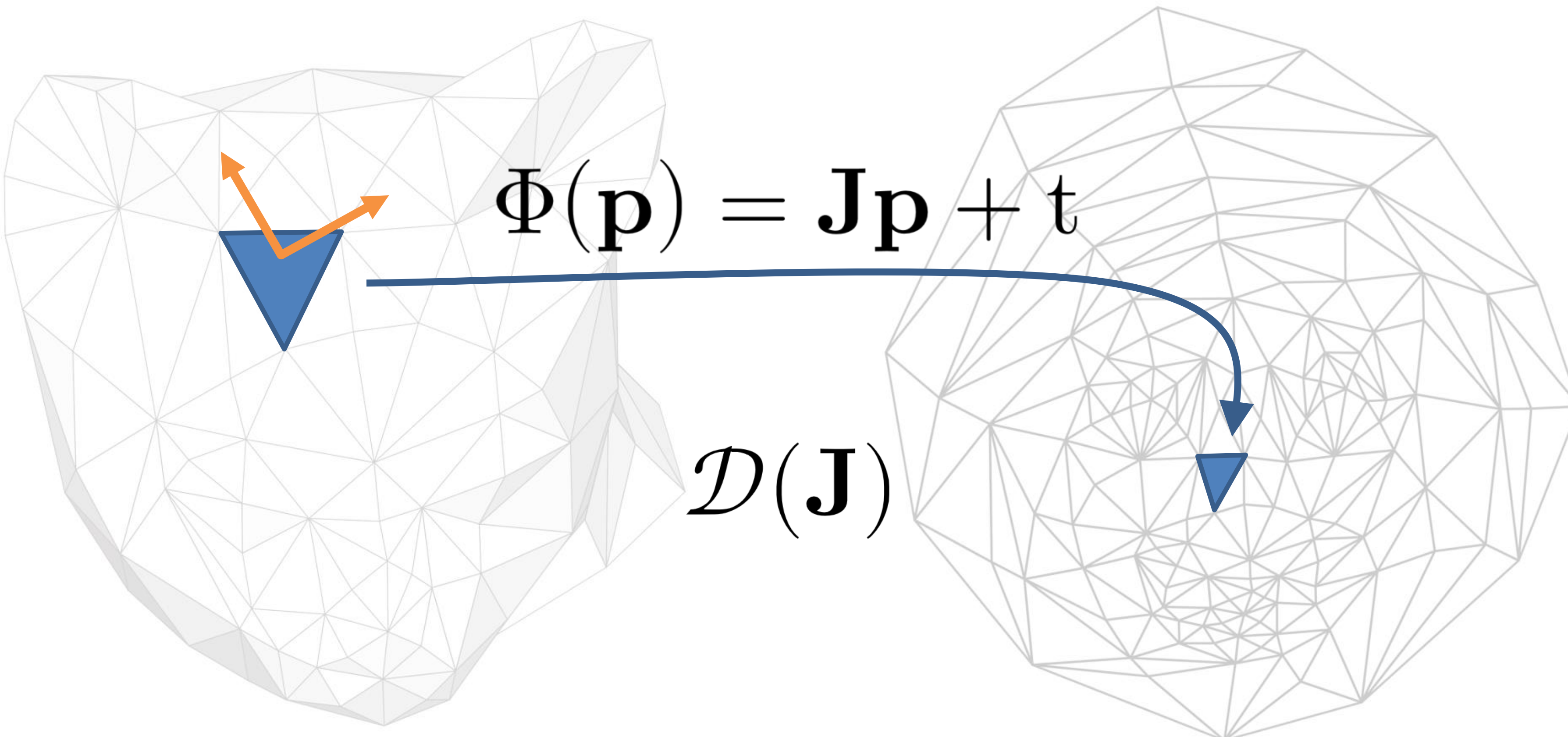
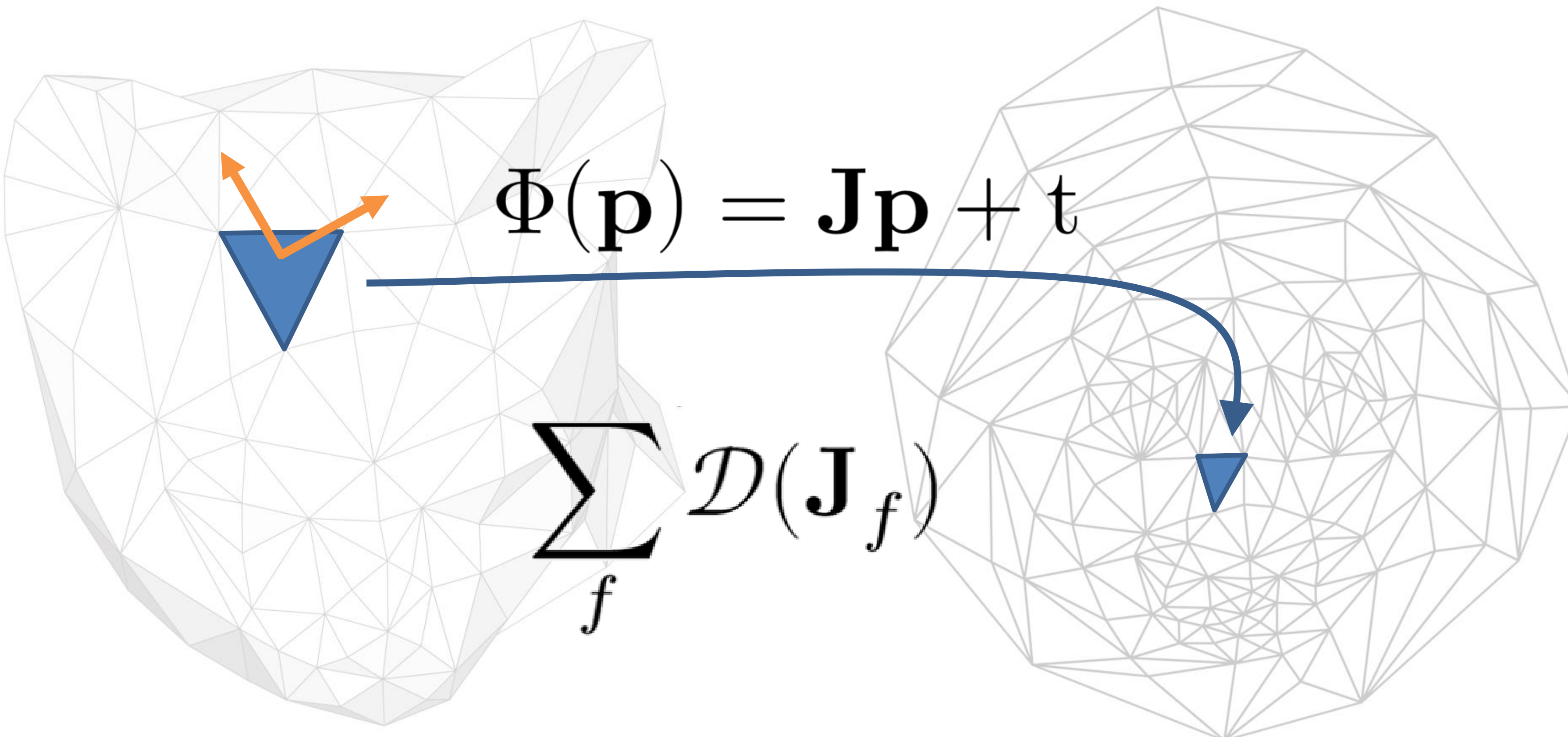
Distortion on Triangle Meshes



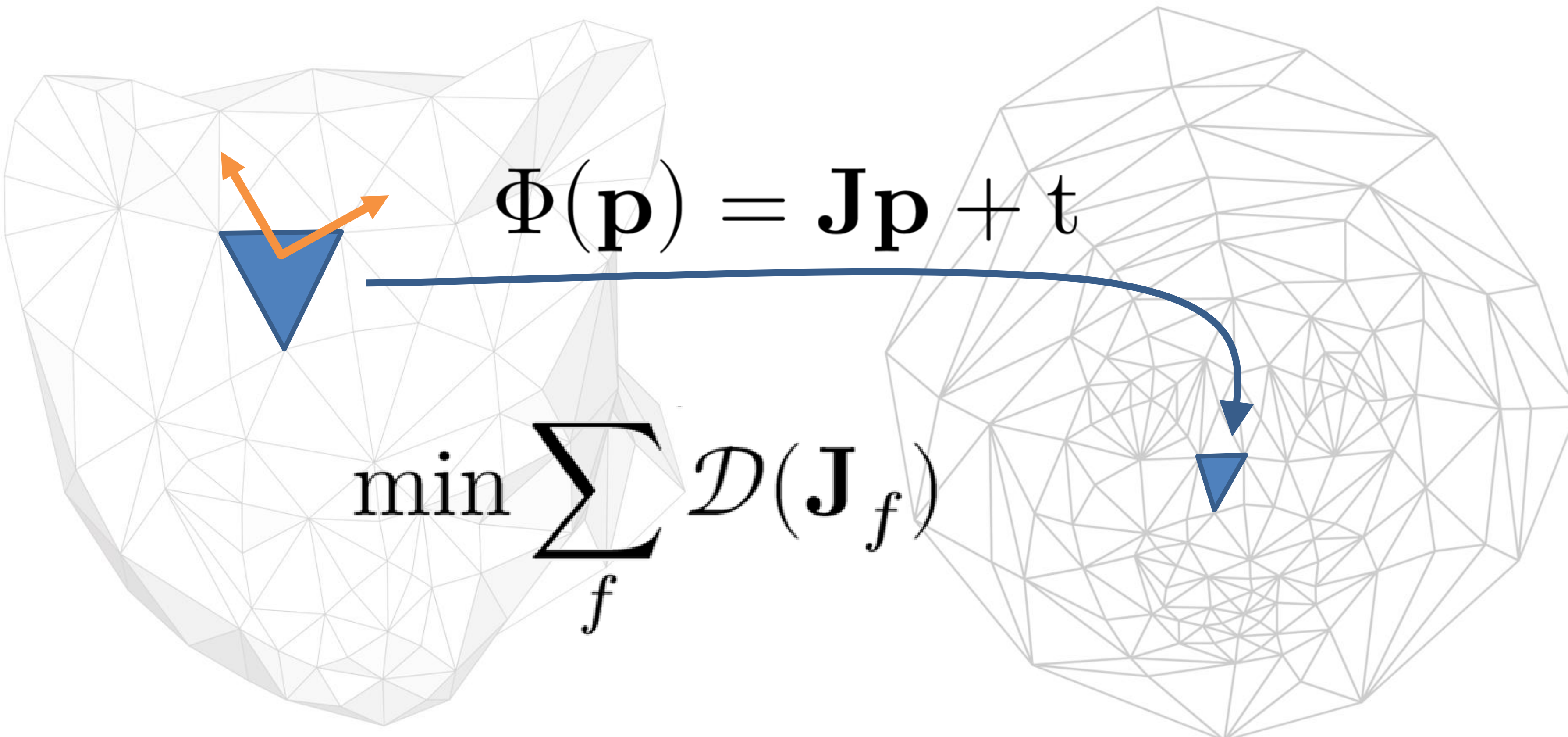
Distortion on Triangle Meshes



$$\Phi(\mathbf{p}) = \mathbf{J}\mathbf{p} + \mathrm{t}$$

$$\sum_f \mathcal{D}(\mathbf{J}_f)$$

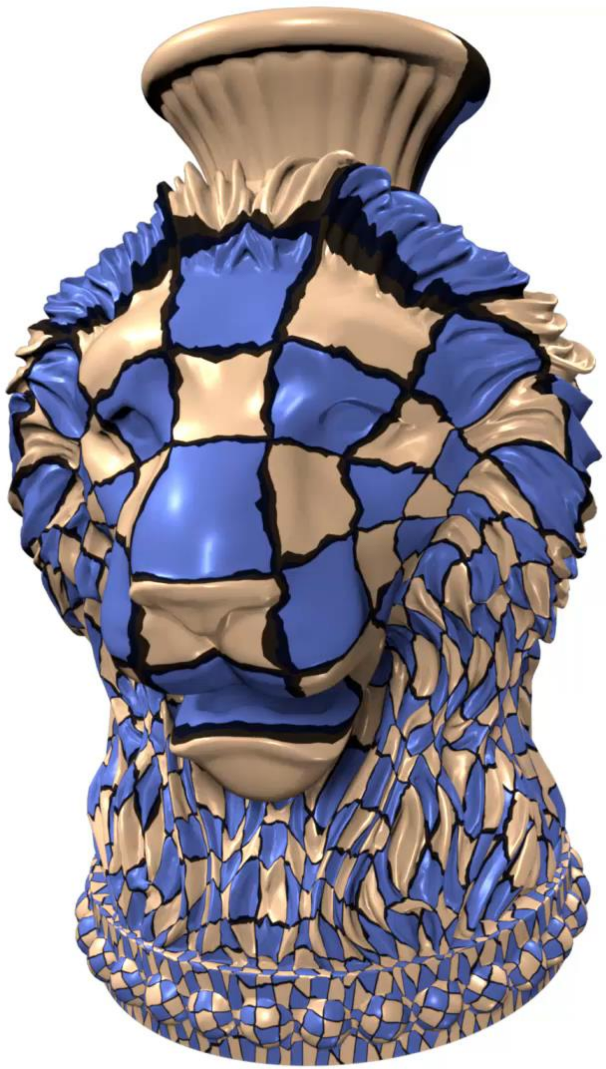
Distortion on Triangle Meshes

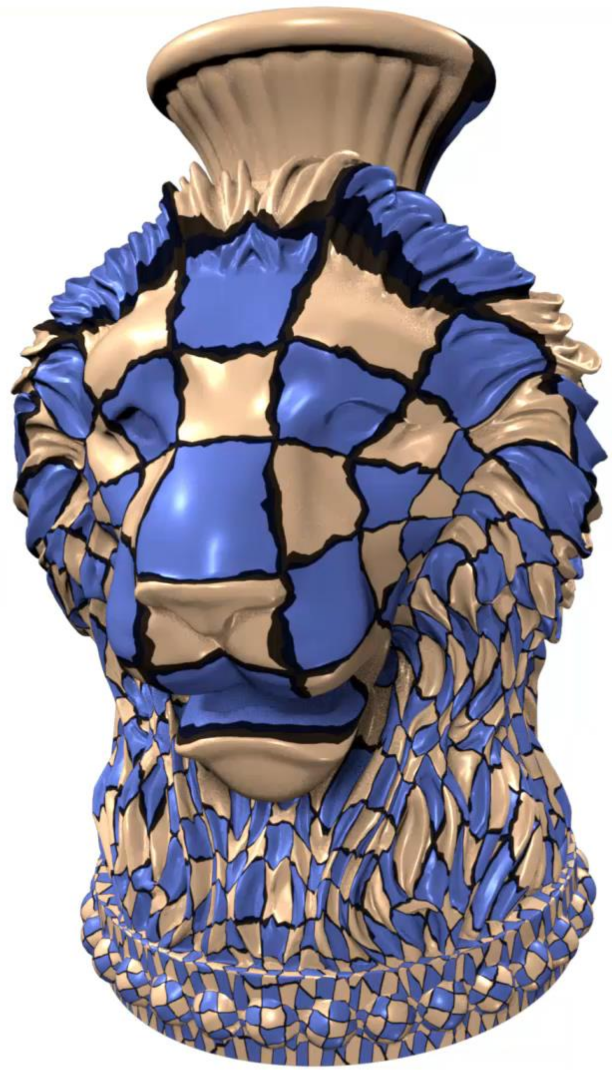


$$\Phi(\mathbf{p}) = \mathbf{J}\mathbf{p} + \mathrm{t}$$

$$\min \sum_{f} \mathcal{D}(\mathbf{J}_f)$$

100K Faces: 3s





Conformal

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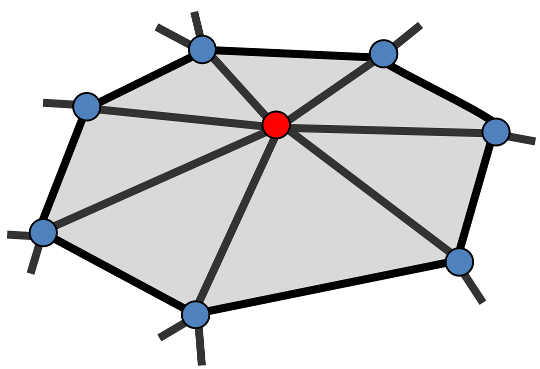
Harmonic Mapping – Idea

- Want to flatten the mesh $\rightarrow$ no curvature $\rightarrow$ Laplace operator gives zero.



$\mathbf{u} = (u,v)$ domain

$$\Delta(\mathbf{u}) = 0$$

need boundary constraints to prevent trivial solution;

which Laplacian operator? (which weights?)

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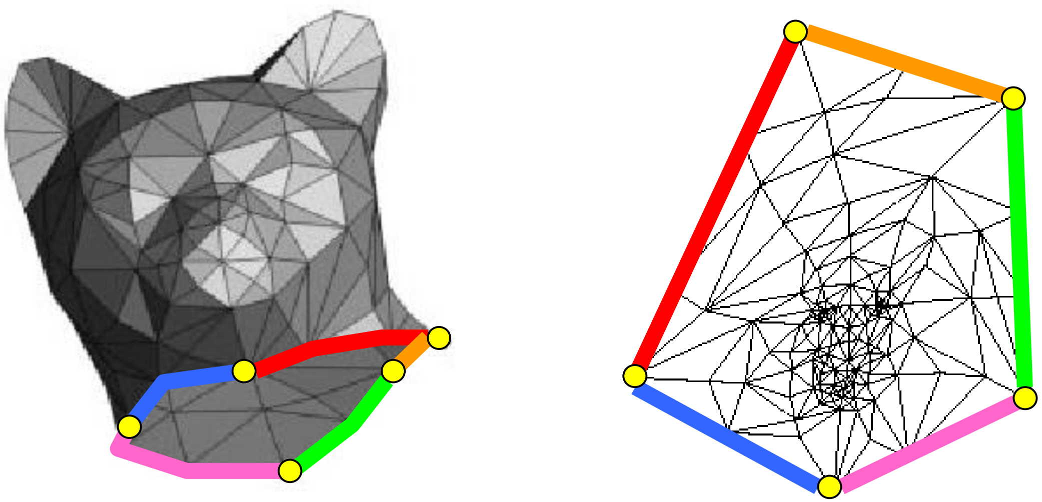
Convex Mapping (Tutte, Floater)

- Boundary vertices are fixed
- Convex weights

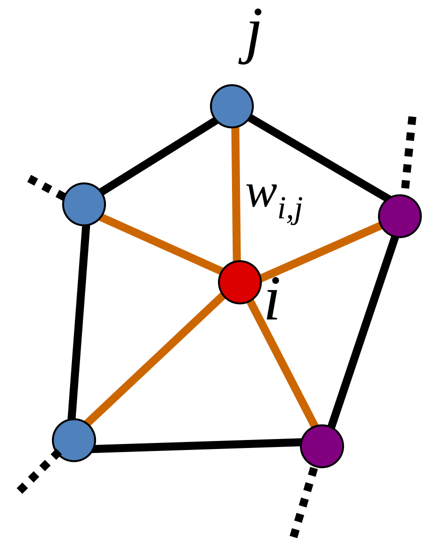


$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ - inner vertices

$\mathbf{v}_{n+1}, \ldots, \mathbf{v}_N$ - boundary vertices

Convex Mapping (Tutte, Floater)

- Boundary vertices are fixed
- Convex weights



$$\Delta(\mathbf{u}_i) = 0, \quad i = 1, \ldots, n$$

$$L(\mathbf{u}_i) = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij}(\mathbf{u}_j - \mathbf{u}_i) = 0, \quad i = 1, \ldots, n$$

$$\boxed{w_{ij} > 0}$$

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Convex Mapping (Tutte, Floater)

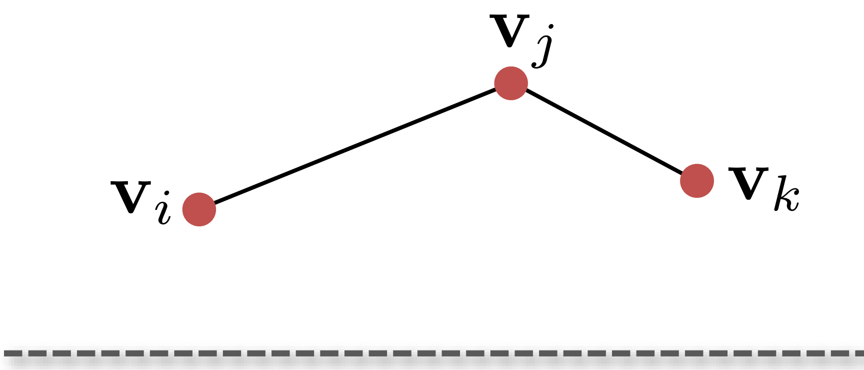
- Solve the linear system

$$L\mathbf{u} = 0 \qquad \mathbf{u} \in \mathbb{R}^{n \times 2}$$

- The values of the boundary vertices are known and thus substituted (transfer to right-hand side)

Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane



Harmonic Mapping

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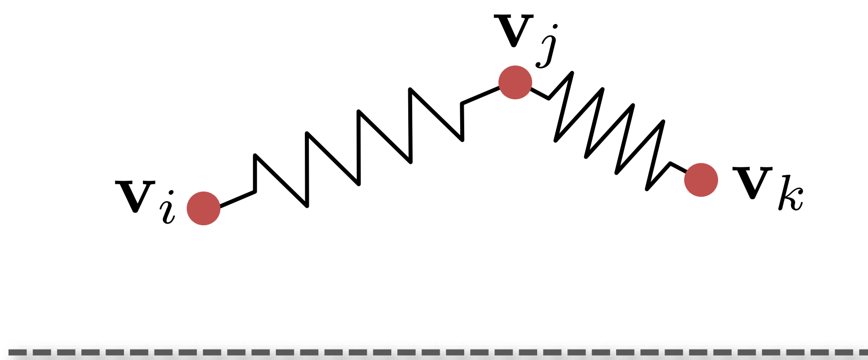

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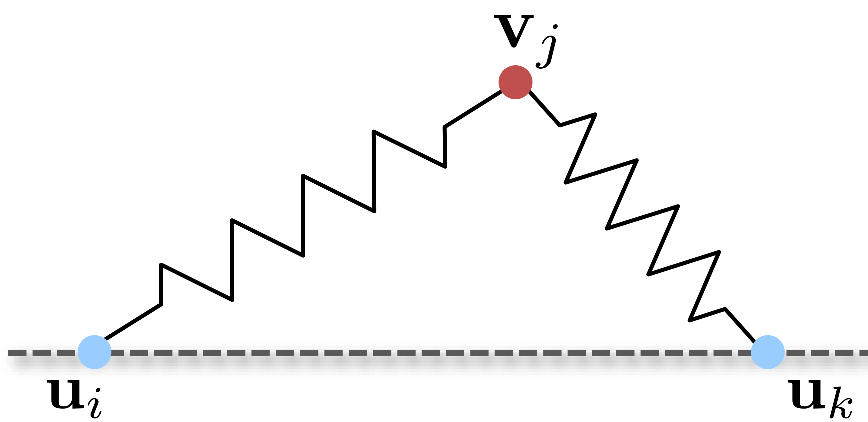

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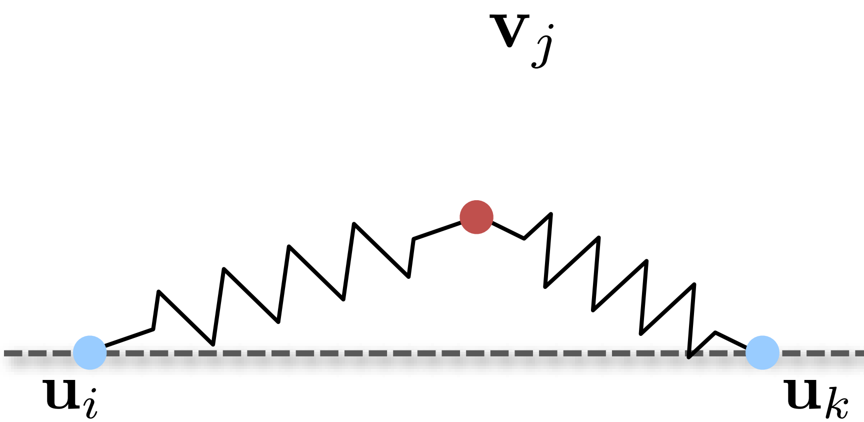

Harmonic Mapping

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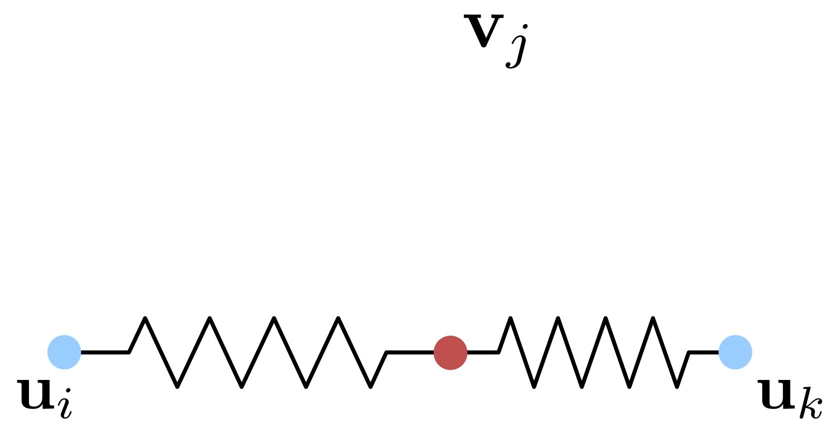

Harmonic Mapping

- Inner mesh edges as springs
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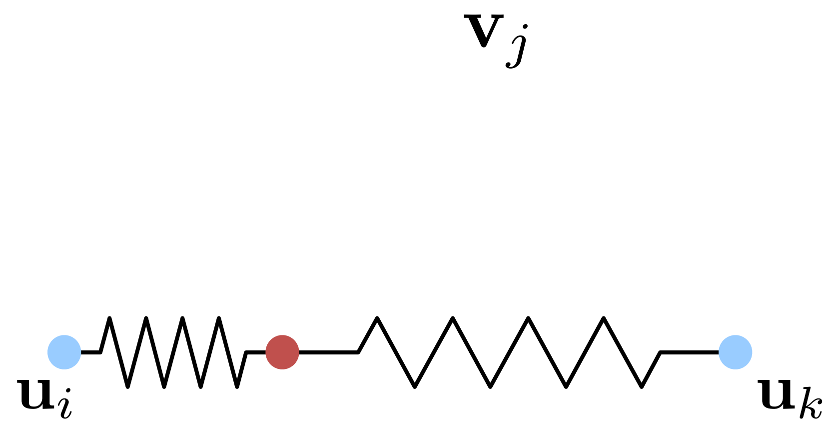

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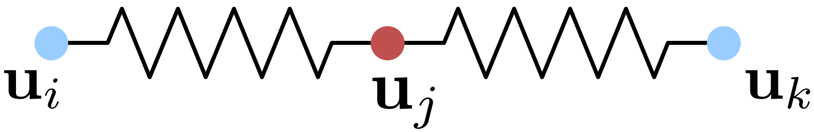
Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane



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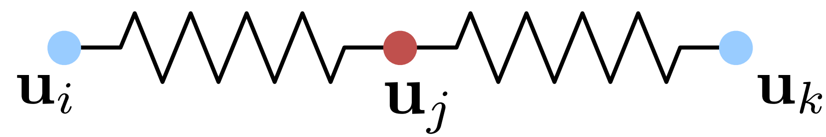
Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane
- Spring energy:

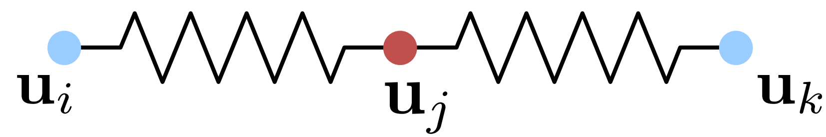


$$\frac{1}{2} k_{i,j} \|\mathbf{u}_i - \mathbf{u}_j\|^2$$

$$\mathbf{u}_i, \mathbf{u}_j \in \mathbb{R}^2$$

Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane



- Total spring energy of the flattened mesh:

$$E(\mathbf{u}_1, \ldots, \mathbf{u}_n) = \sum_{(i,j)\in\mathcal{E}} \frac{1}{2} k_{i,j} \|\mathbf{u}_i - \mathbf{u}_j\|^2$$

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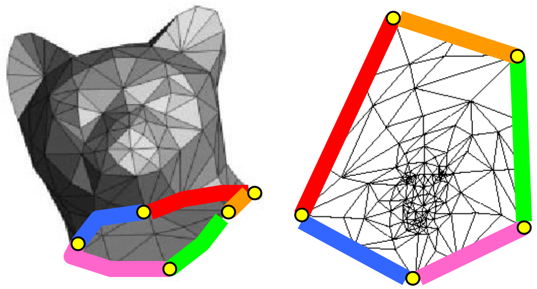
Demo

- Libigl Tutorial 501

Minimizing Spring Energy

$$E(\mathbf{u}_1, \ldots, \mathbf{u}_n) = \sum_{(i,j)\in\mathcal{E}} \frac{1}{2} k_{i,j} \|\mathbf{u}_i - \mathbf{u}_j\|^2$$



$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ - inner vertices

$\mathbf{v}_{n+1}, \ldots, \mathbf{v}_N$ - boundary vertices

$$\frac{\partial E(\mathbf{u}_1, \ldots, \mathbf{u}_n)}{\partial \mathbf{u}_i} = \sum_{j\in\mathcal{N}(i)} k_{i,j}(\mathbf{u}_i - \mathbf{u}_j) = 0$$

$$\sum_{j\in\mathcal{N}(i)\cap\mathcal{B}} k_{i,j}\mathbf{u}_i + \underbrace{\sum_{j\in\mathcal{N}(i)\setminus\mathcal{B}} k_{i,j}(\mathbf{u}_i - \mathbf{u}_j)}_{\text{unknown flat vertex positions}} = \sum_{j\in\mathcal{N}(i)\cap\mathcal{B}} k_{i,j}\underbrace{\mathbf{u}_j}_{\text{known fixed boundary positions}}$$

unknown
flat vertex
positions

known fixed
boundary
positions

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Minimizing Spring Energy

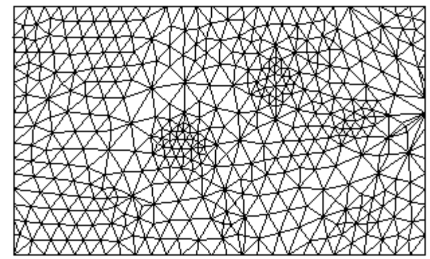
- Sparse linear system of n equations to solve!

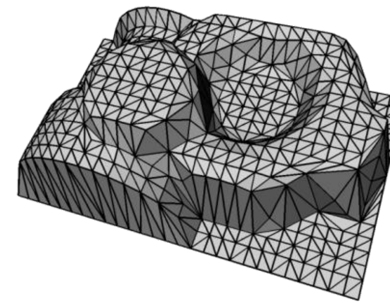
$$\boxed{\sum_{j\in\mathcal{N}(i)\cap\mathcal{B}} k_{i,j}\mathbf{u}_i + \sum_{j\in\mathcal{N}(i)\setminus\mathcal{B}} k_{i,j}(\mathbf{u}_i - \mathbf{u}_j) = \sum_{j\in\mathcal{N}(i)\cap\mathcal{B}} k_{i,j}\mathbf{u}_j}$$

$$\begin{pmatrix} \sum_j k_{i,j} & * & \cdots & -k_{i,j} \\ * & \sum_j k_{i,j} & * & \vdots \\ \vdots & * & \ddots & * \\ -k_{j,i} & \cdots & * & \sum_j k_{i,j} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{u}}_1 \\ \bar{\mathbf{u}}_2 \\ \vdots \\ \bar{\mathbf{u}}_n \end{pmatrix}$$

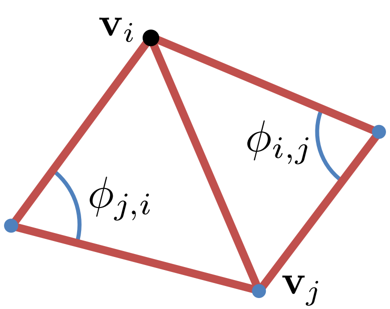
Choice of spring constants $k_{i,j}$

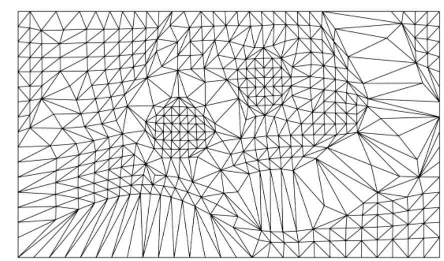
- Uniform $k_{i,j} = 1$





- Cotan $k_{i,j} = \cot \phi_{i,j} + \cot \phi_{j,i}$



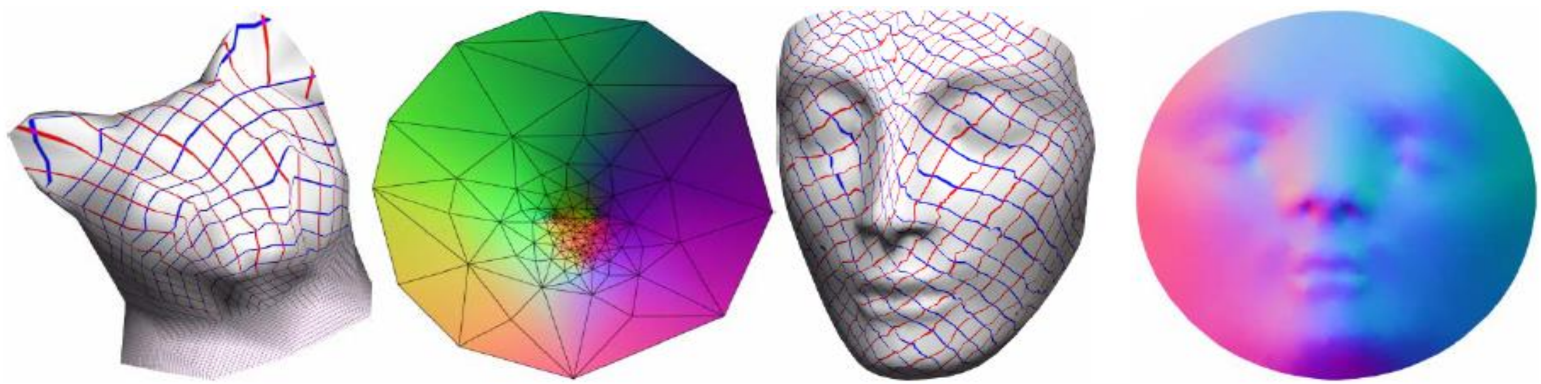


Tutte’s Theorem

- If the weights are **nonnegative**, and the boundary is fixed to a **convex** polygon, the parameterization is **bijective**
- (Tutte’63 proved for uniform weights, Floater’97 extended to arbitrary nonnegative weights)
- W.T. Tutte. “How to draw a graph”. Proceedings of the London Mathematical Society, 13(3):743-768, 1963.

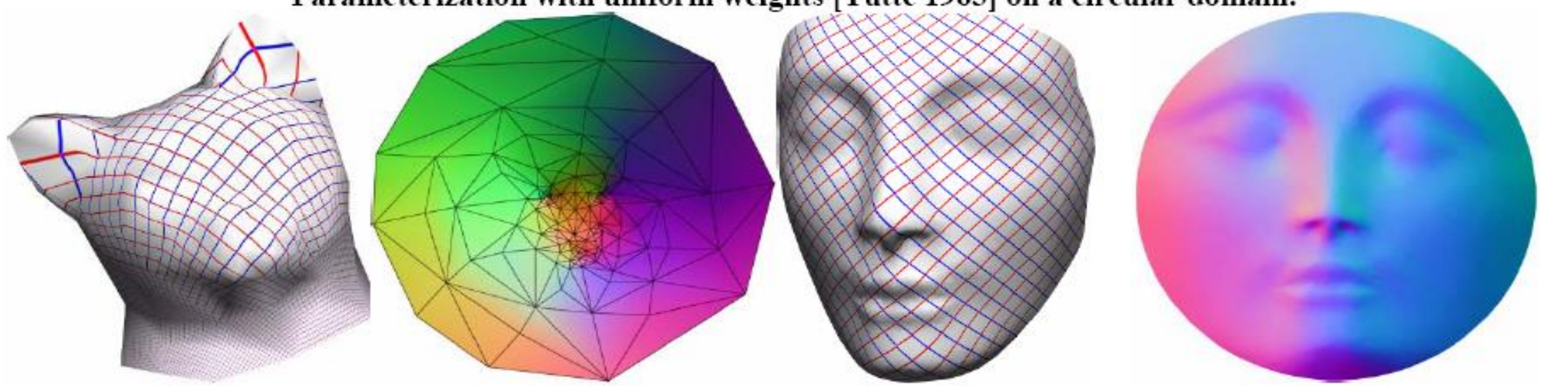
Comparison of Weights

uniform weights



Parameterization with uniform weights [Tutte 1963] on a circular domain.

cotan weights



Parameterization with harmonic weights [Eck et al. 1995] on a circular domain.

Eck et al. 1995, “Multiresolution analysis of arbitrary meshes”, SIGGRAPH 1995

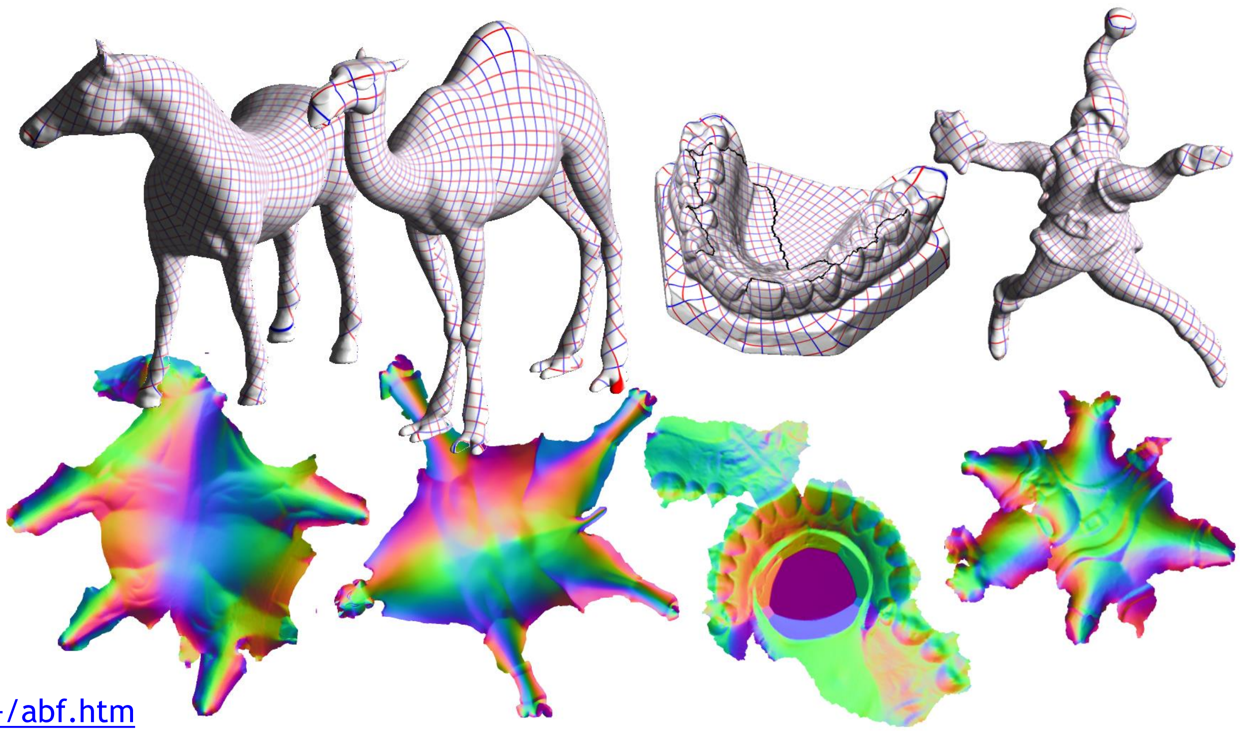
Discussion

- The results of cotan-weights mapping are better than those of uniform convex mapping (local area and angles preservation).
- But: the mapping is not always legal (the cotan weights can be negative for badly-shaped triangles…)

- In any case: sparse system to solve, so robust and efficient numerical solvers exist

Discussion

- Both mappings have the problem of fixed boundary – it constrains the minimization and causes distortion.
- More advanced methods do not require boundary conditions.



ABF++ method,
Sheffer et al. 2005
http://www.cs.ubc.ca/~sheffa/ABF++/abf.htm

Thank You