252-0538-00L, Spring 2018

Shape Modeling and Geometry Processing

Parameterization

21/3/2018

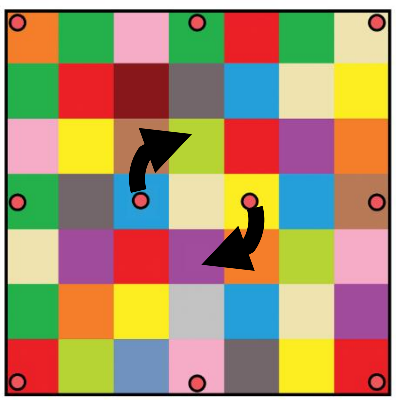
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

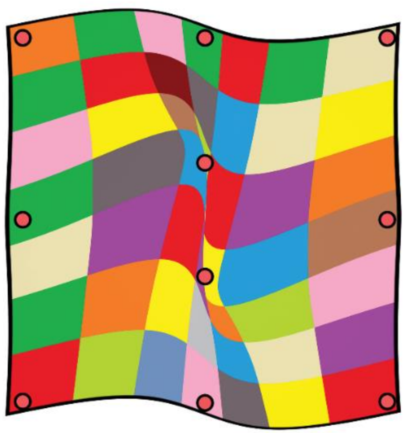
What are good maps?

Local

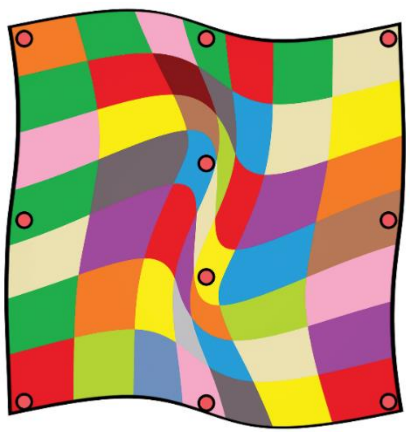
Bijectivity

Low distortion

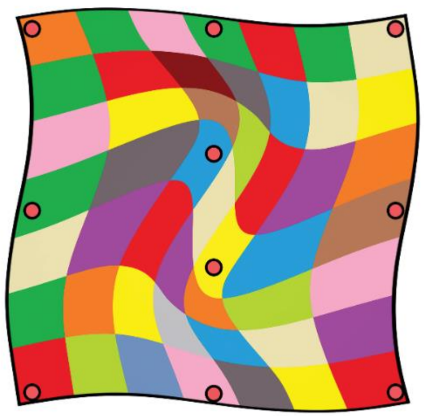




Not Bijective



Bijective



Lower distortion

Distortion - Types

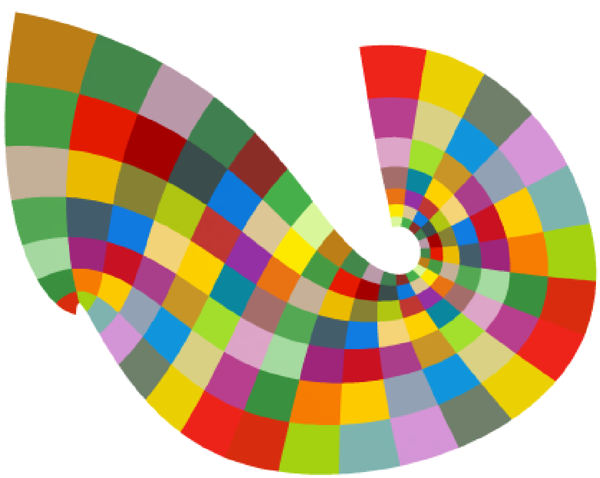
Conformal distortion

Isometric distortion

Distortion - Types

Conformal distortion

Isometric distortion

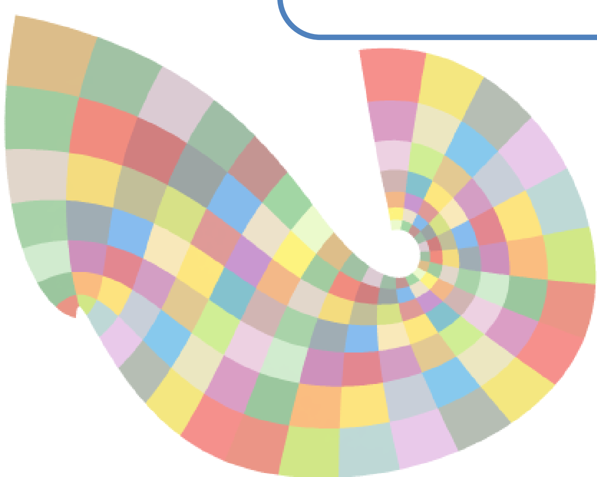


Distortion - Types

Conformal distortion

Isometric distortion

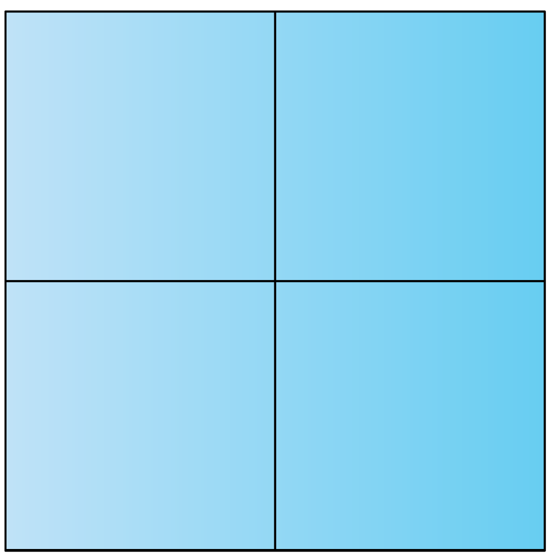
The distortion is a function of the Jacobian at a point



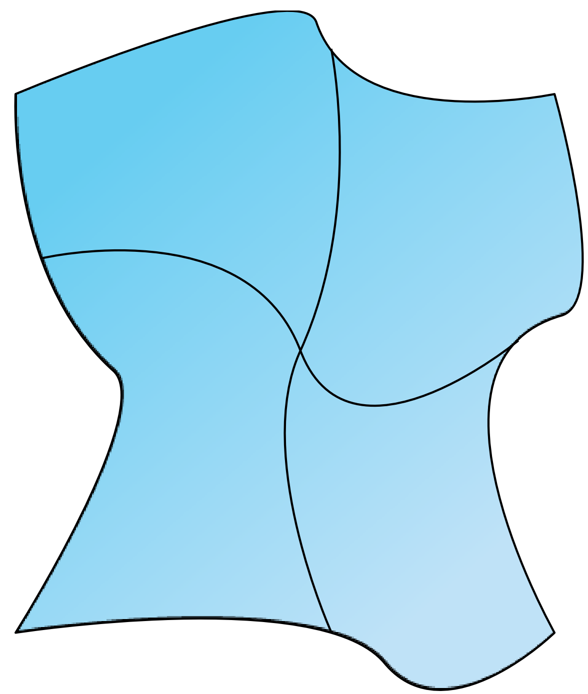
The Jacobian

A mapping $\mathbf{f}(x, y): \mathbb{R}^2 \to \mathbb{R}^2$ is defined by two functions:

$$u(x, y) \qquad v(x, y)$$



$$\mathbf{f}(x, y) = (u, v)$$



The Jacobian

A mapping $\mathbf{f}(x, y) \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by two functions:

$$u(x, y) \qquad v(x, y)$$

The Jabocian $\mathbf{J}$ is defined by

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}$$

The Jacobian

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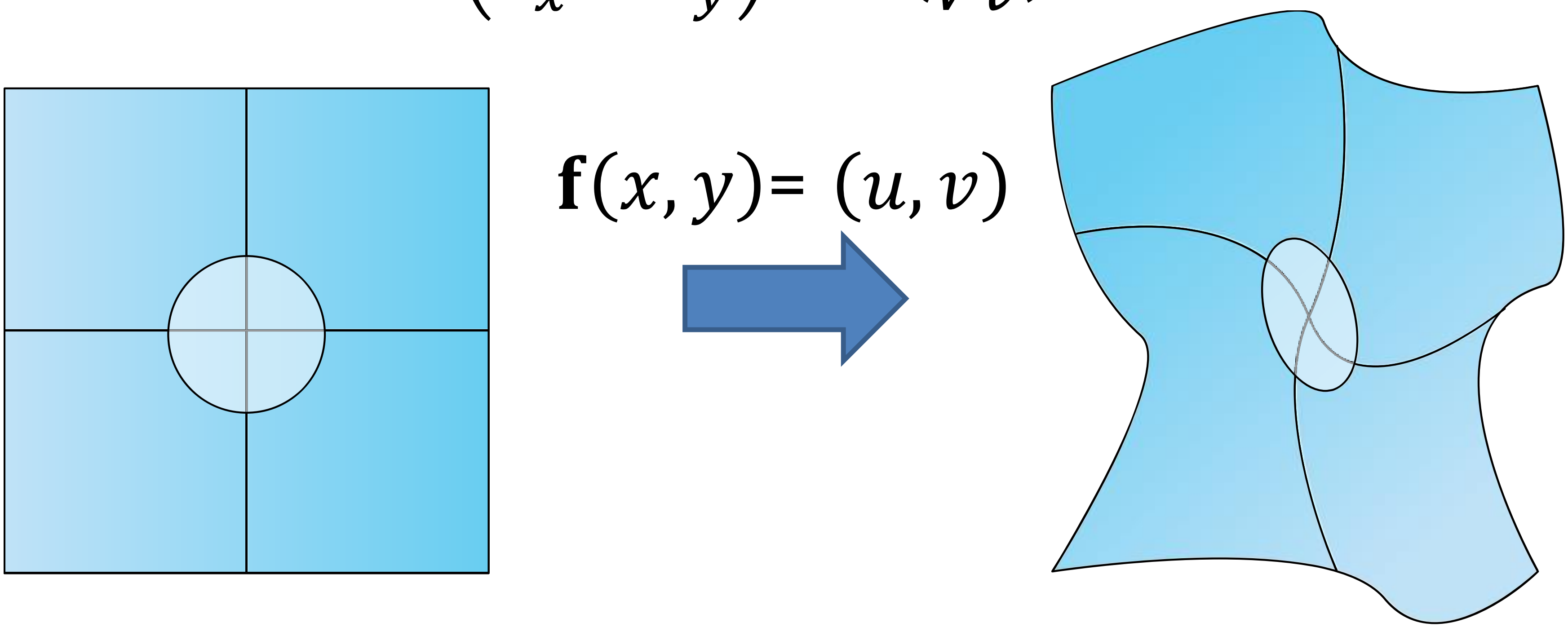

Distortion types

Conformal – angle preserving

Distortion - LSCM [Lévy et al. 2002]

LSCM – **L**east **S**quares **C**onformal **M**ap

We want the Jacobian

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

to be a similarity matrix

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\begin{aligned} \partial_x u &= \partial_y v \\ \partial_y u &= -\partial_x v \end{aligned}$$

Cauchy-Riemann Equations

Distortion - LSCM

LSCM – **L**east **S**quares **C**onformal **M**ap

We want the Jacobian to be a similarity matrix

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \qquad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\mathcal{D}_{\mathrm{LSCM}} = (\partial_x u - \partial_y v)^2 + (\partial_y u + \partial_x v)^2$$

Quick Notation Change

$$\mathcal{J}\mathbf{f} \longrightarrow \mathbf{J}$$

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{D}_{\text{LSCM}} = (\partial_x u - \partial_y v)^2 + (\partial_y u + \partial_x v)^2$$

$$(a - d)^2 + (b + c)^2$$

Distortion types

Conformal – angle preserving

$$\mathcal{D}_{\mathrm{LSCM}} = (\quad a \quad - \quad d \quad)^2 + (\quad b \quad + \quad c \quad)^2$$

Distortion types

Conformal – angle preserving

$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} + \mathbf{J}^{\mathrm{T}} - 2(tr\,\mathbf{J})\,\mathbf{I} \right\|_F^2$$

Distortion types

Conformal – angle preserving

$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} + \mathbf{J}^{\mathrm{T}} - 2(tr\,\mathbf{J})\,\mathbf{I} \right\|_F^2$$

Isometric – length preserving

$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} - \mathrm{R} \right\|_F^2$$

Distortion types

Conformal – angle preserving

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Isometric – length preserving

$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} - \mathrm{R} \right\|_F^2$$

Authalic - area preserving

$$\mathcal{D}(\mathbf{J}) = (\det \mathbf{J} - 1)^2$$

Singular value of the Jacobian

$$J = USV^T = U\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} V^T$$

$$\sigma_1 > \sigma_2$$

σ_1, σ_2 determine the local principle stretches

Singular value of the Jacobian

σ_1, σ_2 determine the local principle stretches

Conformal	$\sigma_1 = \sigma_2$
Isometric	$\sigma_1, \sigma_2 = 1$
Authalic	$\sigma_1 \sigma_2 = 1$

Distortion types

Conformal – angle preserving

$$\sigma_1 = \sigma_2 \qquad \mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} + \mathbf{J}^{\mathrm{T}} - 2(tr\,\mathbf{J})\,\mathbf{I} \right\|_F^2$$

Isometric – length preserving

$$\sigma_1, \sigma_2 = 1 \qquad \mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} - \mathrm{R} \right\|_F^2$$

Authalic - area preserving

$$\sigma_1 \sigma_2 = 1 \qquad \mathcal{D}(\mathbf{J}) = (\det \mathbf{J} - 1)^2$$

Distortion types

Conformal – angle preserving

$$\sigma_1 = \sigma_2 \quad \mathcal{D}(\mathbf{J})$$

Isometric – length preserving

$$\sigma_1, \sigma_2 = 1 \quad \mathcal{D}(\mathbf{J})$$

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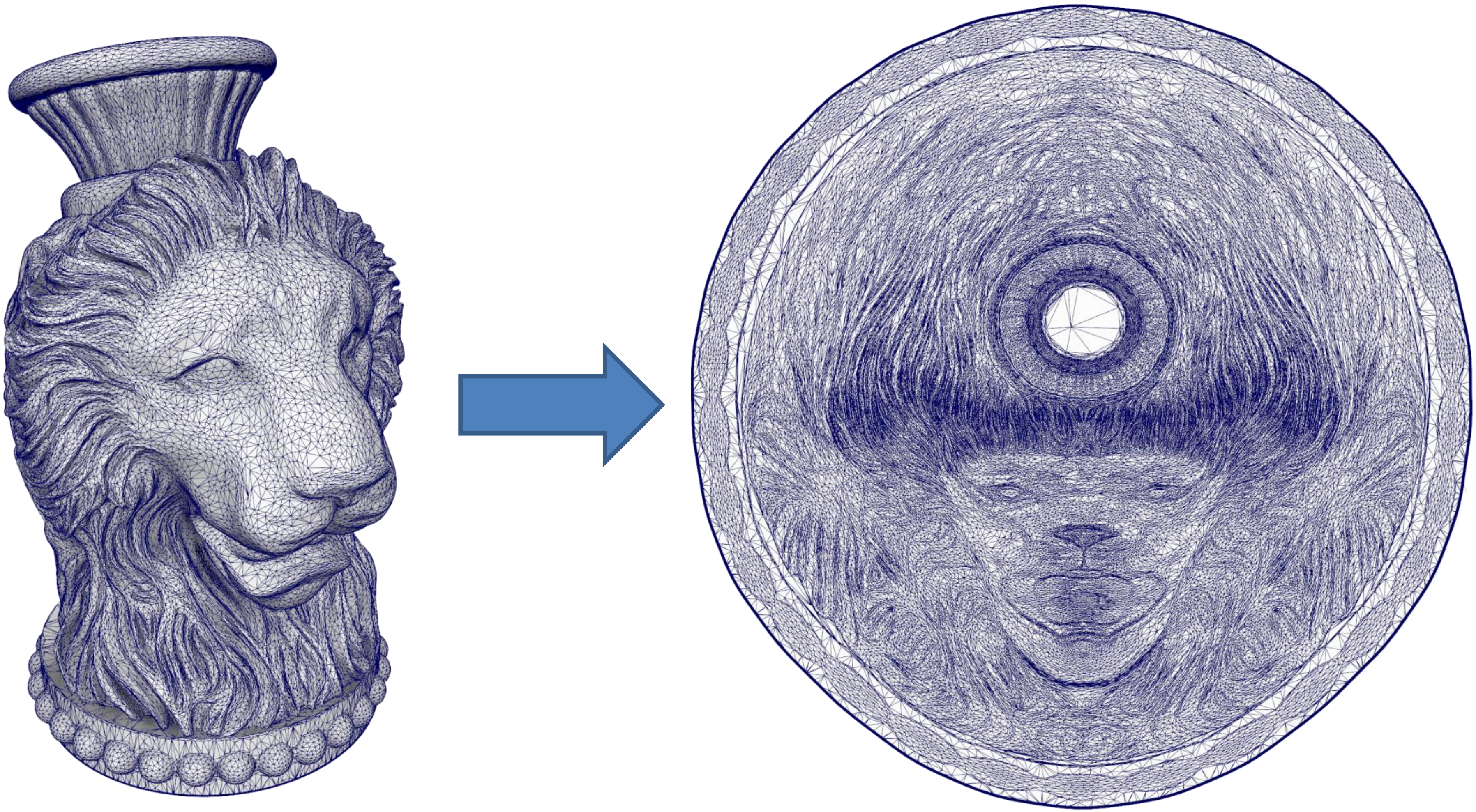
$$\sigma_1 \sigma_2 = 1 \quad \mathcal{D}(\mathbf{J})$$

What Is a Parameterization?



What Is a Parameterization?

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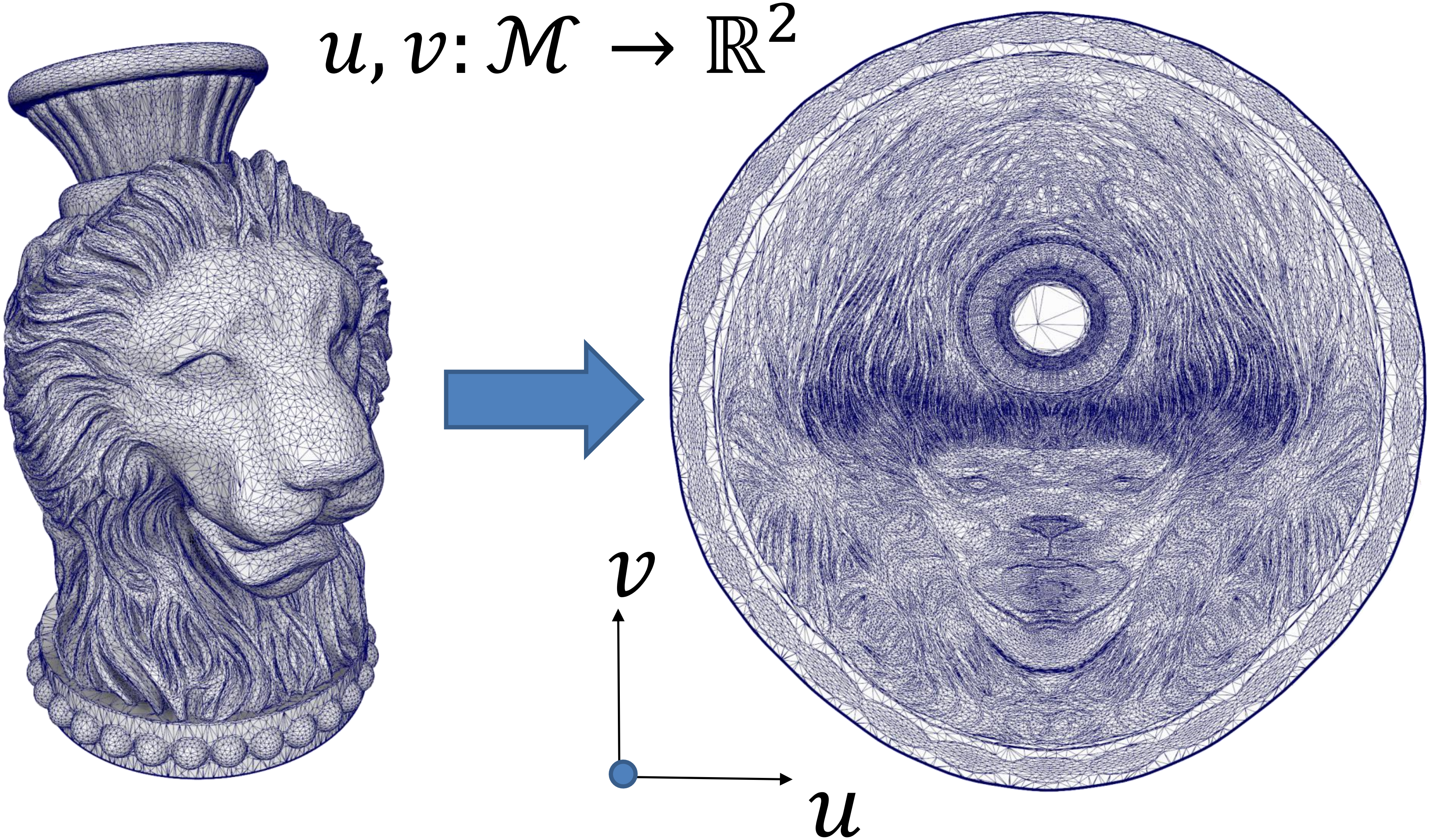
$$u, v\colon \mathcal{M} \rightarrow \mathbb{R}^2$$



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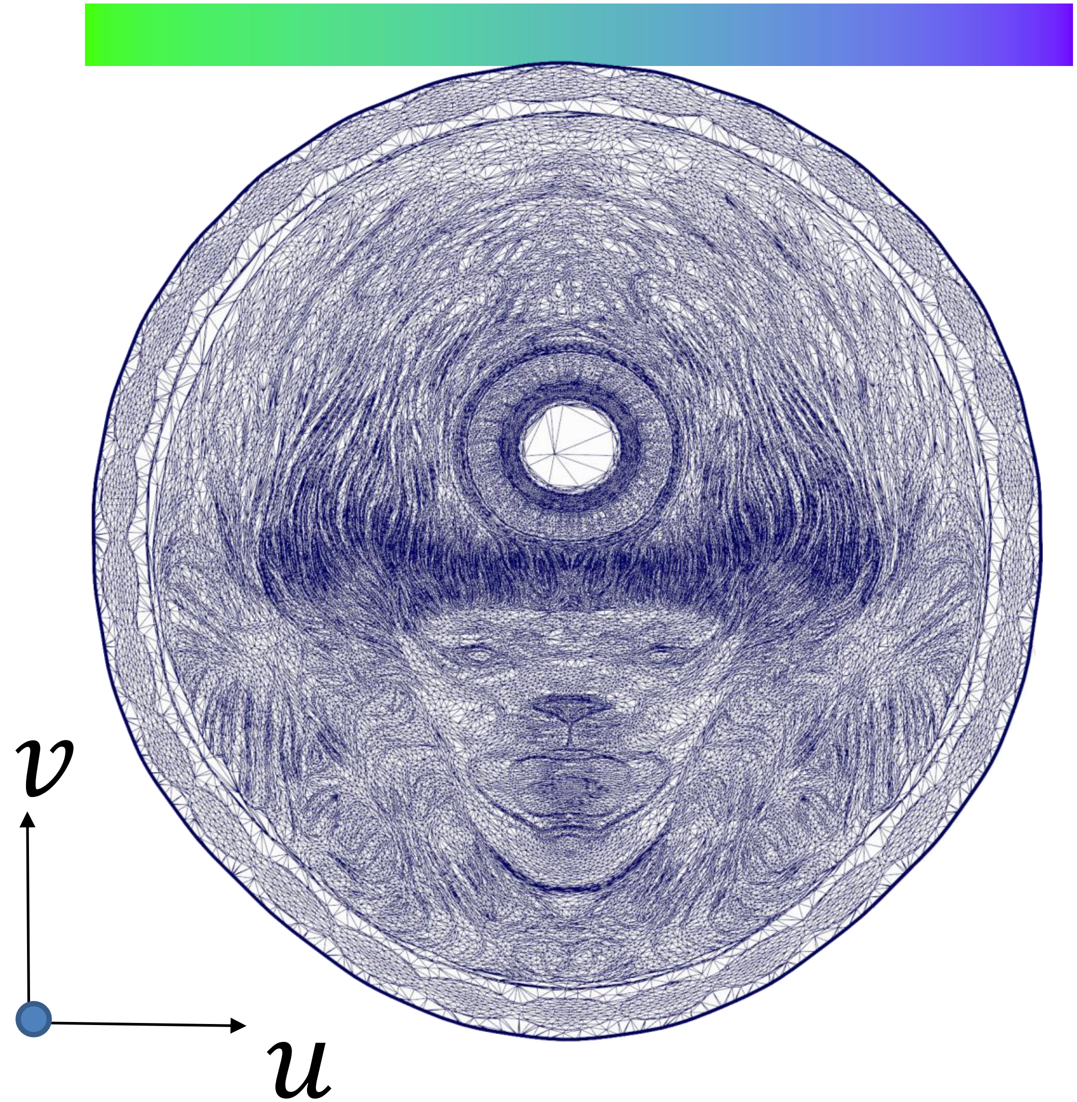

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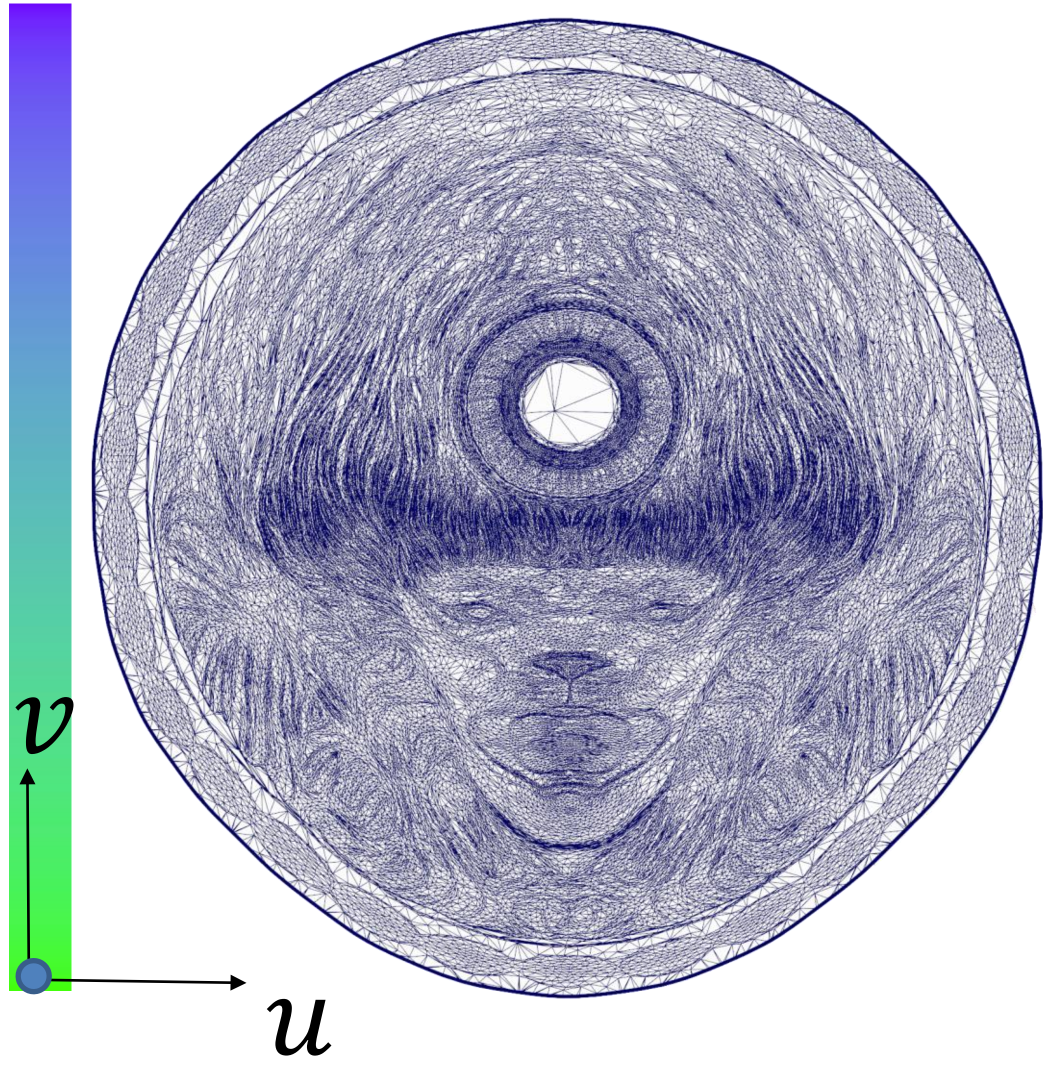
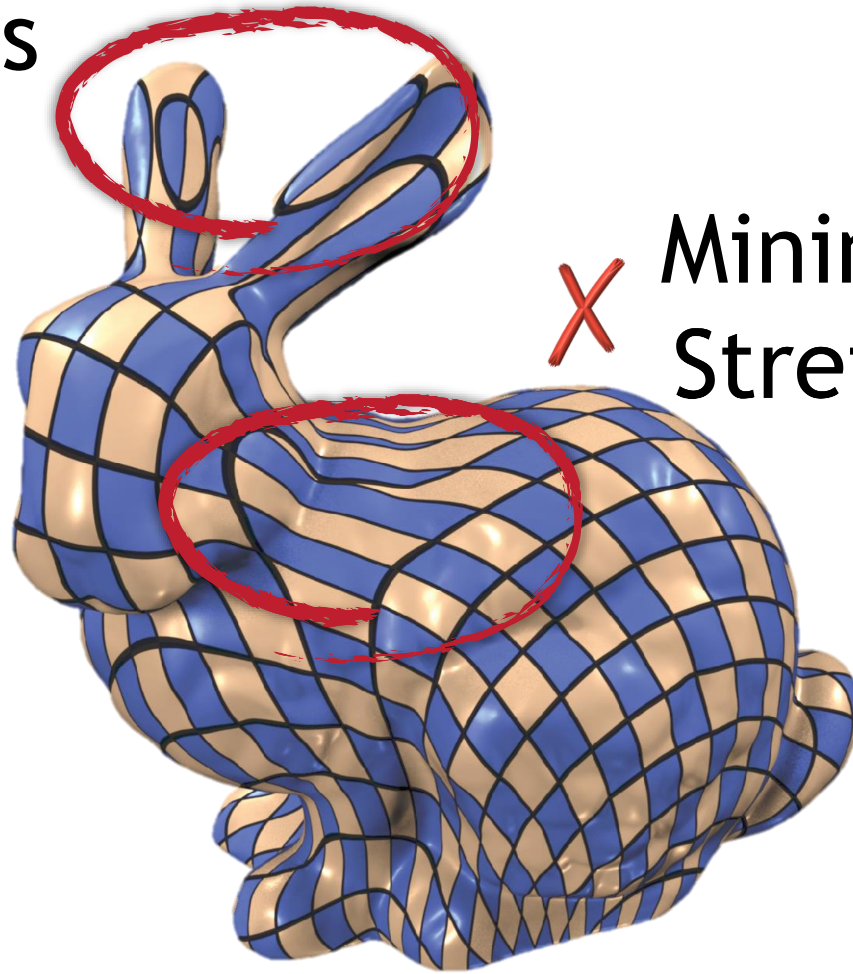


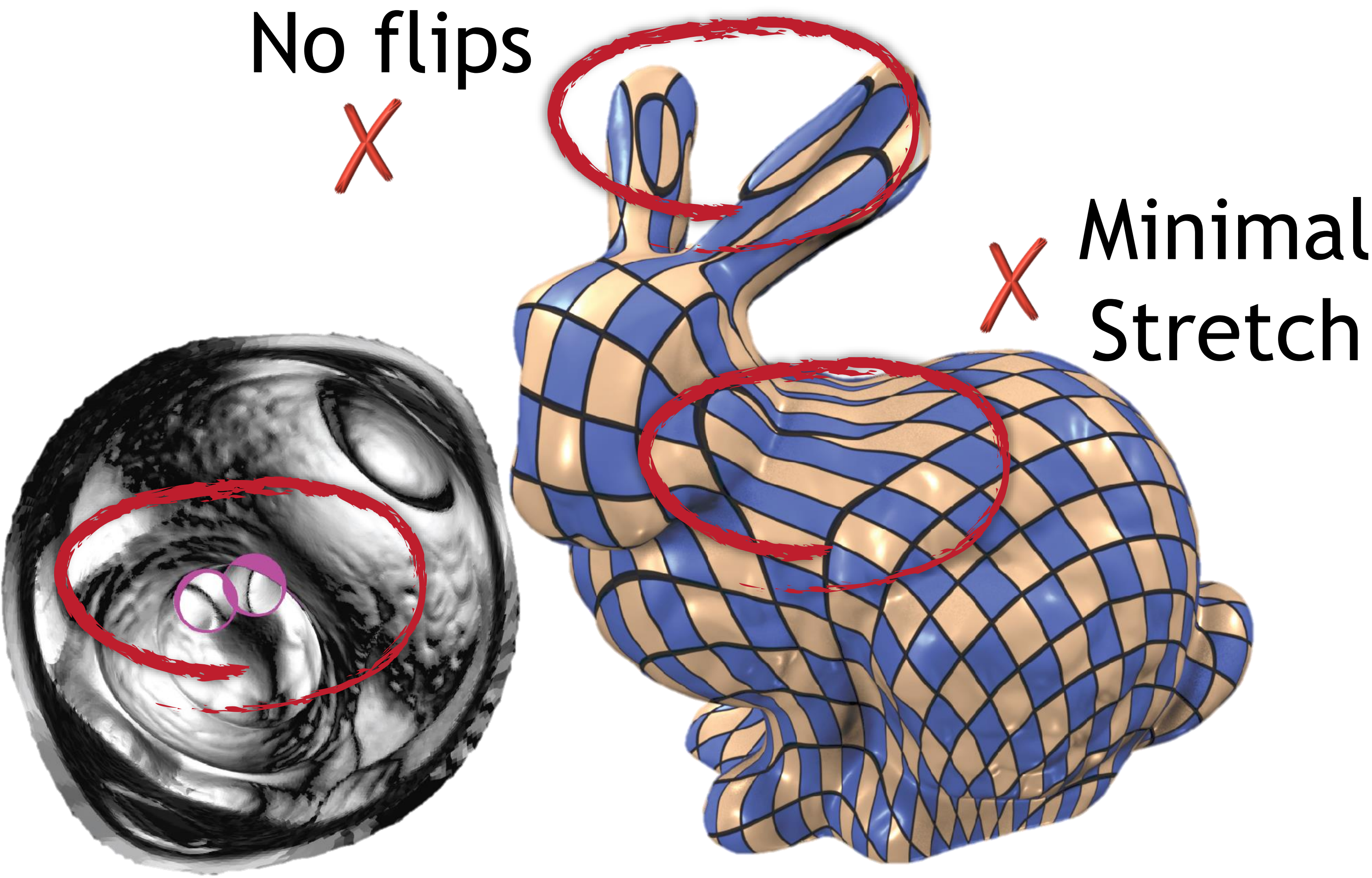
Good Parameterization

No flips ✗

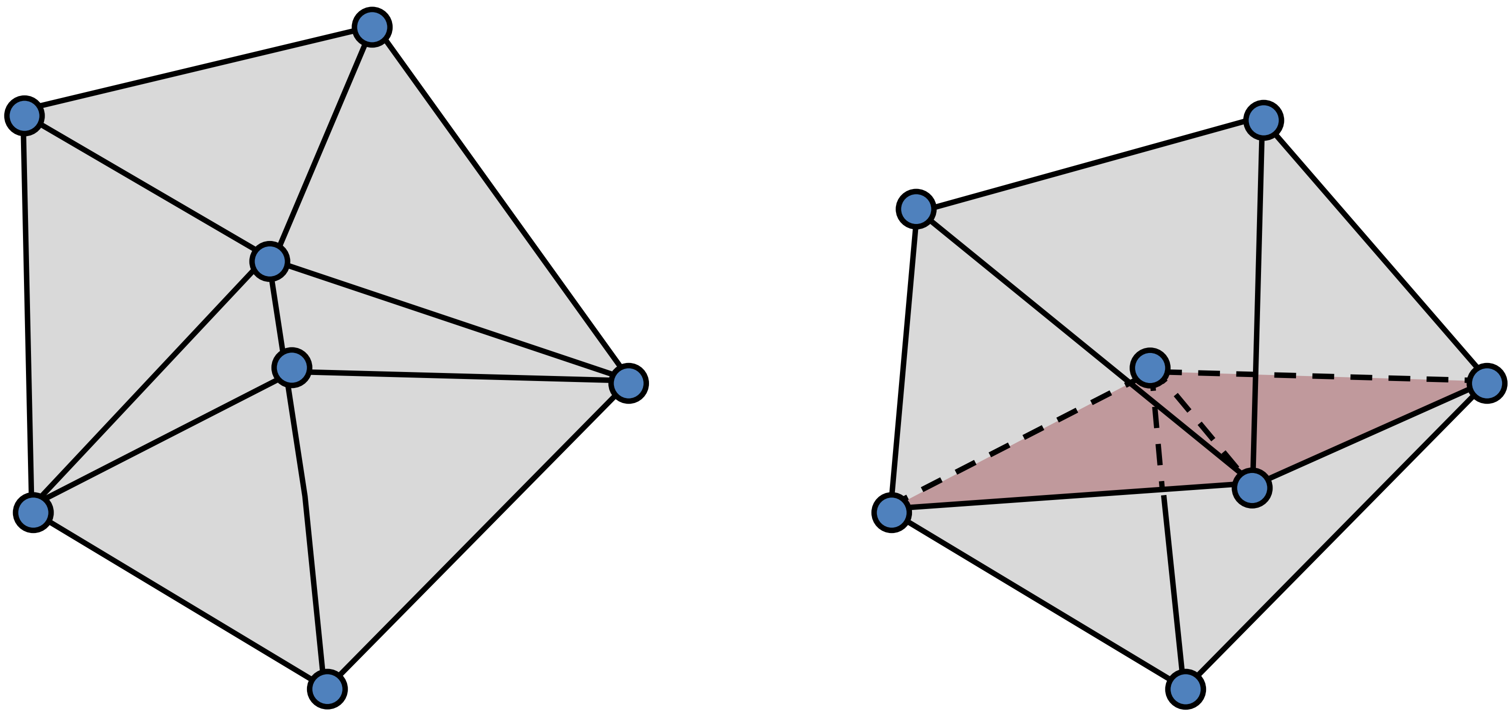
Minimal Stretch ✗



Good Parameterization



Local flips



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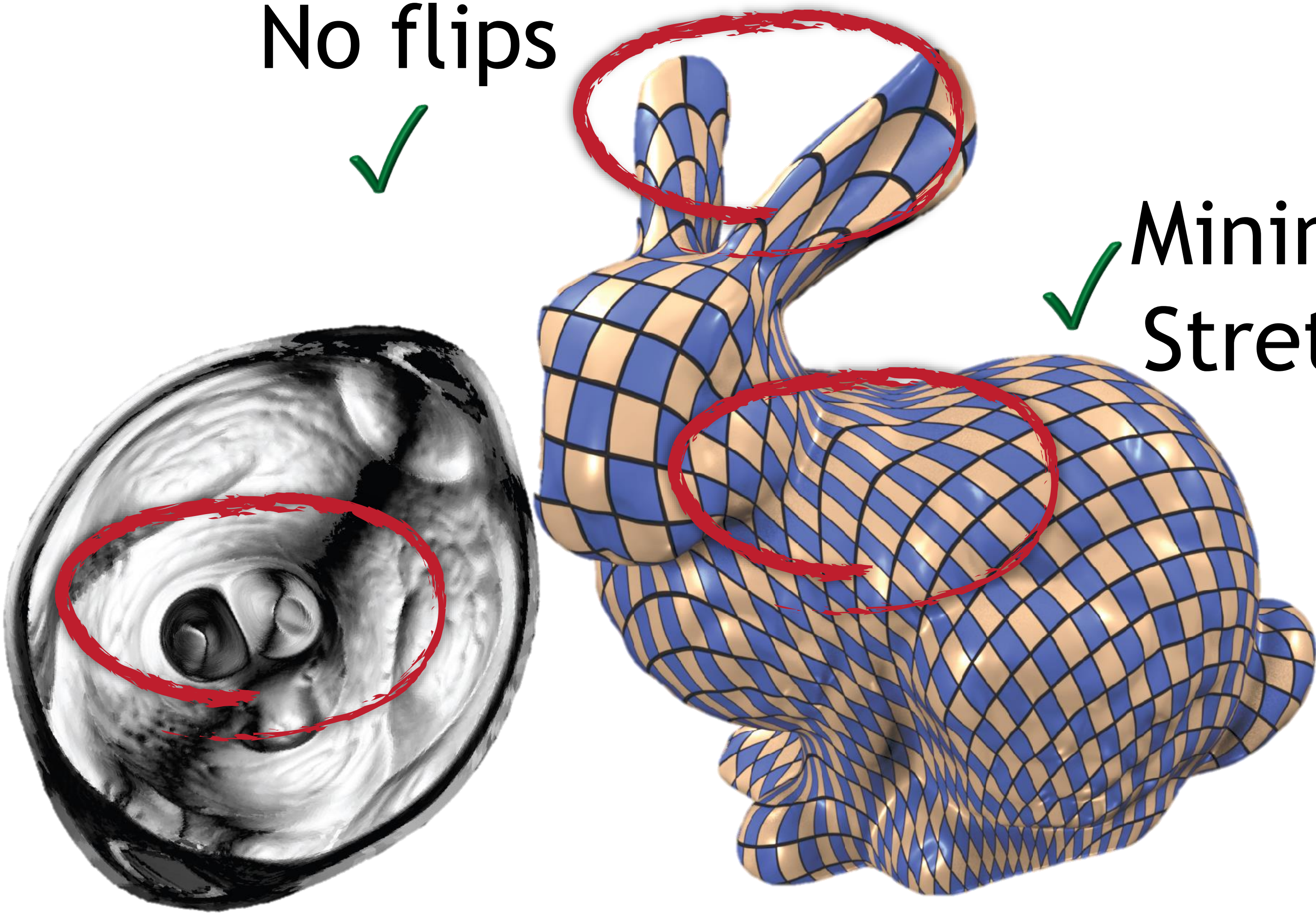
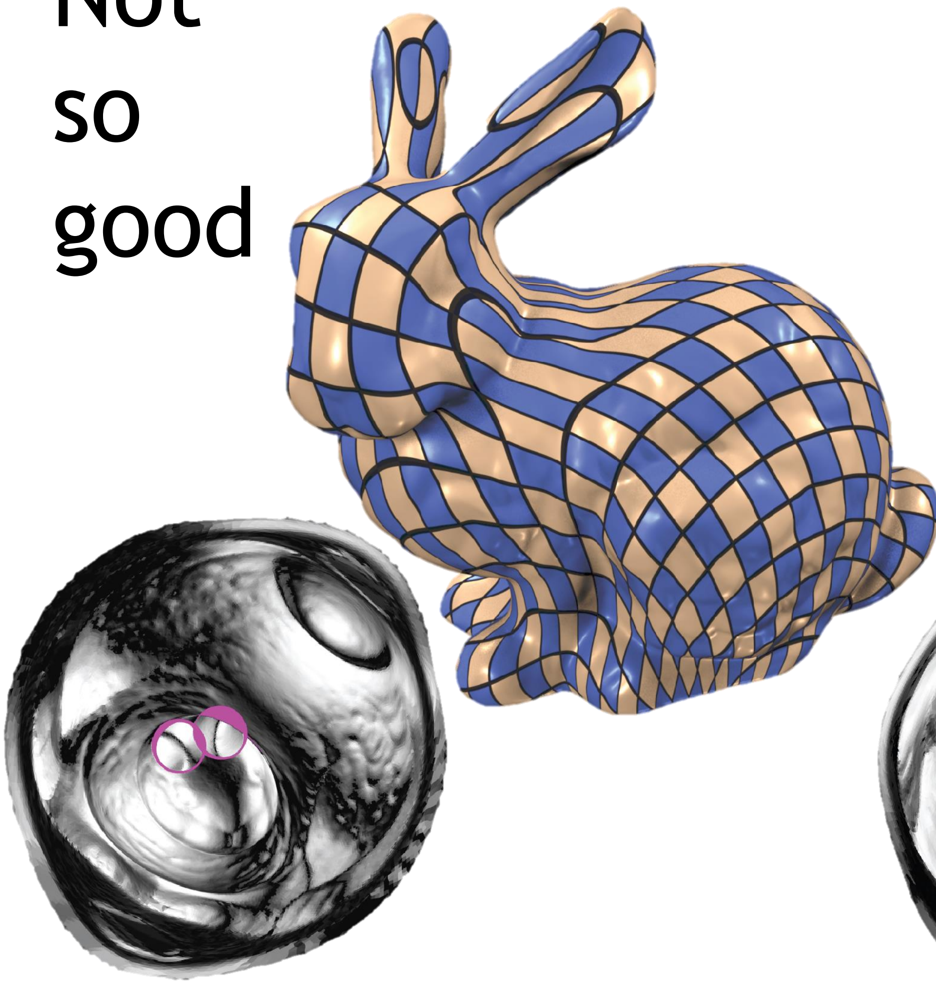
Good Parameterization

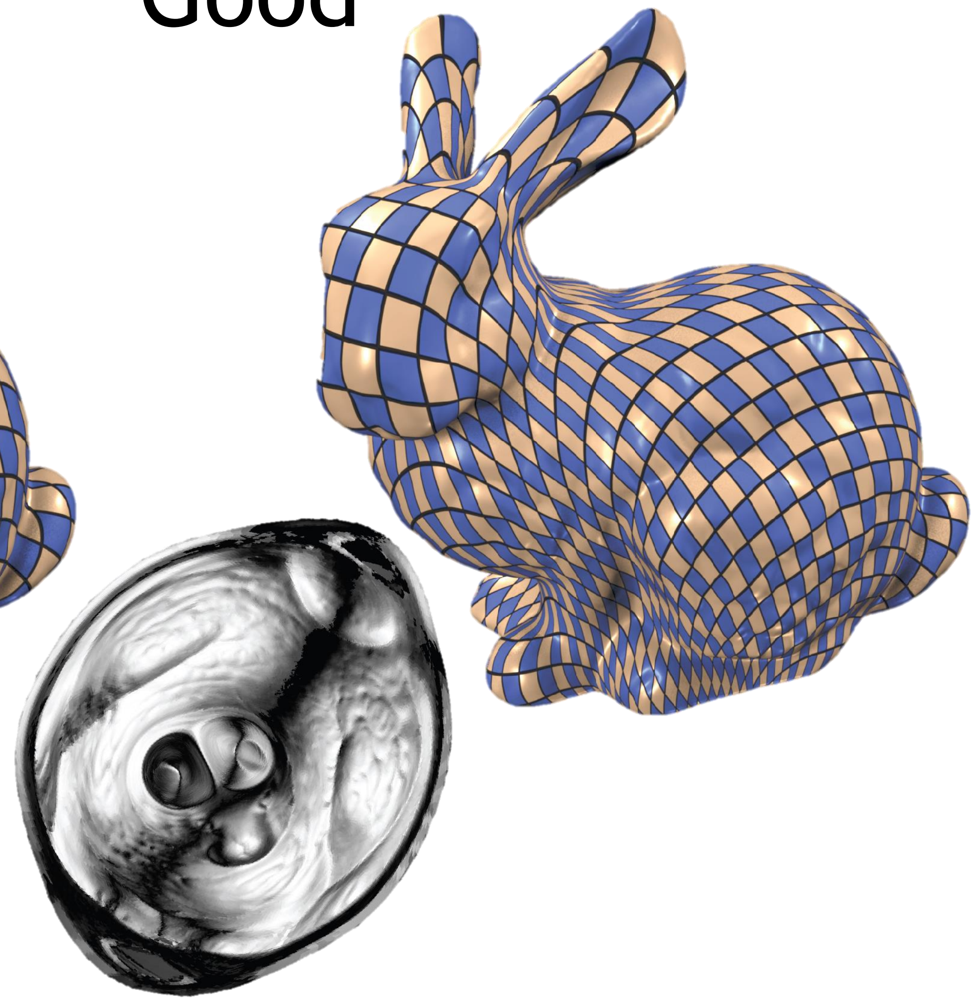


Good Parameterization

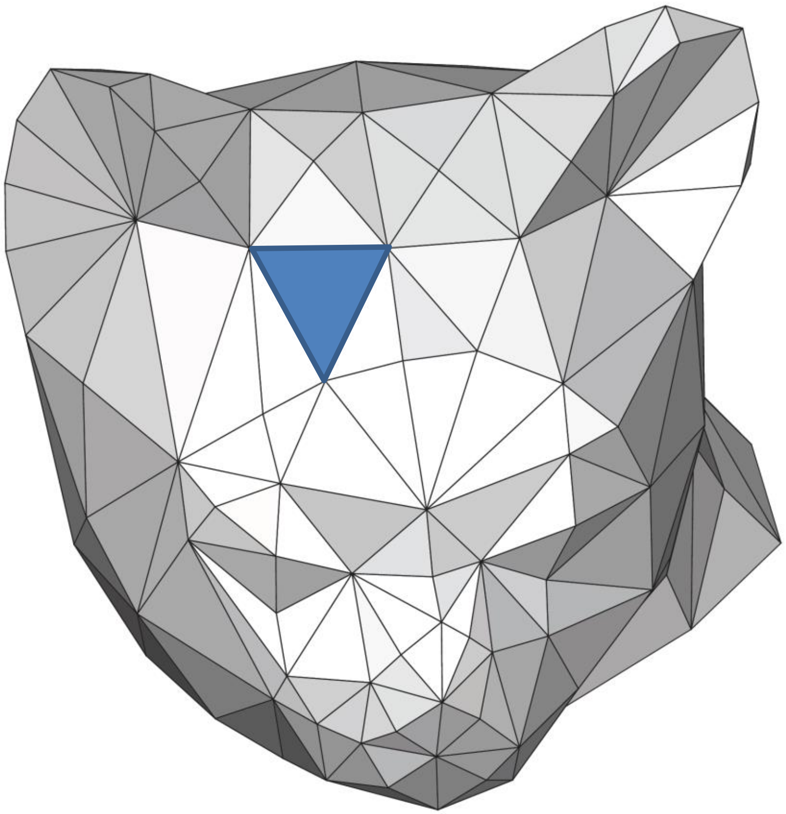
Not so good

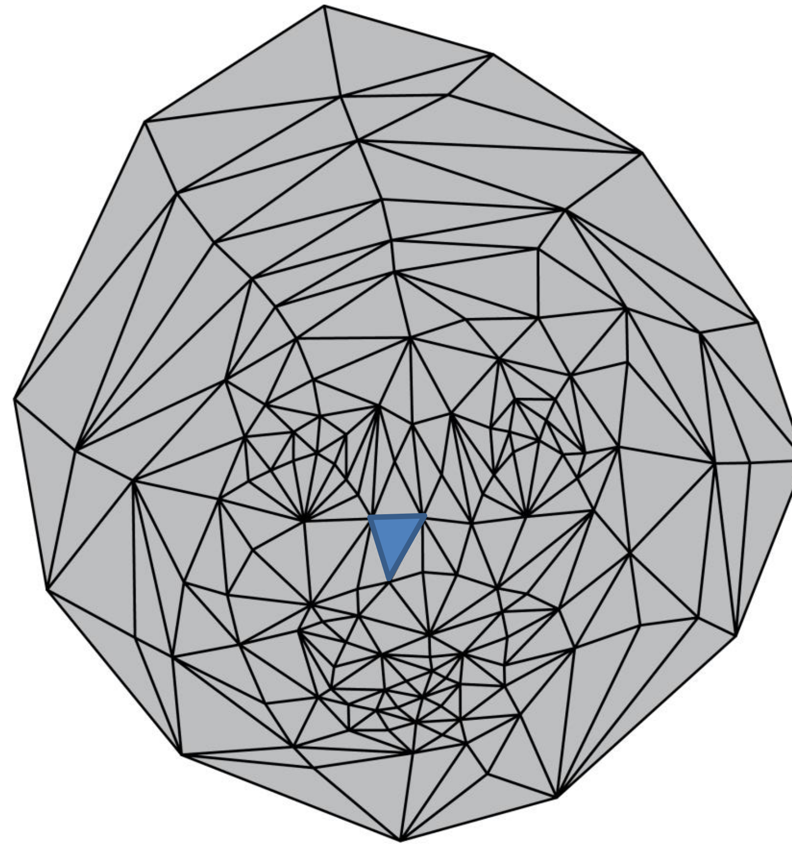


Good



Distortion on Triangle Meshes





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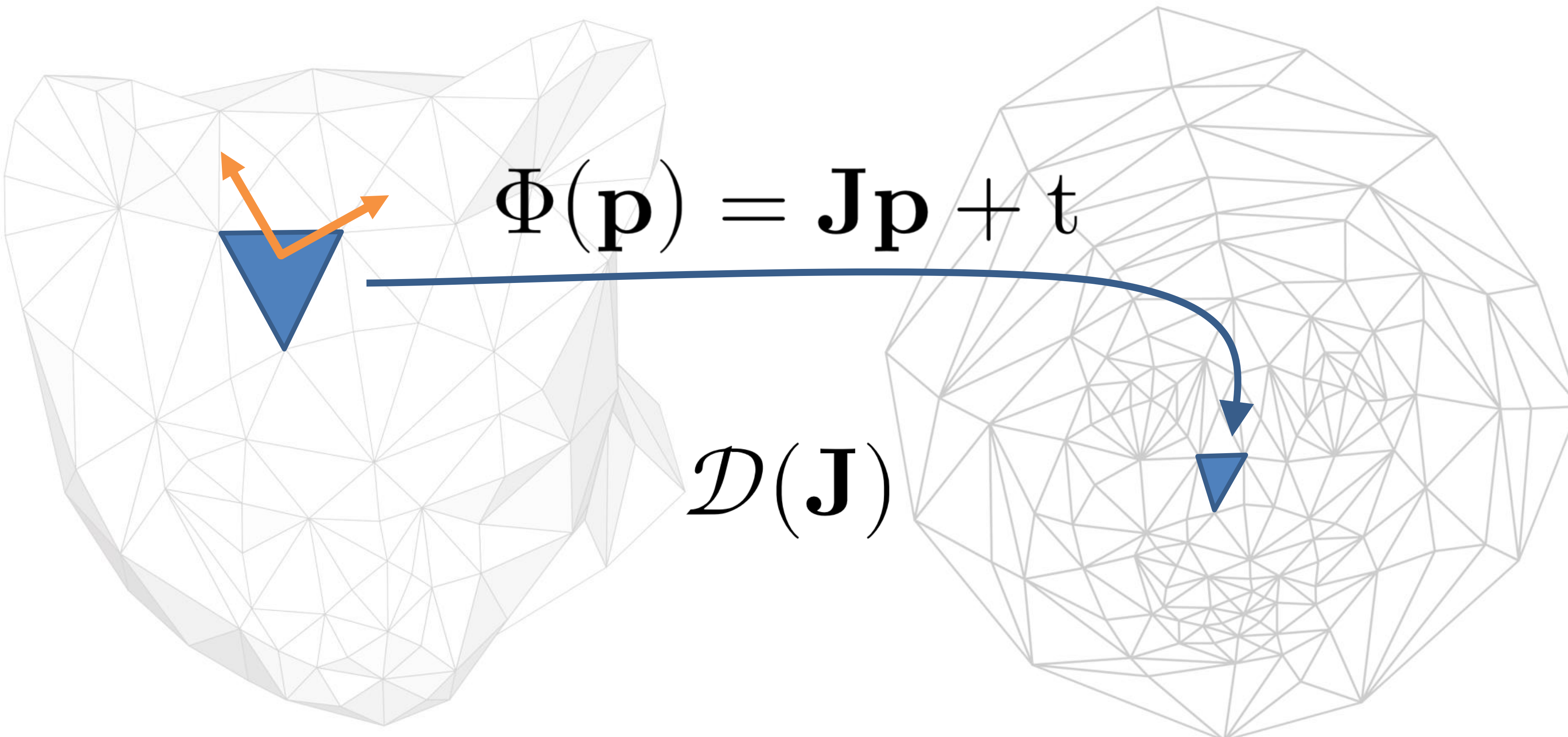
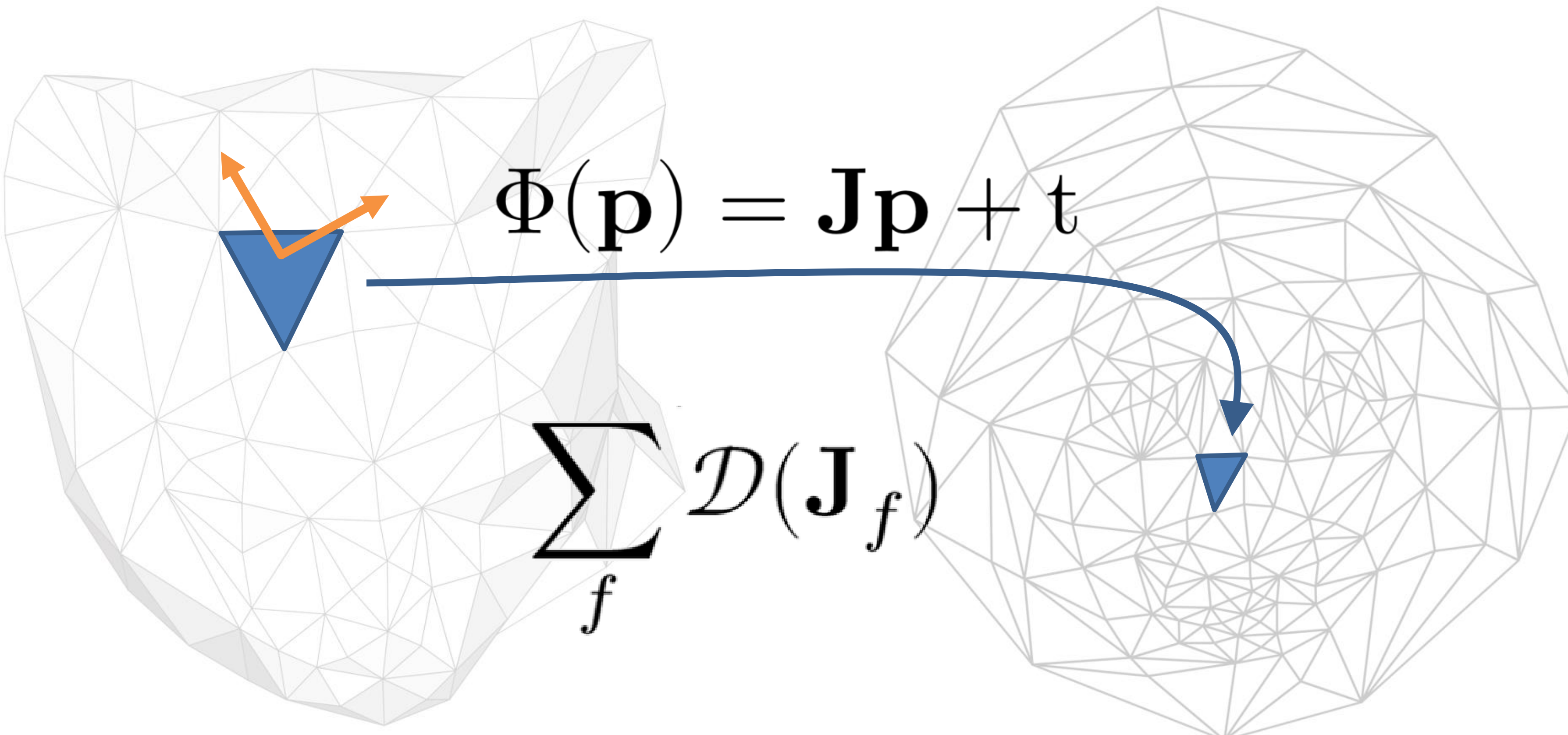
Distortion on Triangle Meshes



$$\Phi(\mathbf{p}) = \mathbf{J}\mathbf{p} + \mathrm{t}$$

$$\mathcal{D}(\mathbf{J})$$

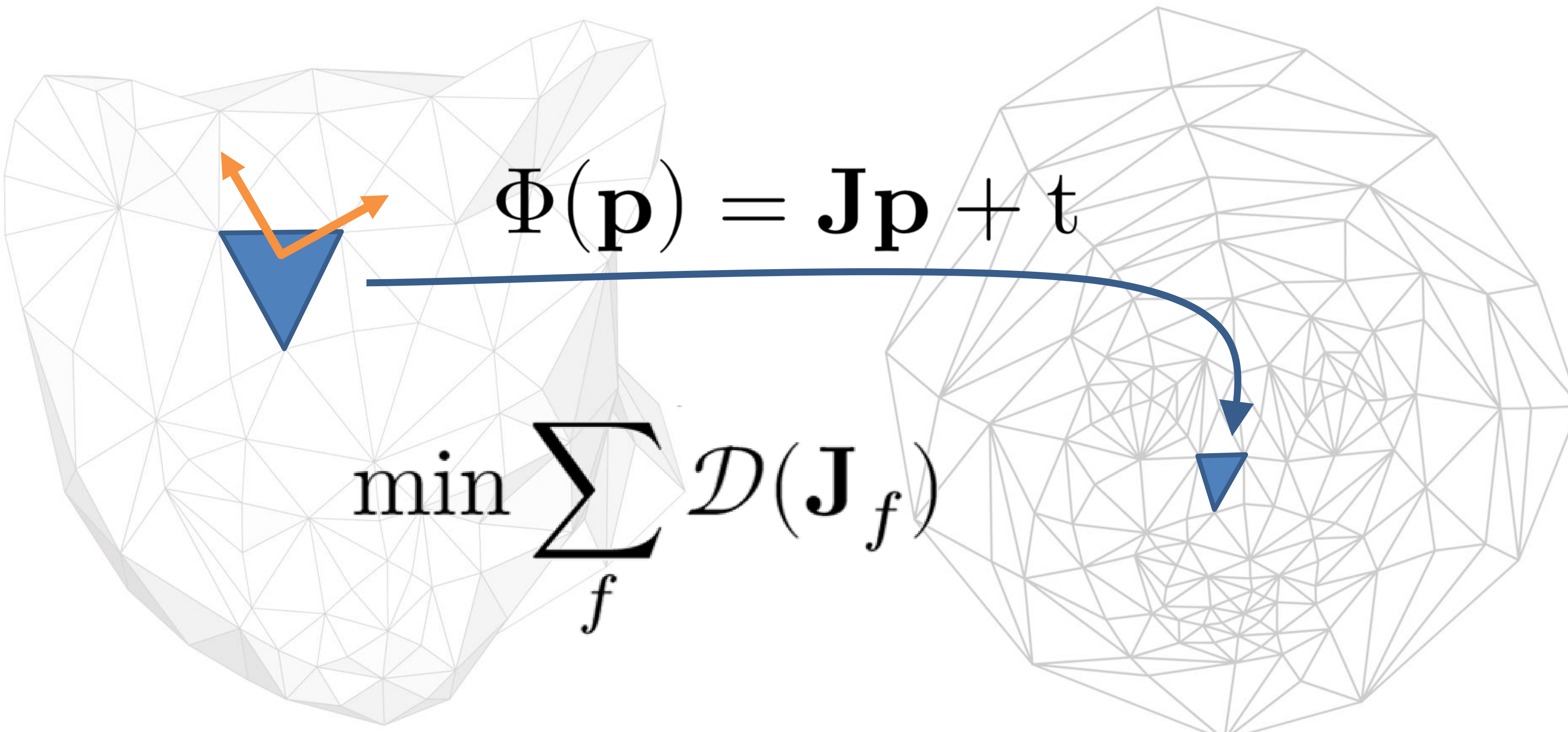
Distortion on Triangle Meshes



$$\Phi(\mathbf{p}) = \mathbf{J}\mathbf{p} + \mathrm{t}$$

$$\sum_f \mathcal{D}(\mathbf{J}_f)$$

Distortion on Triangle Meshes

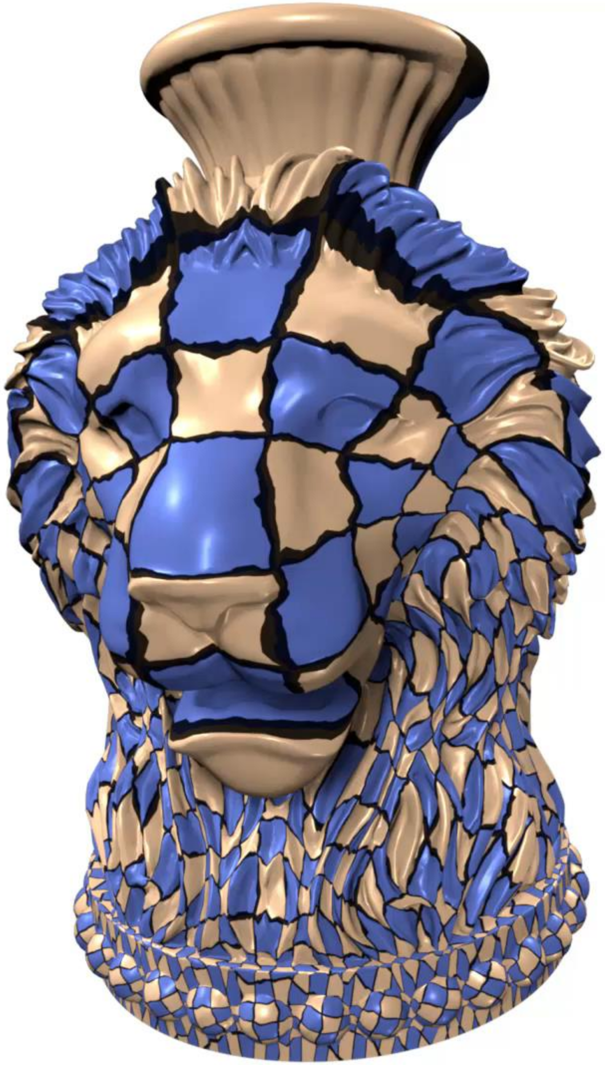


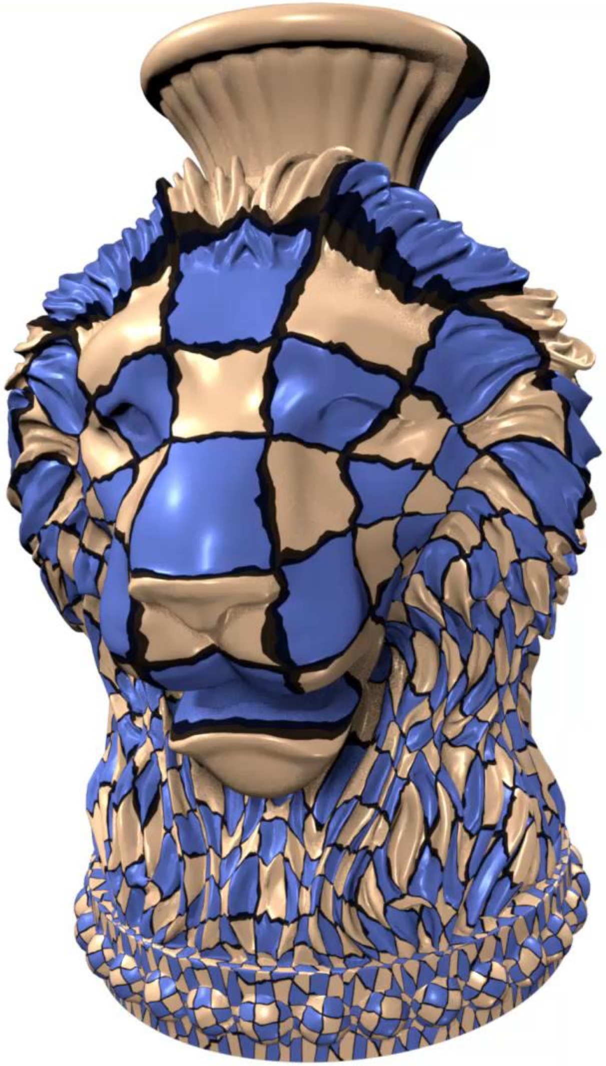
$$\Phi(\mathbf{p}) = \mathbf{J}\mathbf{p} + \mathrm{t}$$

$$\min \sum_f \mathcal{D}(\mathbf{J}_f)$$

Distortion types

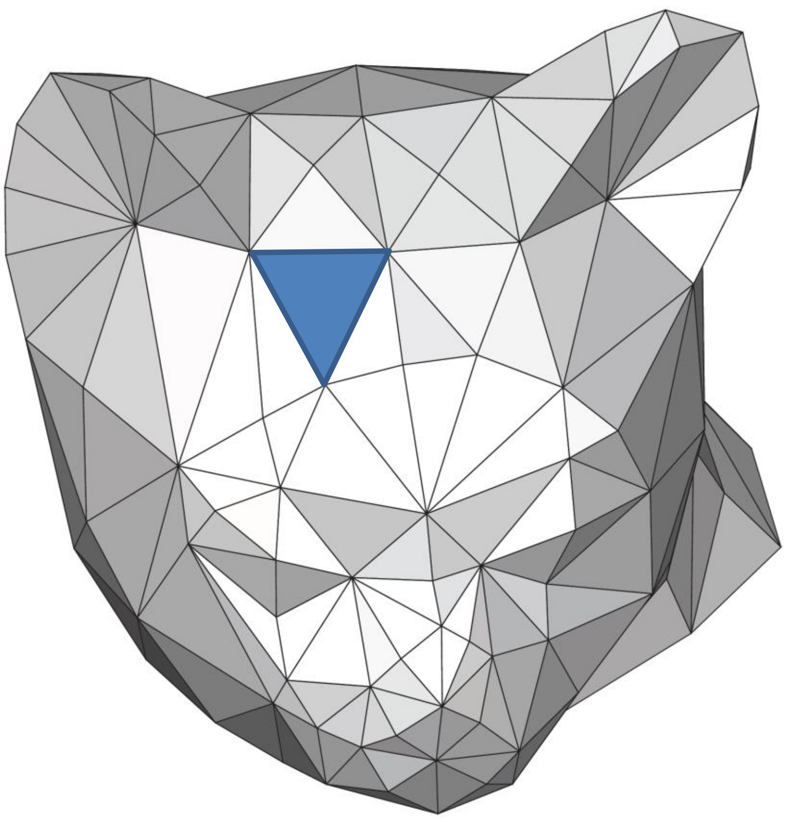
100K Faces: 3s

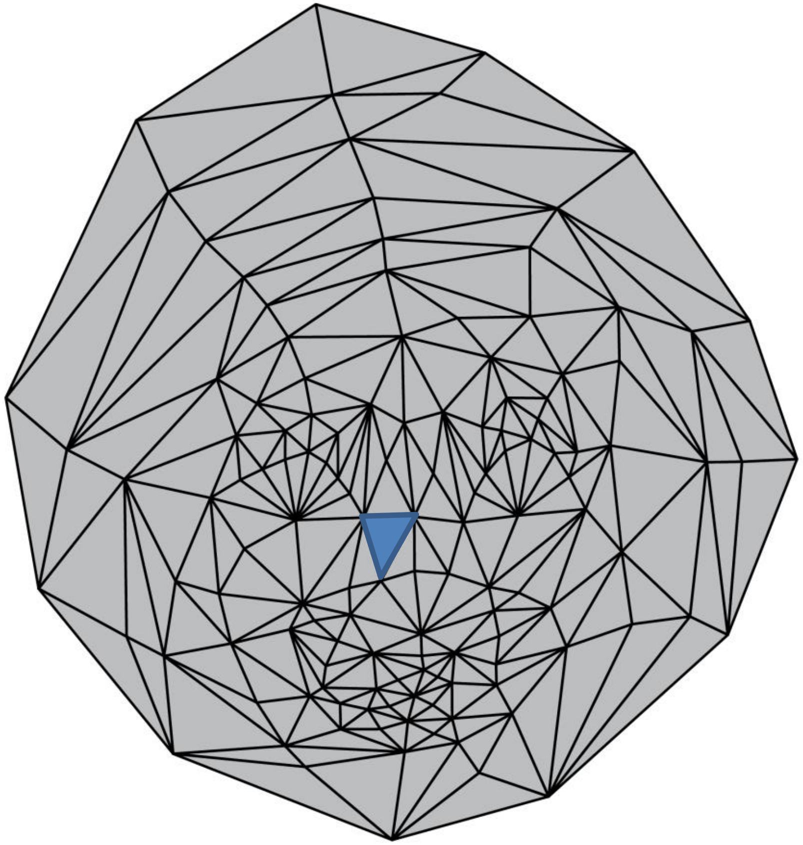




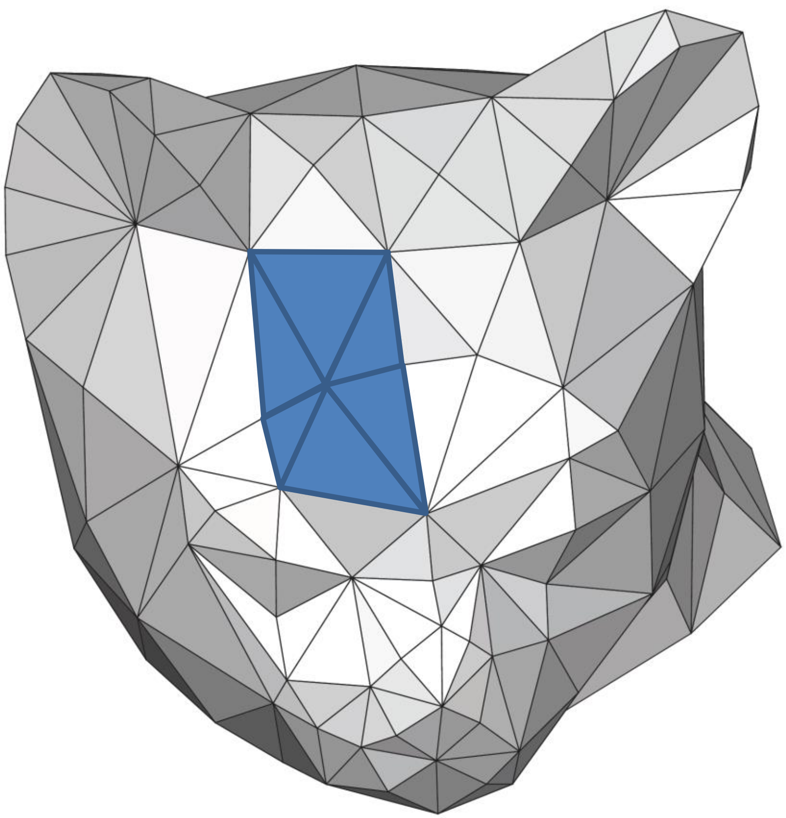
Conformal

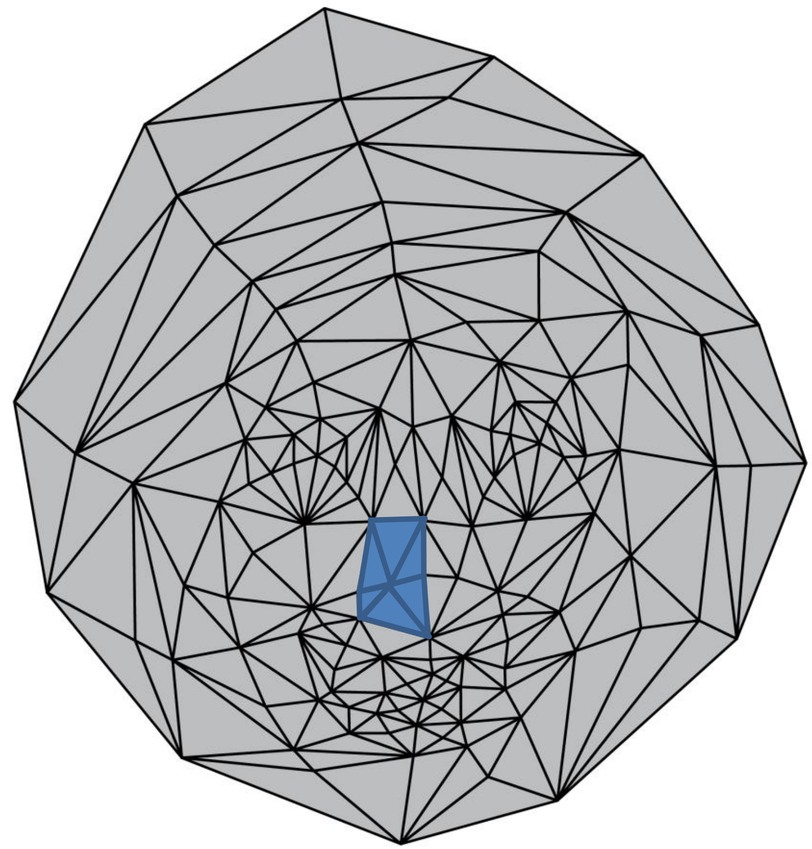
Convex Combination Maps





Convex Combination Maps

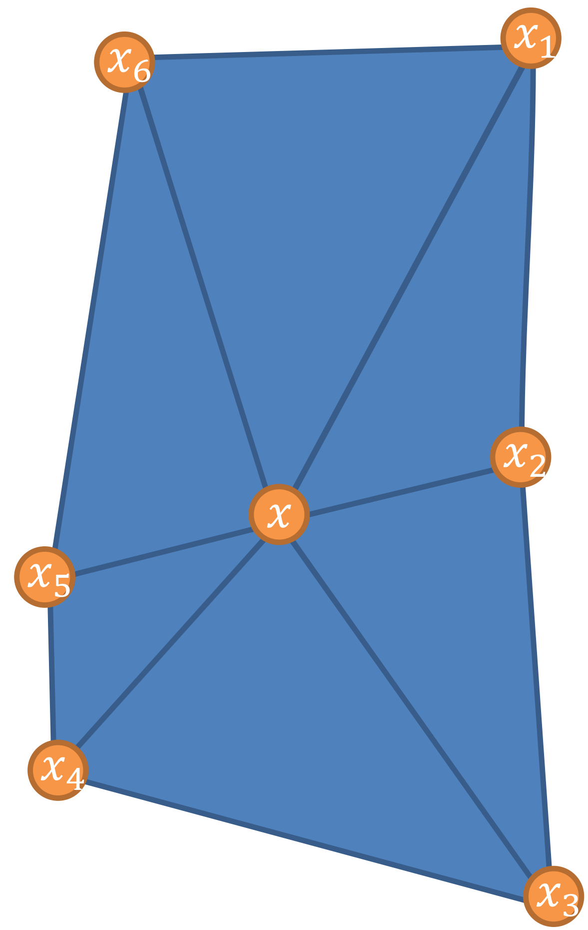




Convex Combination Maps

$$x = \sum w_i x_i$$

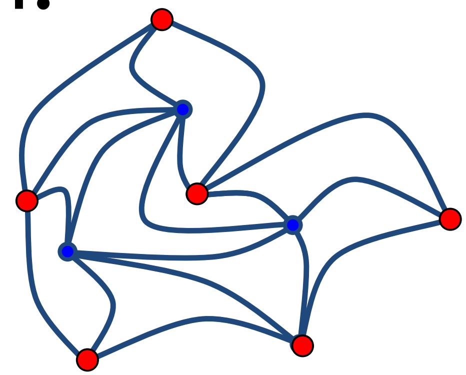
$$w_i > 0$$

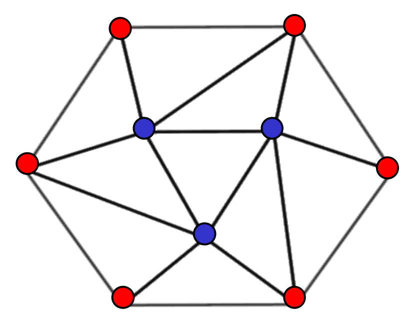


Convex Combination Maps

Fix 2D boundary to **convex** polygon.

Define drawing as a solution of





$$Wx = b_x \qquad Wy = b_y$$

$$w_{ij} = \begin{cases} > 0 & (i,j) \in E \\ -\sum_{j \neq i} w_{ij} & (i,i), i \notin B \\ 1 & (i,i), i \in B \\ 0 & otherwise \end{cases}$$

$$W \text{ is } symmetric: \; w_{ij} = w_{ji}$$

Weights w_{ij} control triangle shapes

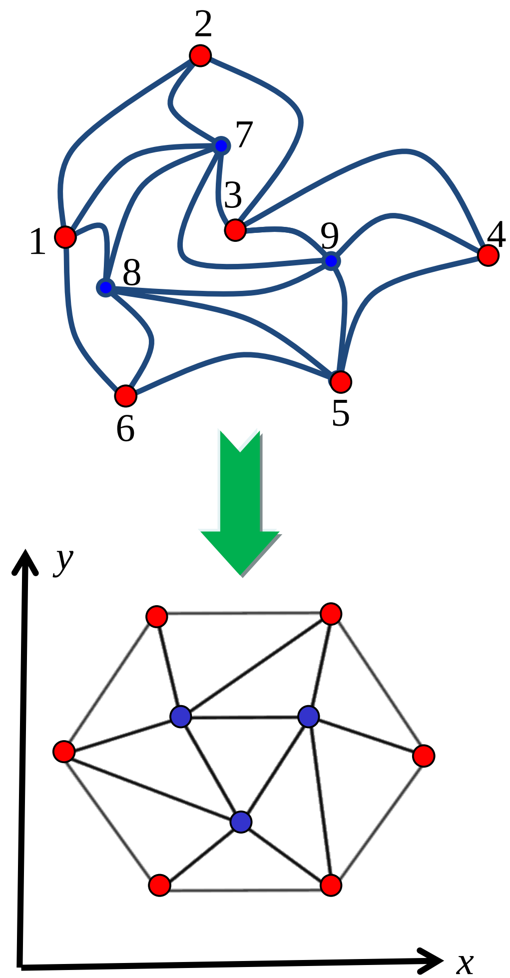
Guaranteed bijectivity

Example

$$w_{ij} = 1$$

Uniform Laplacian Matrix

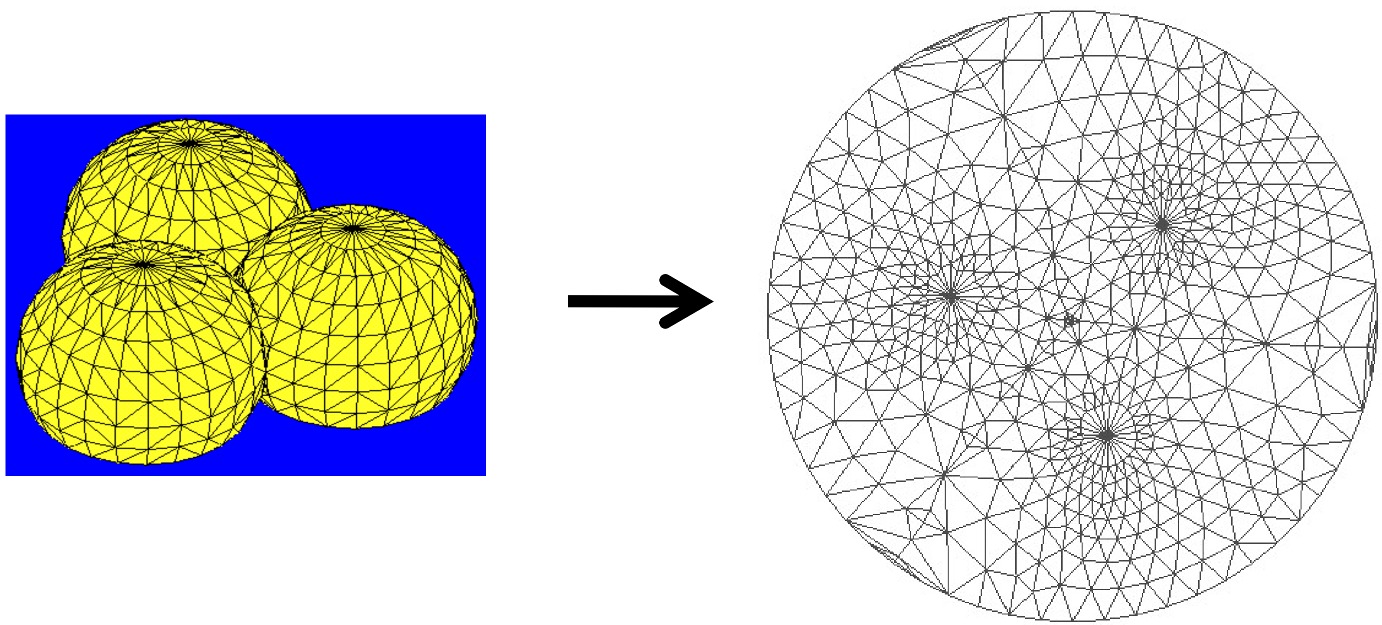
$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -5 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & -5 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -5 \end{pmatrix} \quad b_x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad b_y = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



Why it Works

Theorem (Tutte, Floater)

If $G \;=< V,E >$ is a 3-connected planar graph (triangular mesh) then any **convex combinataion map** has no intersections.



Harmonic Maps

f is *harmonic* if it satisfies (for each coordinate):

$$\Delta_M f = 0$$

Minimizes the Dirichlet energy

$$E_D(f) = \frac{1}{2}\int_S \|\nabla_S f\|^2$$

Easier to compute than conformal

Does not preserve angles

May not be bijective

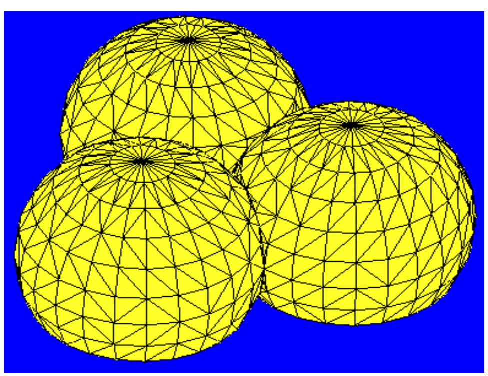
Uniform Weights

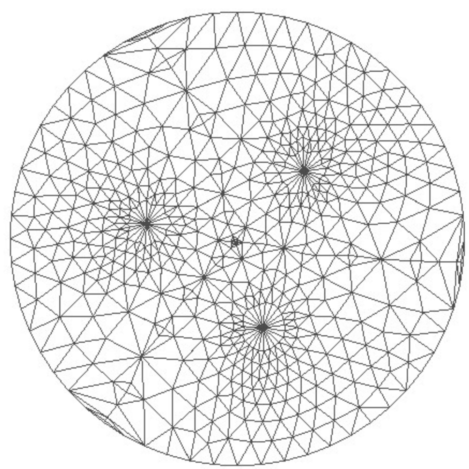
$$\boxed{w_{ij} = 1}$$

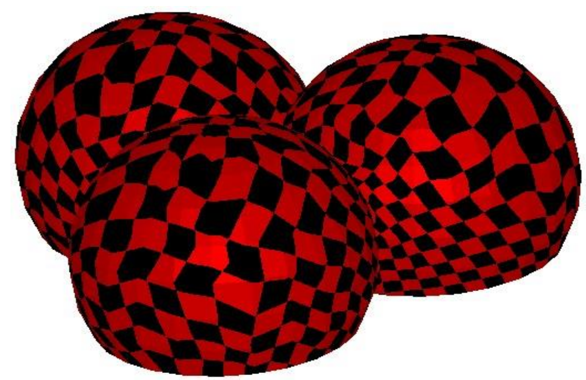
No shape information – “equilateral” triangles

Fastest to compute and solve

Not 2D reproducible

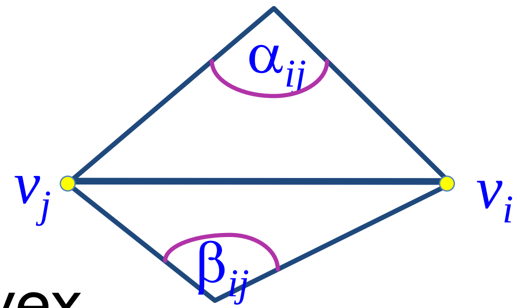






Harmonic Weights

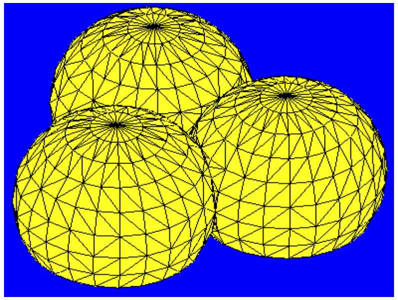
$$w_{ij} = \frac{\cot(\alpha_{ij}) + \cot(\beta_{ij})}{2}$$

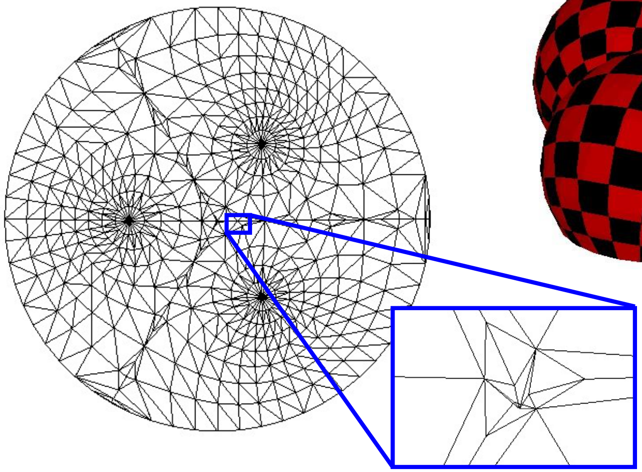


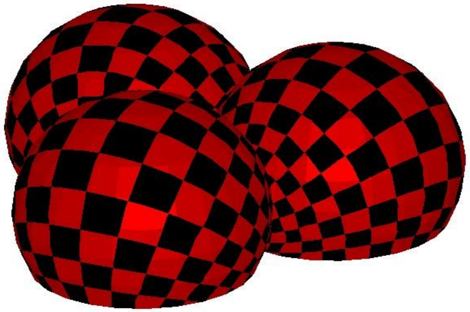
Weights can be negative – not always convex

Weights depend only on angles - close to conformal

2D reproducible







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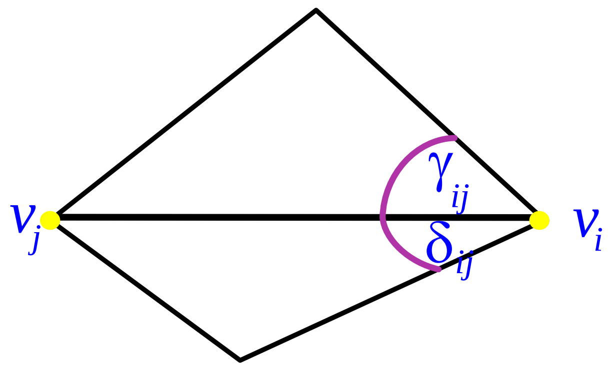
Mean-Value Weights

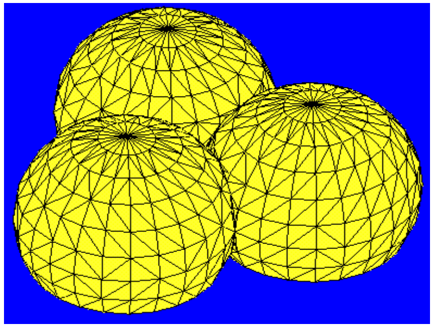
$$w_{ij} = \frac{\tan(\gamma_{ij}/2) + \tan(\delta_{ij}/2)}{2\,\|v_i - v_j\|}$$

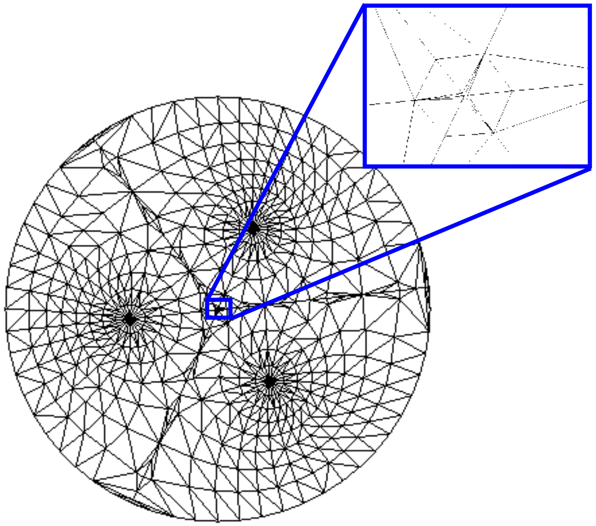


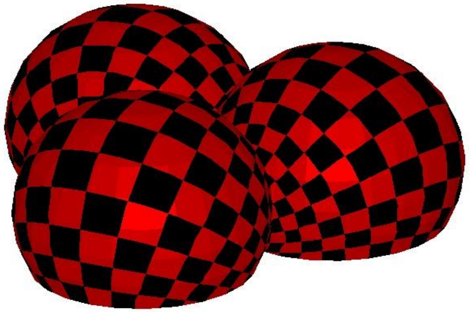
Result visually similar to harmonic

No negative weights – **always** valid

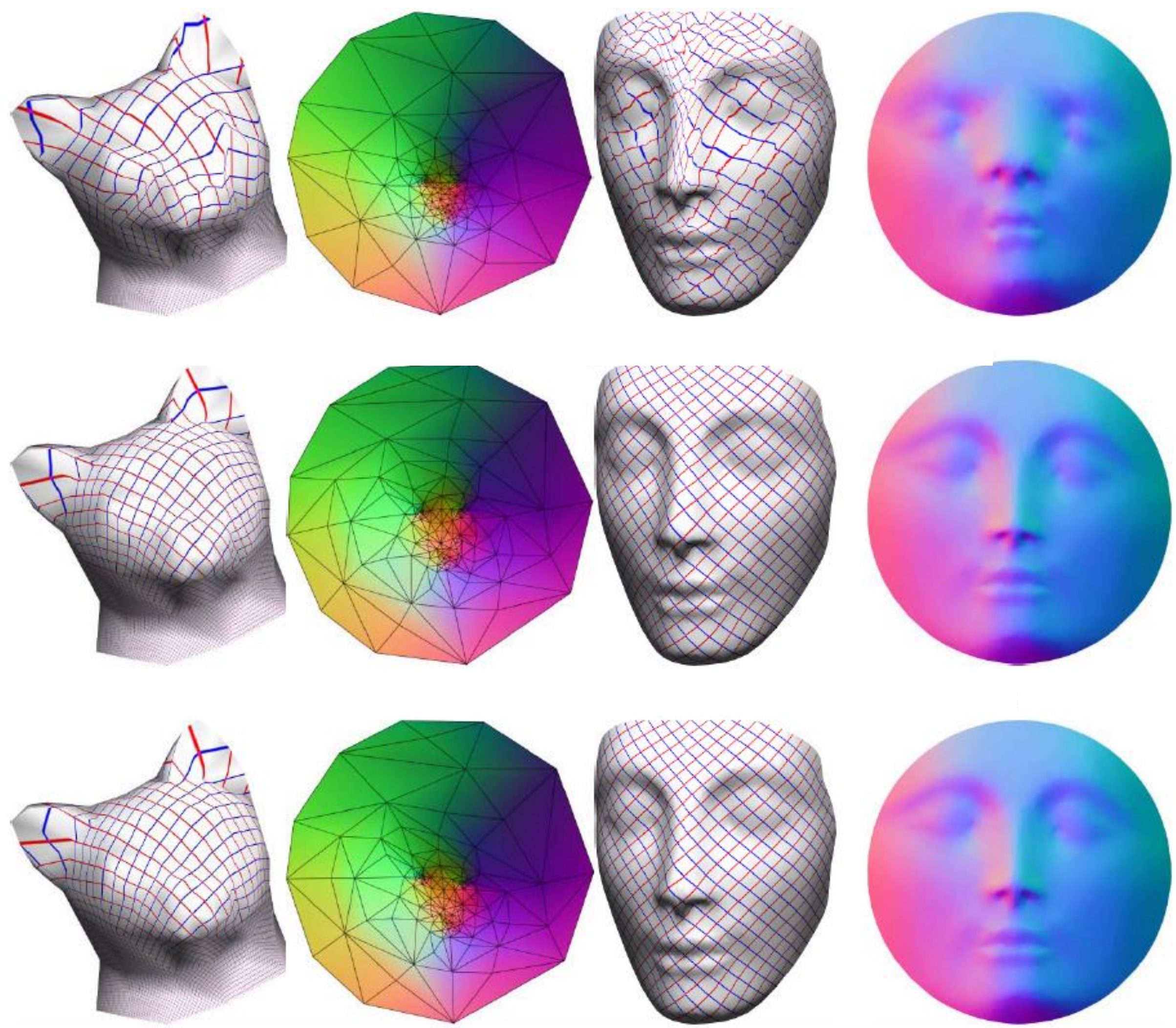
2D reproducible







Example



uniform

harmonic

mean-value

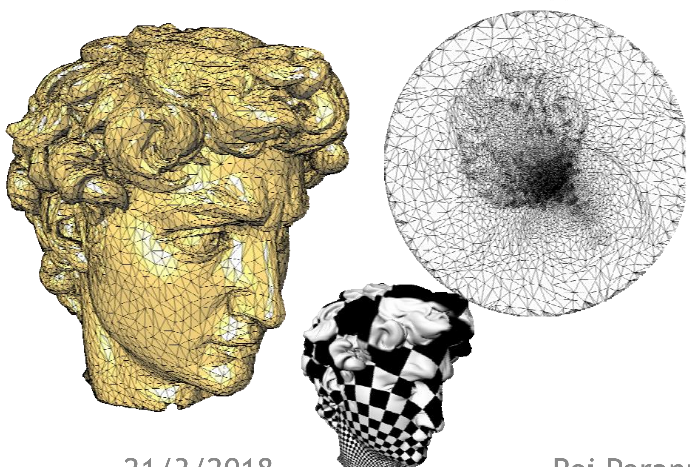
Fixing the Boundary

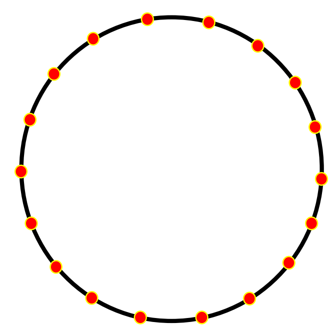
Simple convex shape (triangle, square, circle)

Distribute points on boundary

- Use chord length parameterization

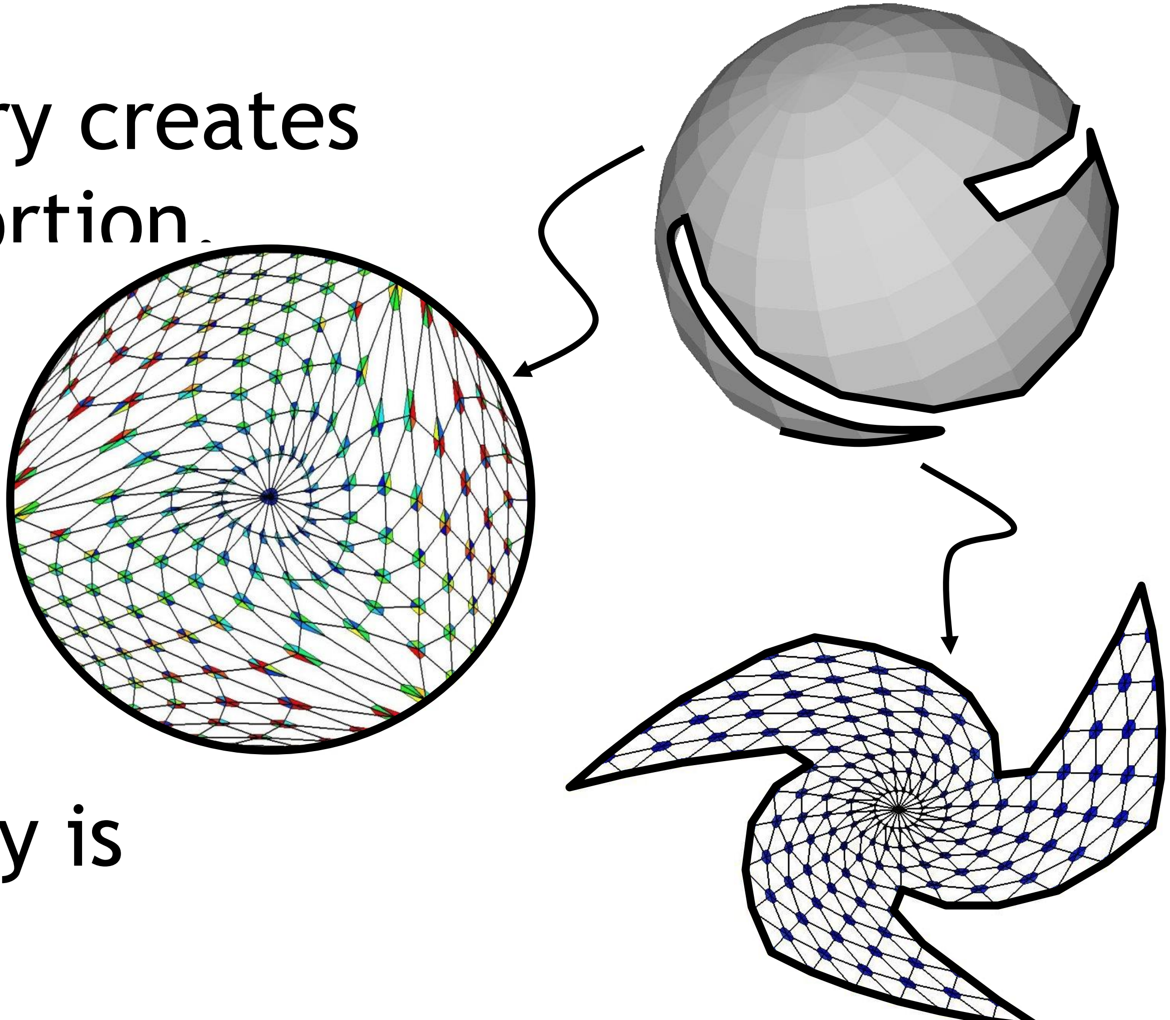
Fixed boundary can create high distortion





Non-Convex Boundary

Convex boundary creates significant distortion.



"Free" boundary is better.

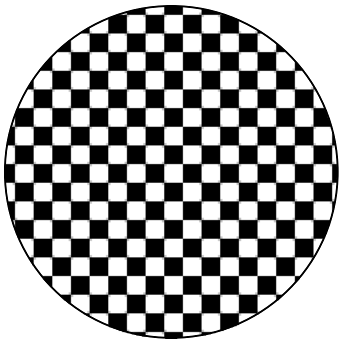
Relevant literature

Least Squares Conformal Maps
(**LSCM**), Lévy et al., SIGGRAPH 2002

A Local/Global Approach to Mesh Parameterization
(**ARAP**), Liu et al., SGP 2008

LSCM - “Conformal” ARAP - “Isometric”

Practical Distortion Minimization

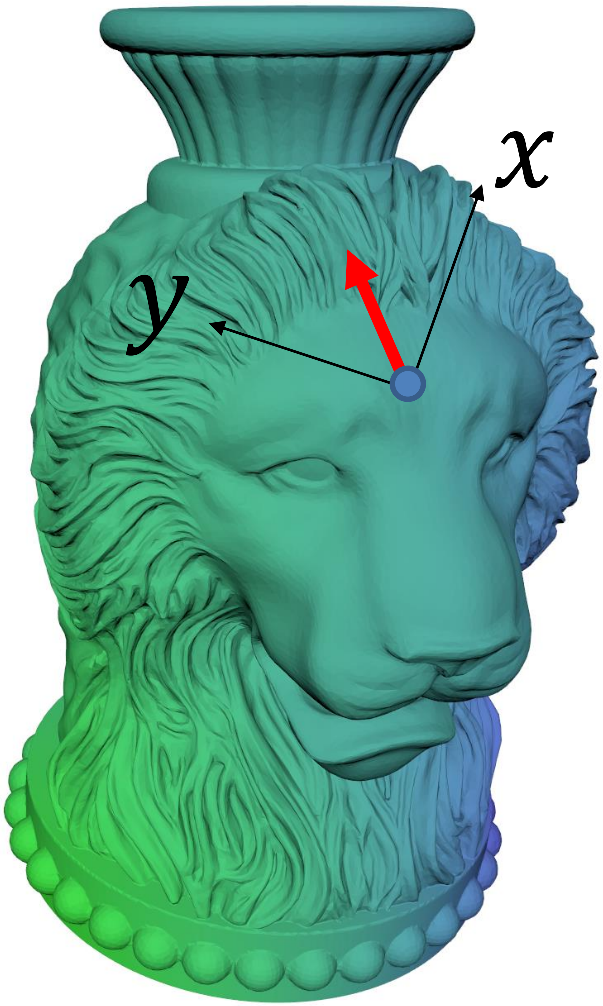

Texture map

$$\underset{(u_1,v_1),\ldots,(u_n,v_n)}{\operatorname{argmin}} \sum_T E(T)$$

Gradients on surfaces

- Like the Euclidean gradient
- Arrow pointing in the steepest direction
- libIGL tutorial 204

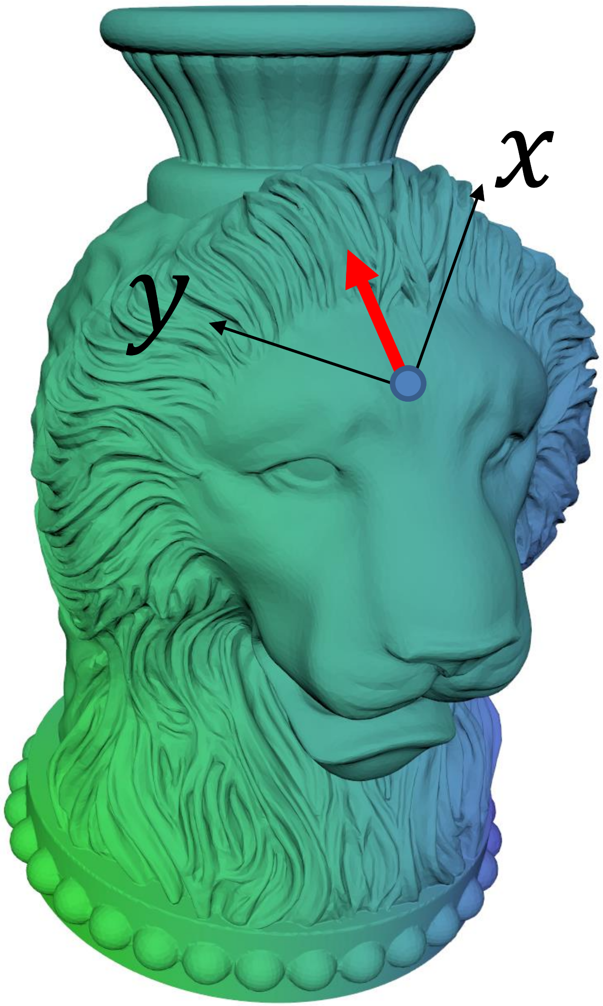
Gradients on surfaces



Arrows are tangent to the surface

described in *local frames*

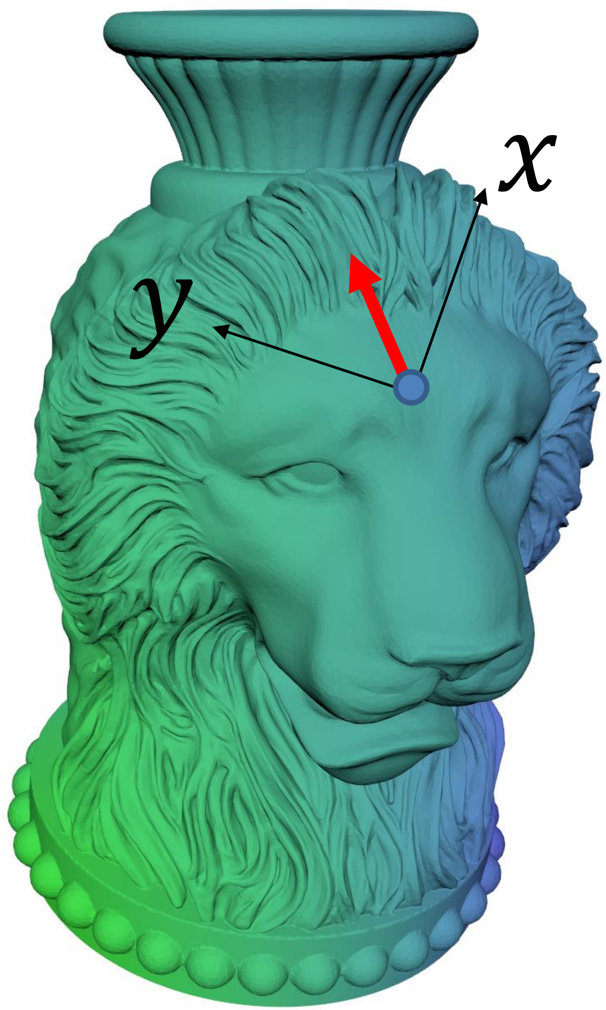
Gradients on surfaces



Arrows are tangent to the surface

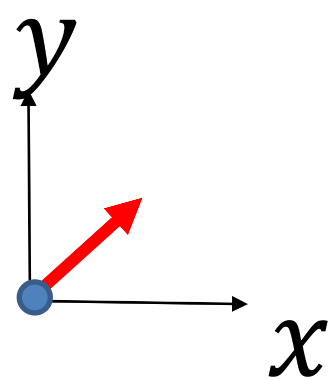
described in *local frames*

Gradients on surfaces



Arrows are tangent to the surface

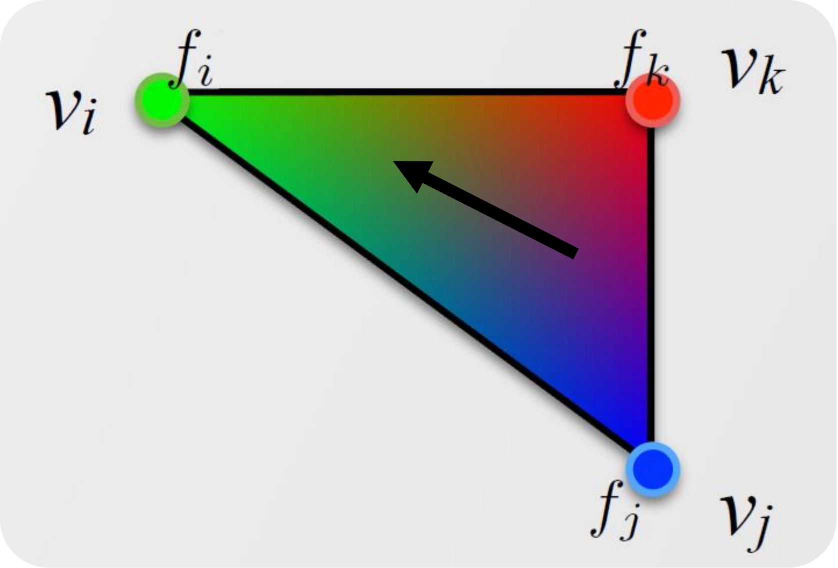
described in *local frames*

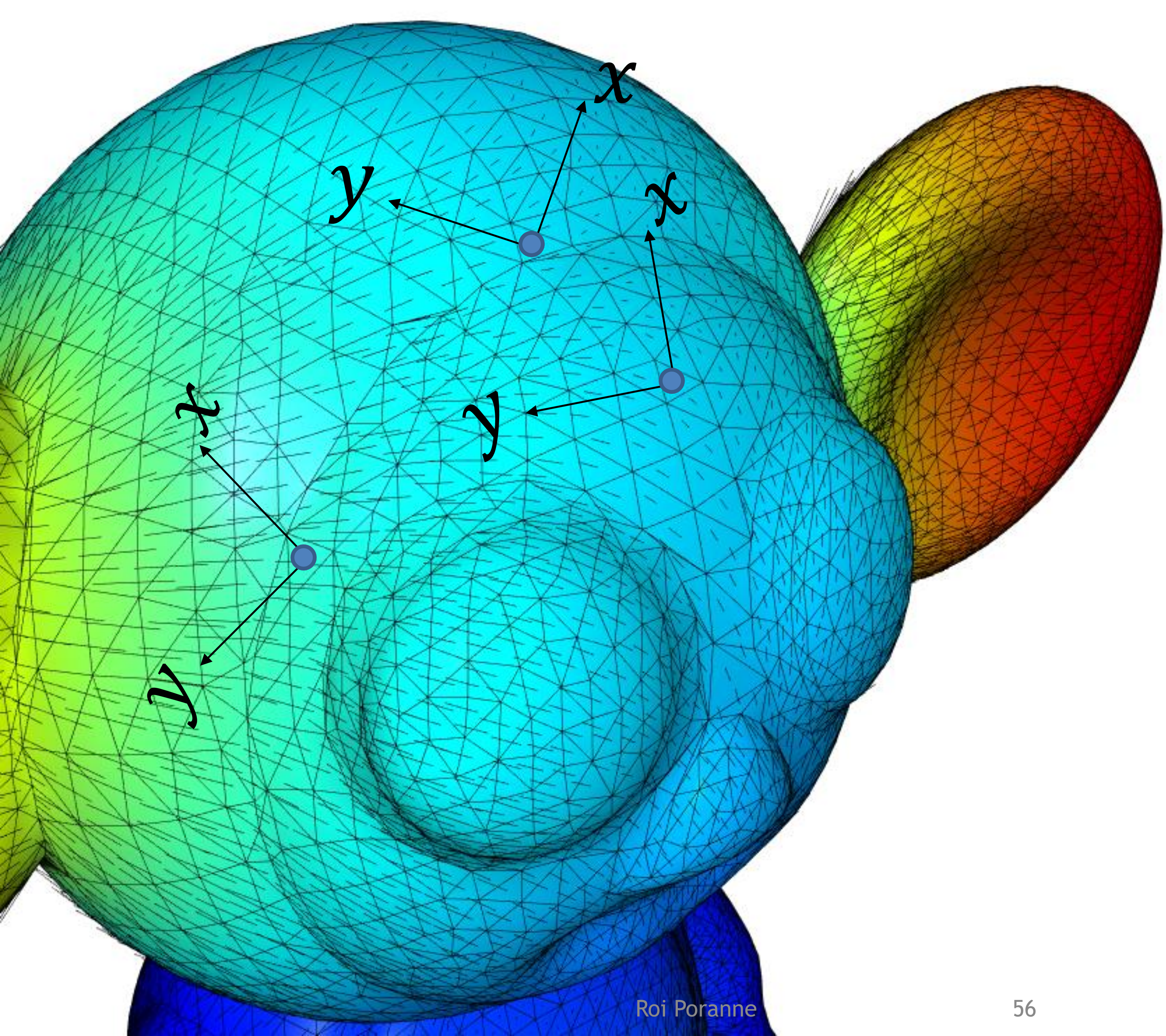


Discrete Gradients

Functions are interpolated linearly on each face

The gradient is an arrow on a face





x
y
x
y
x
y

The Gradient Matrices

$$D_x \qquad D_y$$

Given a local frame per face compute the directional derivative w.r.t. the frames

$$\underbrace{\nabla \overbrace{f}^{\#V \times 1}}_{\#F \times 2} = \underbrace{\left(D_x f, \overbrace{D_y}^{\#F \times \#V} f\right)}_{\#F \times 2}$$

Example: Dirichlet energy

$$\min \sum_f \mathcal{D}(\mathbf{J}_f) \qquad \mathcal{D}(\mathbf{J}) = A_f \|\mathbf{J}\|_F^2$$

$$\sum_f A_f \|\mathbf{J}\|_F^2 =$$

$$= \Big\| \underbrace{A^{0.5}\big(D_x u, D_y u, D_x v, D_y v\big)}_{\#F \times 4} \Big\|_F^2$$

$$A = \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{\#F} \end{bmatrix}$$

Example: Dirichlet energy

$$= \left\| A^{0.5}\left(D_x u, D_y u, D_x v, D_y v\right) \right\|_F^2$$

$$= \left\| \qquad\qquad \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2$$

Example: Dirichlet energy

$$= \left\| A^{0.5} \left(D_x u, D_y u, D_x v, D_y v \right) \right\|_F^2$$

$$= \left\| \begin{pmatrix} A^{0.5} D_x & 0 \\ A^{0.5} D_y & 0 \\ 0 & A^{0.5} D_x \\ 0 & A^{0.5} D_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2$$

$$= \left\| \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2$$

Example: Dirichlet energy

$$\left\| \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2$$

$$\min_{u,v} \left\| \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2$$

Example: Dirichlet energy

$$\left\|\mathcal{A}\begin{pmatrix}u\\v\end{pmatrix}\right\|^2$$

$$\min_{u,v}\left\|\mathcal{A}\begin{pmatrix}u\\v\end{pmatrix}\right\|^2 \rightarrow \text{solve } \mathcal{A}^{\mathrm{T}}\mathcal{A}\begin{pmatrix}u\\v\end{pmatrix}=0$$

$$\mathcal{A}^{\mathrm{T}}\mathcal{A}=$$
$$\begin{pmatrix} D_x^T A D_x + D_y^T A D_y & 0 \\ 0 & D_x^T A D_x + D_y^T A D_y \end{pmatrix}$$

Example: Dirichlet energy

$$\mathcal{A}^{\mathrm{T}}\mathcal{A} = \begin{pmatrix} D_x^T A D_x + D_y^T A D_y & 0 \\ 0 & D_x^T A D_x + D_y^T A D_y \end{pmatrix}$$

$$\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$$

Cotangent Laplacian

In the assignment

- Conformal – LSCM

$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} + \mathbf{J}^{\mathrm{T}} - (tr\,\mathbf{J})\,\mathbf{I} \right\|_F^2$$

- Isometric – ARAP

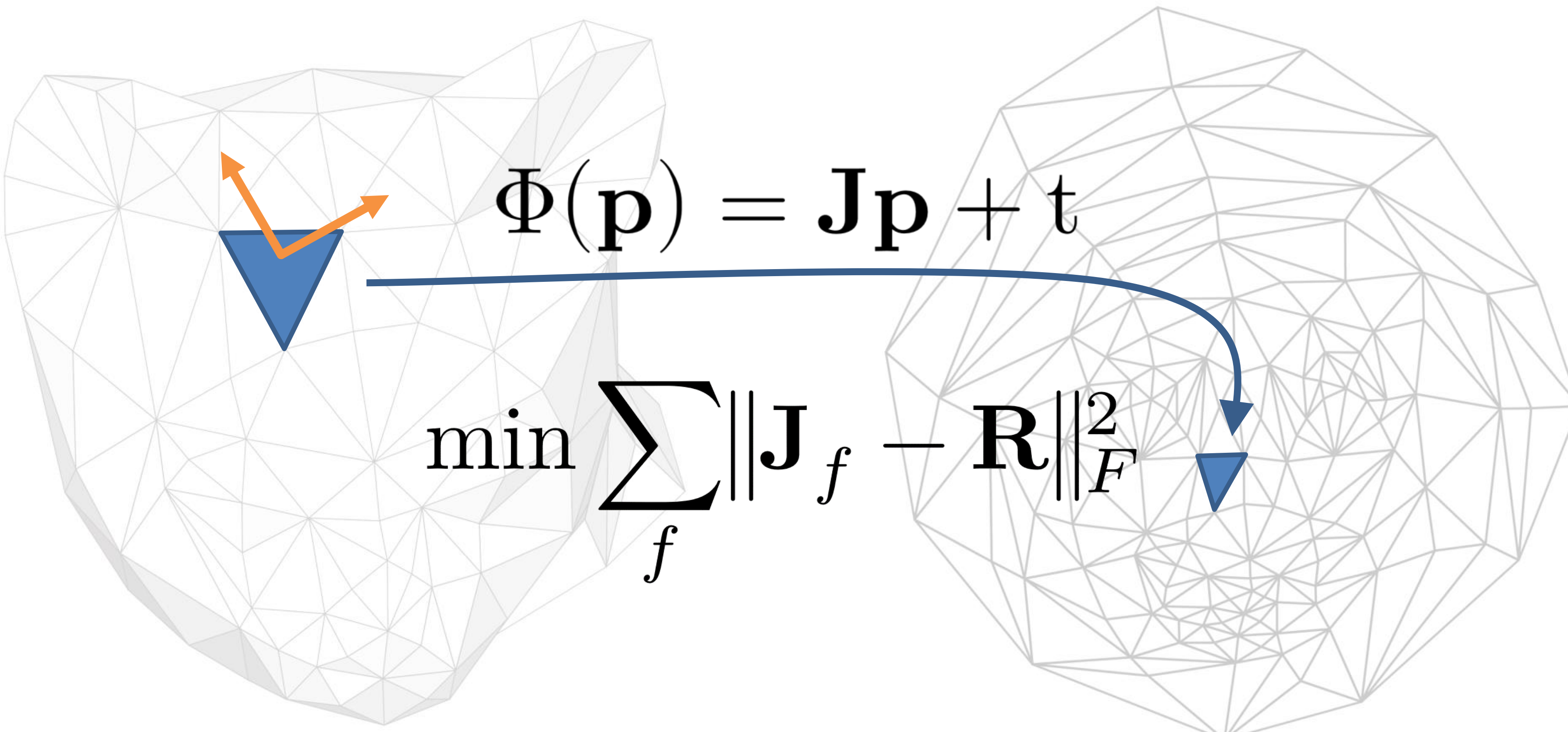
$$\mathcal{D}(\mathbf{J}) = \left\| \mathbf{J} - \mathrm{R} \right\|_F^2$$

Bonus

- ~~Authalic - area presereing~~

$$\cancel{\mathcal{D}(\mathbf{J}) = (\det \mathbf{J} - 1)^2}$$

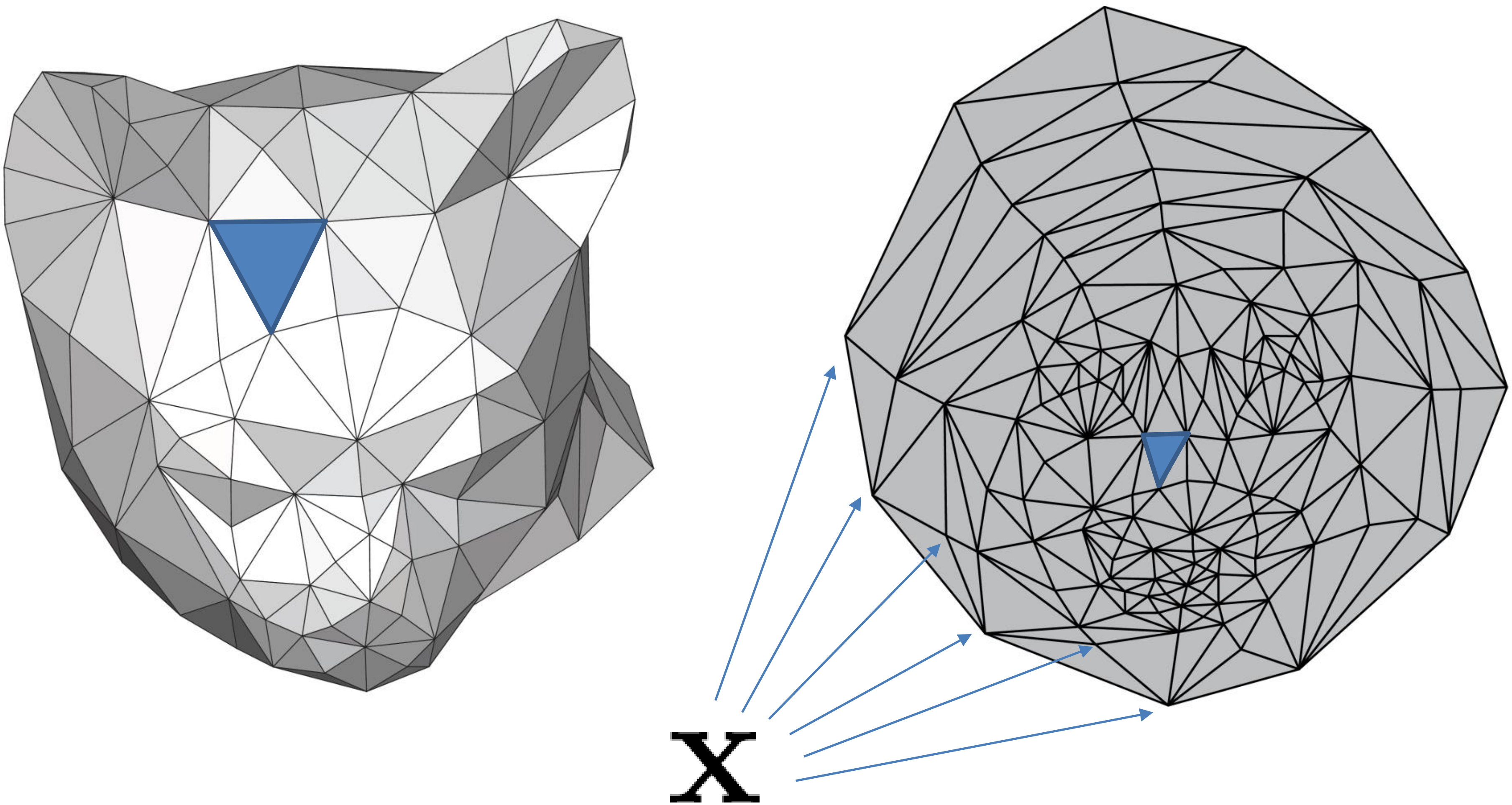
As Rigid As Possible



$$\Phi(\mathbf{p}) = \mathbf{J}\mathbf{p} + t$$

$$\min \sum_f \|\mathbf{J}_f - \mathbf{R}\|_F^2$$

As Rigid As Possible Measures



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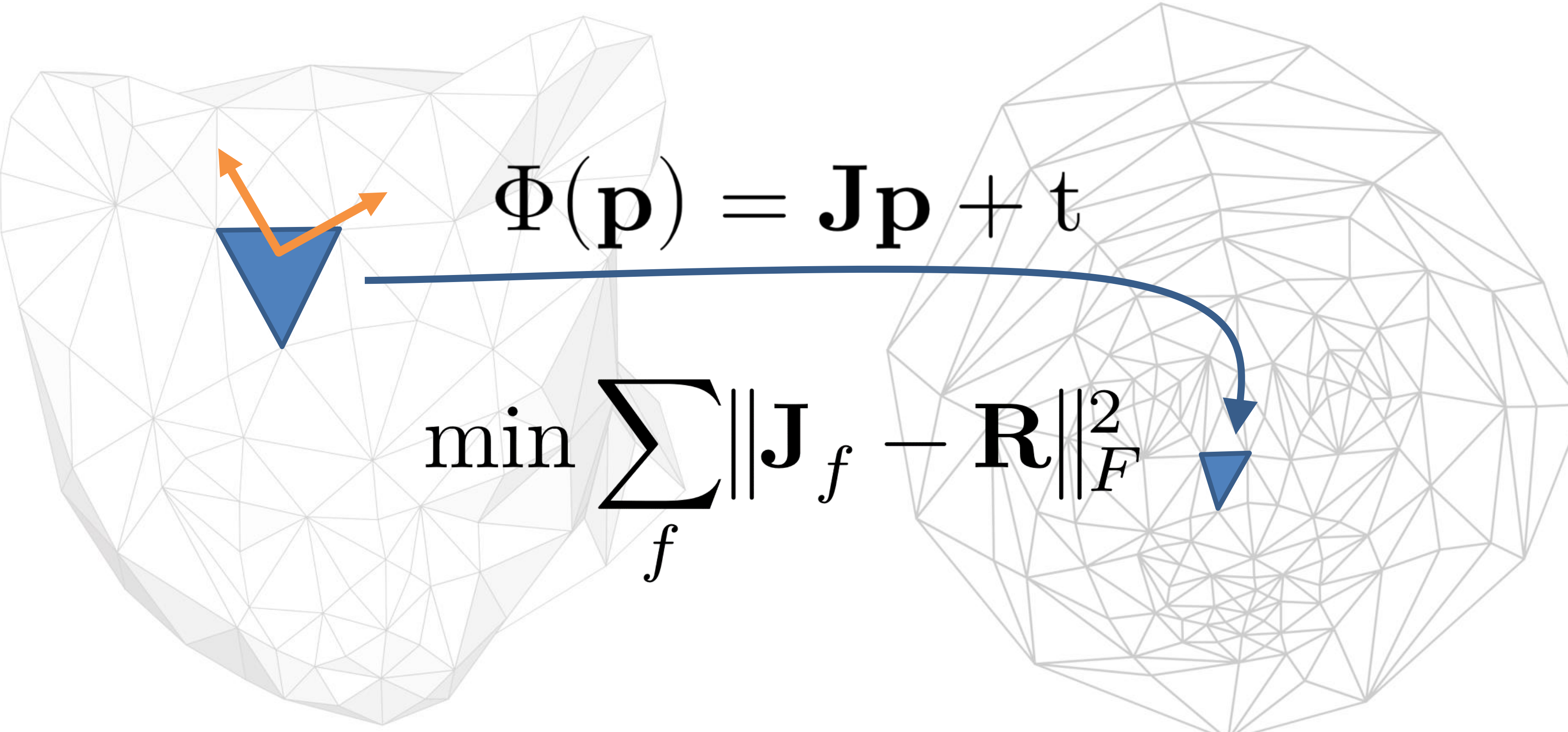
As Rigid As Possible



$$\Phi(\mathbf{p}) = \mathbf{J}\mathbf{p} + \mathrm{t}$$

$$\min \sum_f \|\mathbf{J}_f - \mathbf{R}\|_F^2$$

Local/Global for ARAP

$$\min \sum_f \|\mathbf{J}_f - \mathbf{R}\|_F^2$$

Local/Global for ARAP

$$\min \sum_f \|\mathbf{J}_f(\mathbf{x}) - \mathbf{R}\|_F^2$$

Local/Global for ARAP

$$\min_{\mathbf{x}} \sum_{f} \| \underbrace{\mathbf{J}_f(\mathbf{x})}_{\text{Linear}} - \underbrace{\mathbf{R}\left(\mathbf{J}_f(\mathbf{x})\right)}_{\text{Non Linear}} \|_F^2$$

Local/Global for ARAP

$$\min_{\mathbf{x}} \sum_f \Big\|\underbrace{\mathbf{J}_f(\mathbf{x})}_{\text{Linear}} - \underbrace{\mathbf{R}\left(\mathbf{J}_f(\mathbf{x})\right)}_{\text{Non Linear}}\Big\|_F^2$$

Local/Global for ARAP

$$\mathbf{R}^k = \mathbf{R}\left(\mathbf{J}_f(\mathbf{x})\right)$$

$$\min_{\mathbf{x}} \sum_f \|\underbrace{\mathbf{J}_f(\mathbf{x})}_{\text{Linear}} - \underbrace{\qquad\qquad\qquad}_{\text{Non Linear}}\|_F^2$$

Local/Global for ARAP

$$\mathbf{R}^k = \mathbf{R}\left(\mathbf{J}_f(\mathbf{x})\right)$$

$$\min_{\mathbf{x}} \sum_f \| \underbrace{\mathbf{J}_f(\mathbf{x})}_{\text{Linear}} - \underbrace{\mathbf{R}^k}_{\text{Non Linear}} \|_F^2$$

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Local/Global for ARAP

$$\mathbf{R}^k = \mathbf{R}\left(\mathbf{J}_f(\mathbf{x})\right)$$

$$\min_{\mathbf{x}} \sum_f \| \underbrace{\mathbf{J}_f(\mathbf{x}) - \mathbf{R}^k}_{\text{Linear}} \|_F^2$$

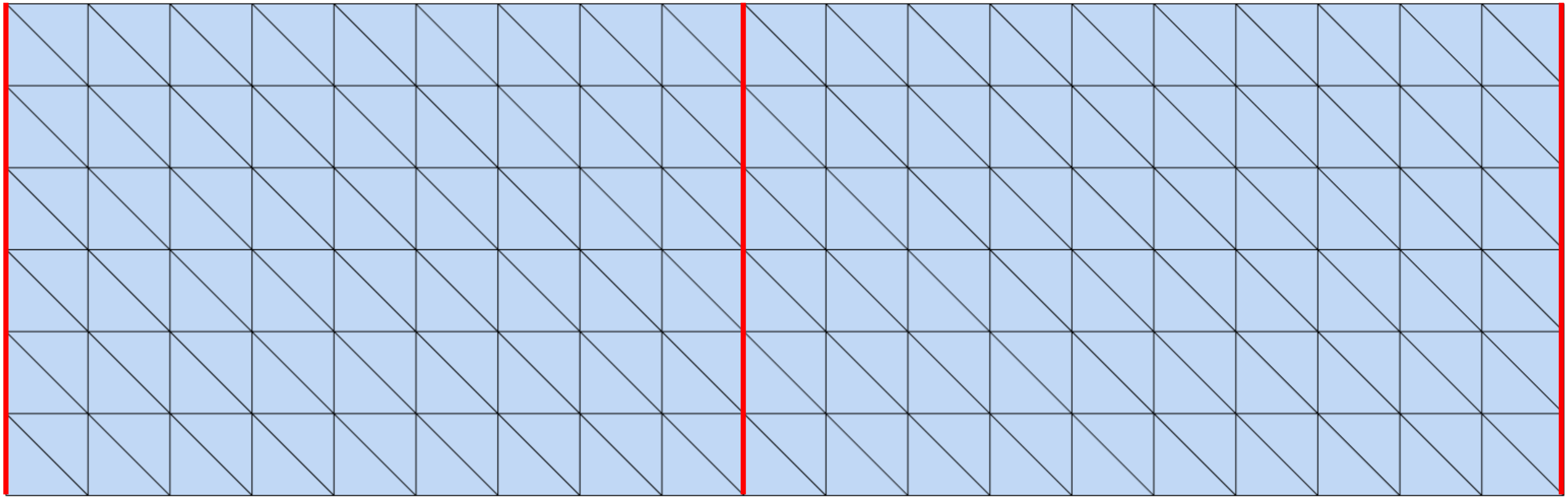
Global/Local for ARAP

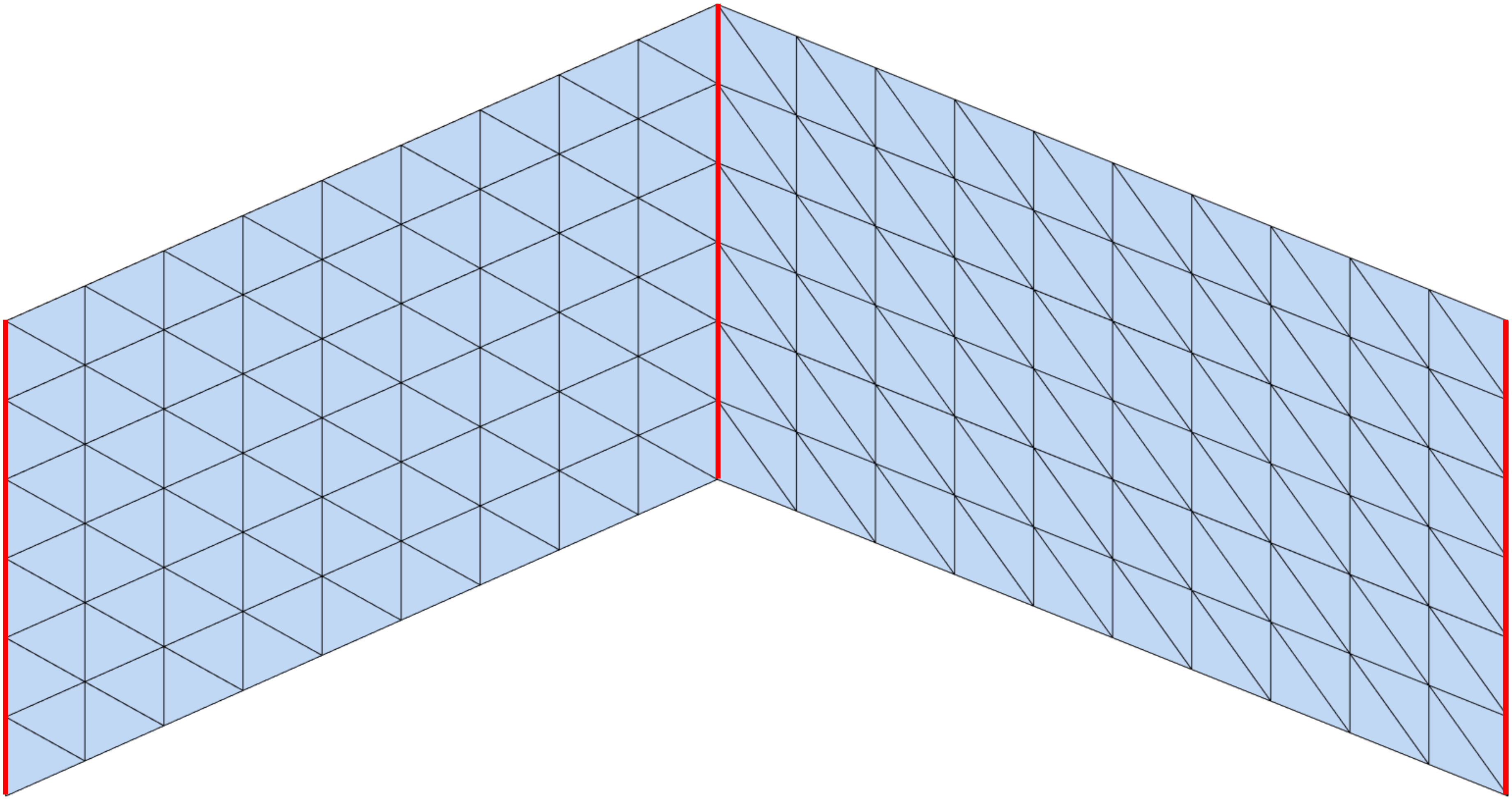
$$\min_{\mathbf{x}} \sum_f \left\| \mathbf{J}_f(\mathbf{x}) - \mathbf{R}\left(\mathbf{J}_f(\mathbf{x})\right) \right\|_F^2$$

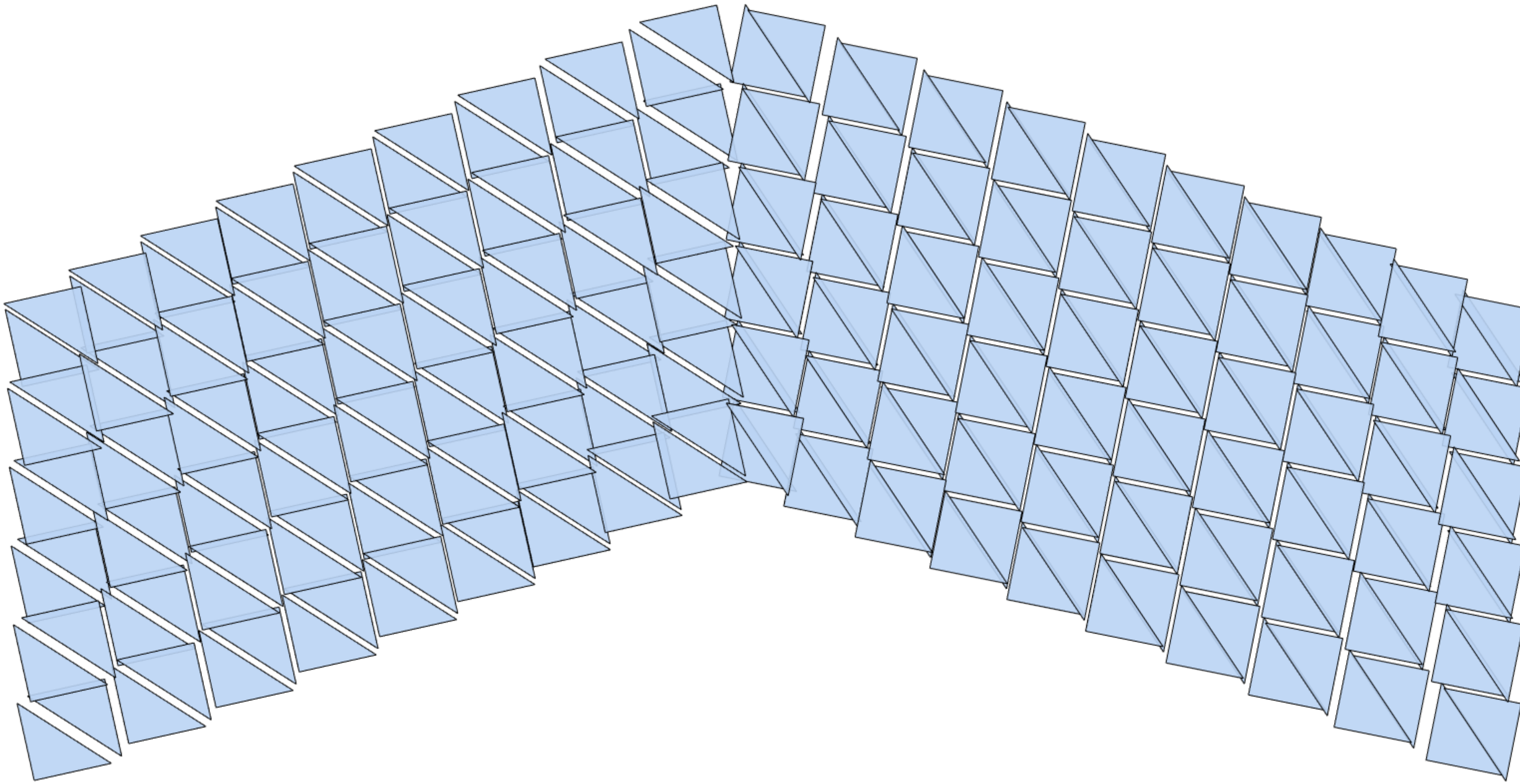
Iterate

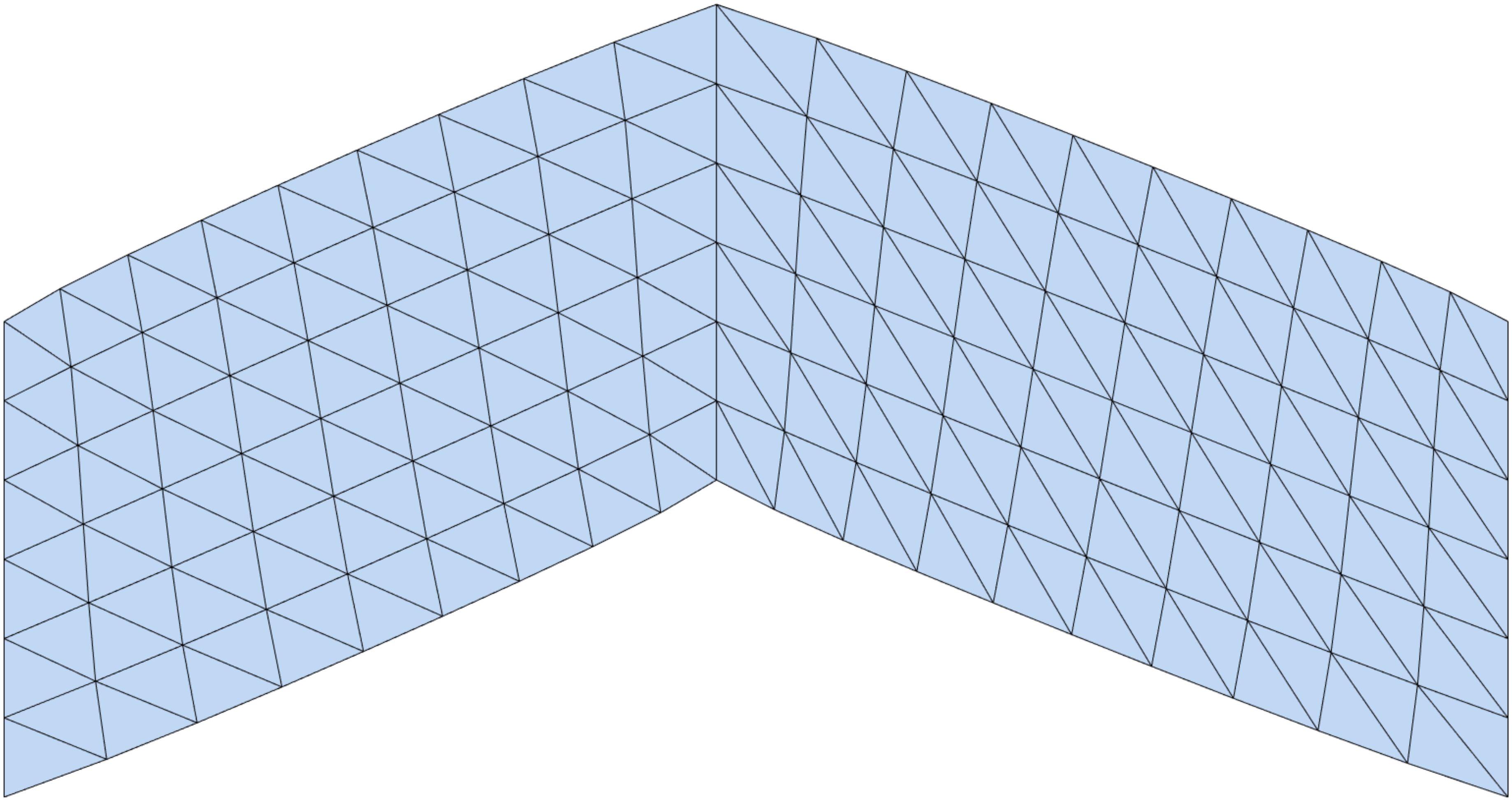
Local Fix $\mathbf{R}^k$

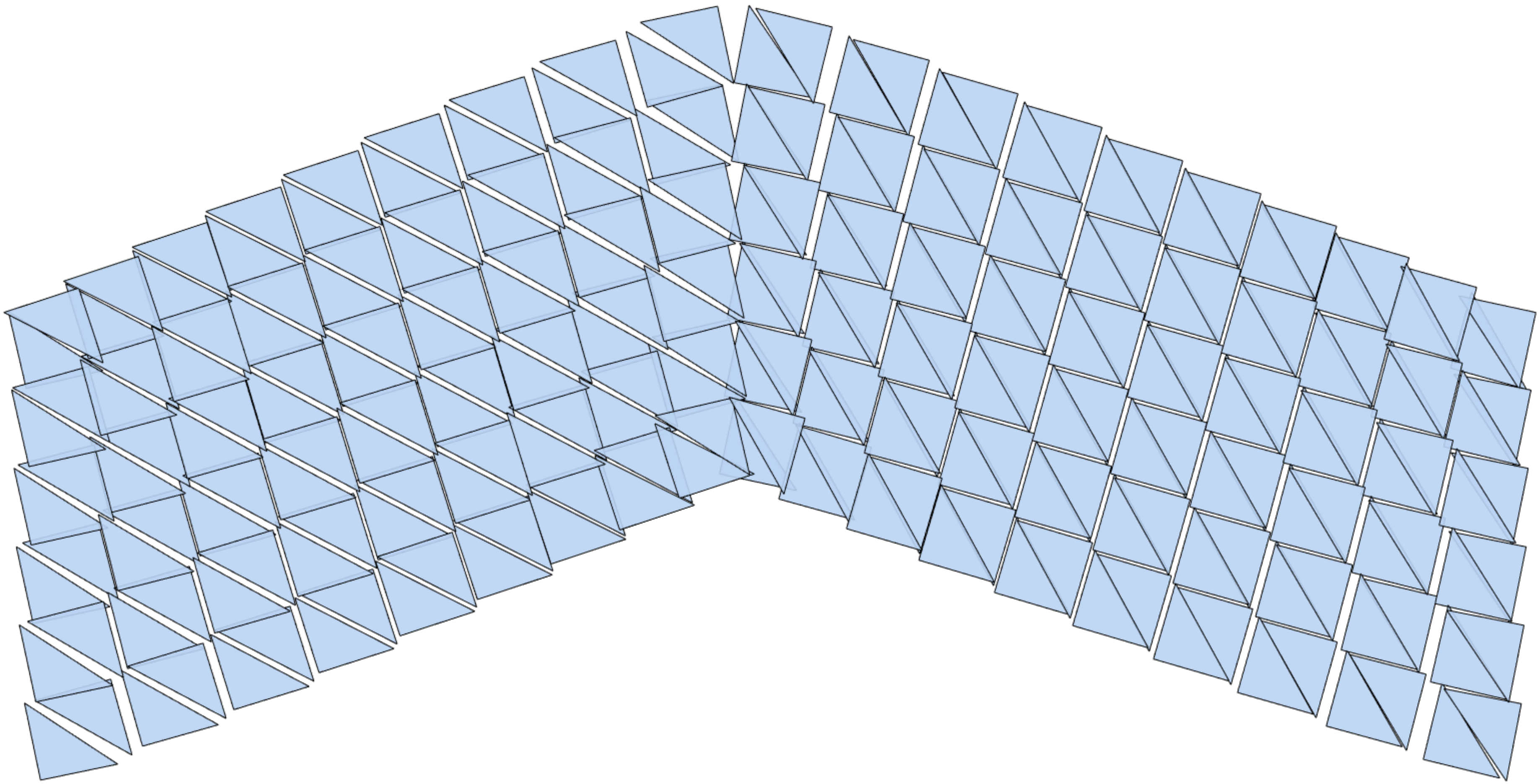
Global Solve with $\mathbf{R}^k$ fixed

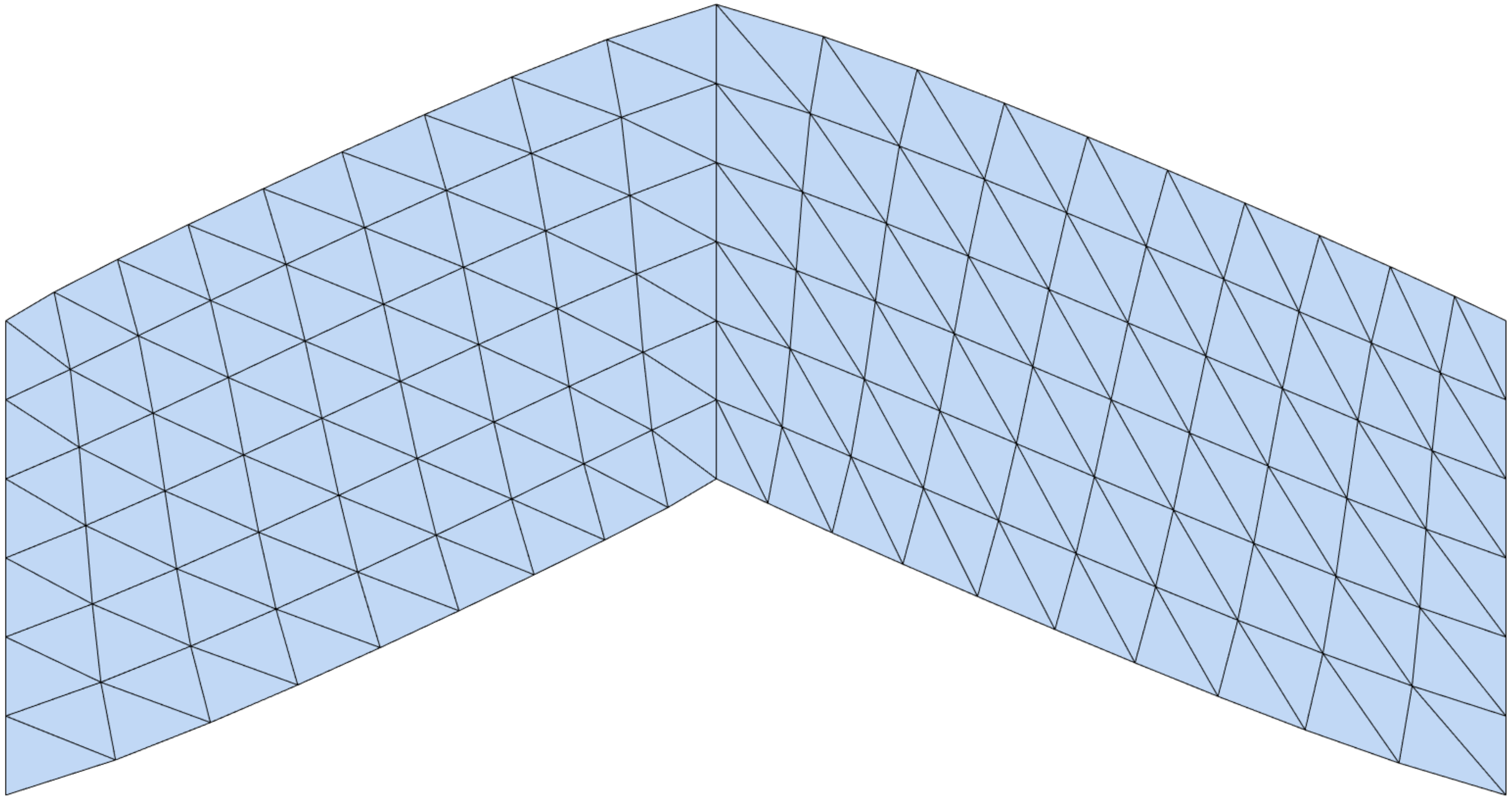












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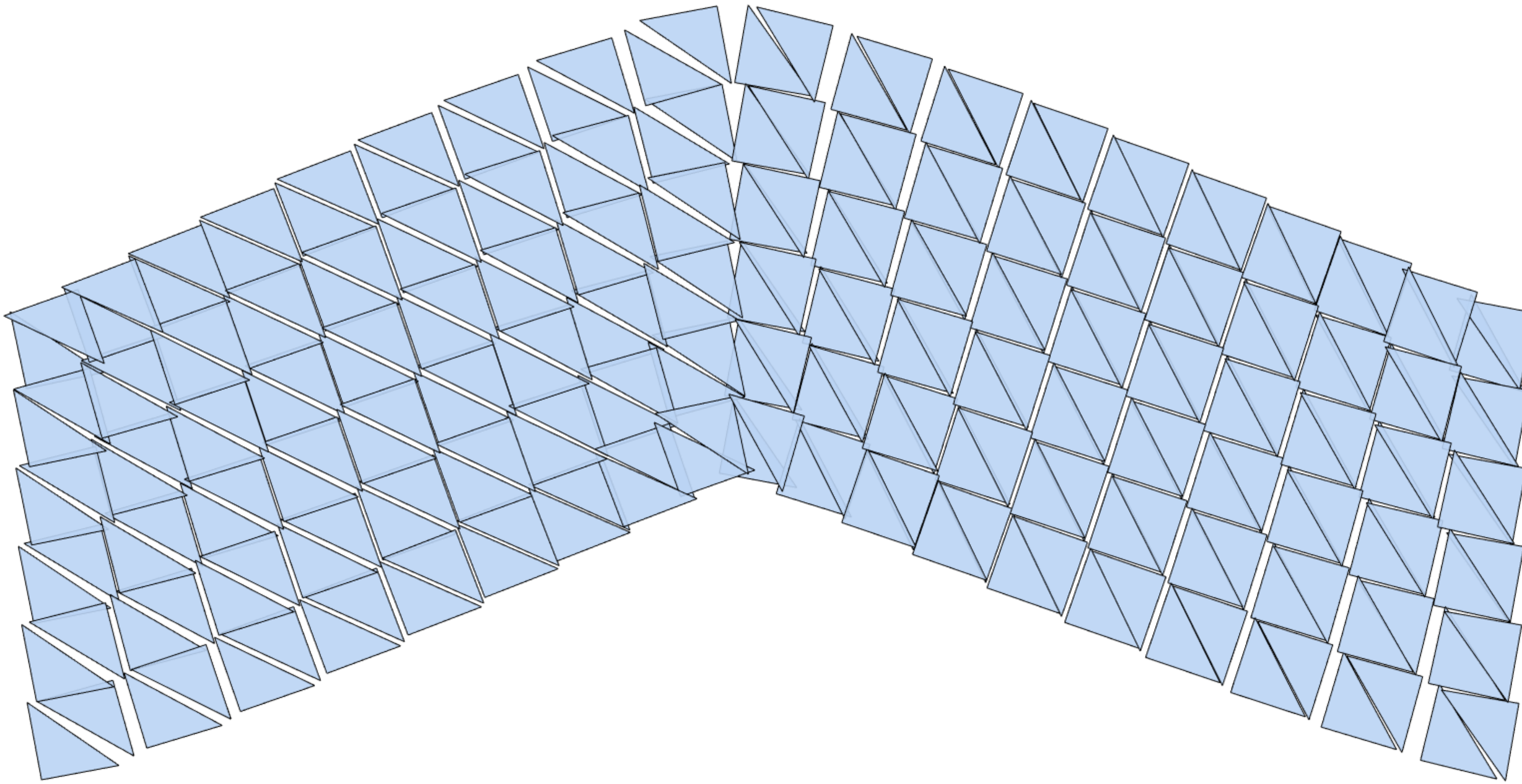

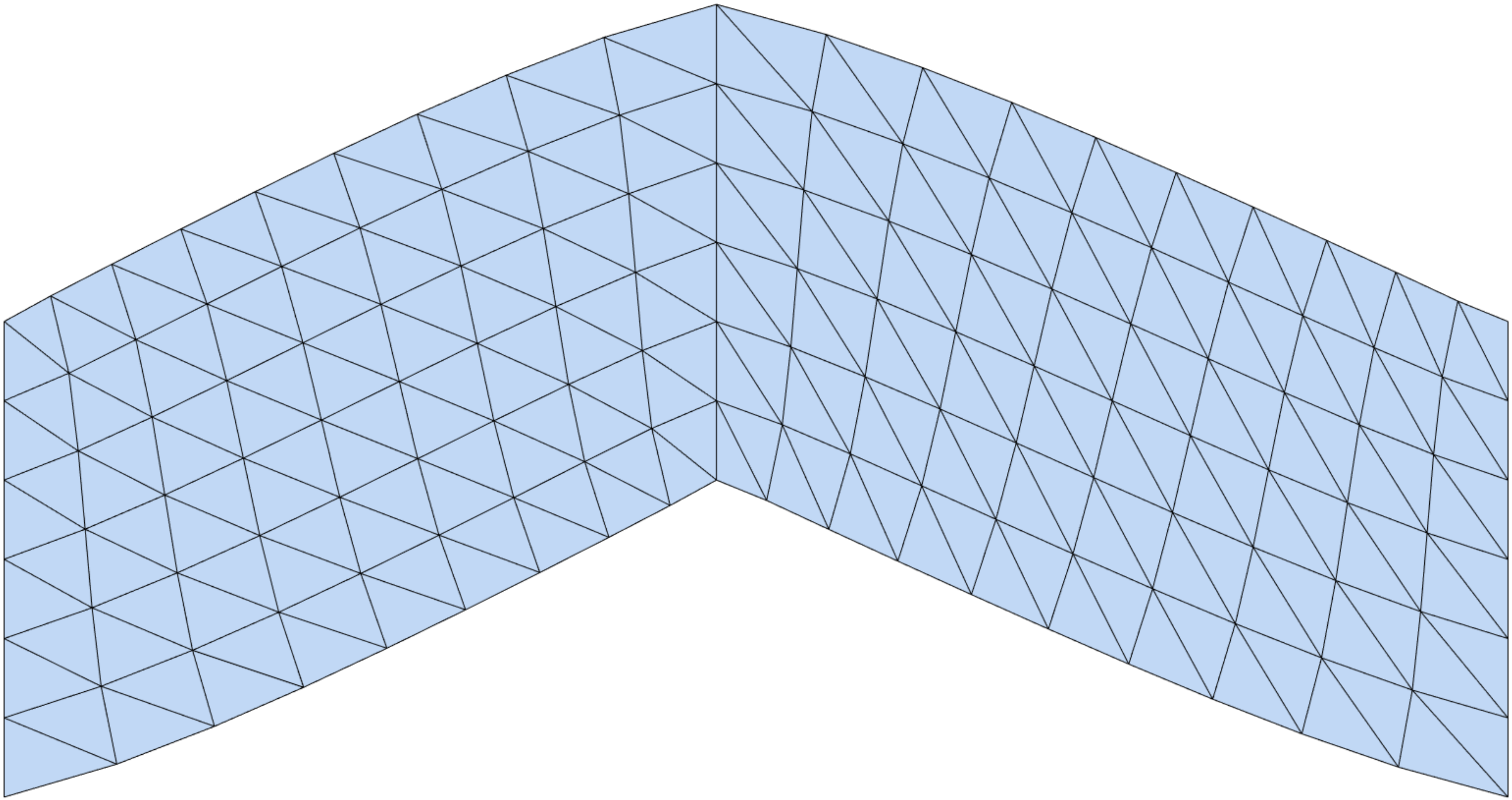

Global/Local for ARAP

$$\min_{\mathbf{x}} \sum_f \left\| \mathbf{J}_f(\mathbf{x}) - \mathbf{R}\left(\mathbf{J}_f(\mathbf{x})\right) \right\|_F^2$$

Iterate

Local $\mathbf{R}^k = \mathbf{R}\left(\mathbf{J}_f(\mathbf{x}^k)\right)$

Global Solve with R^k fixed

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Global/Local for ARAP

$$\min_{\mathbf{x}} \sum_f \left\| \mathbf{J}_f(\mathbf{x}) - \mathbf{R}\left(\mathbf{J}_f(\mathbf{x})\right) \right\|_F^2$$

Iterate

Local $\quad \mathbf{R}^k = \mathbf{R}\left(\mathbf{J}_f(\mathbf{x}^k)\right)$

Global $\quad \mathbf{x}^{k+1} = \min_{\mathbf{x}} \sum_f \left\| \mathbf{J}_f(\mathbf{x}) - \mathbf{R}^k \right\|_F^2$

Global/Local for ARAP

$$\min_{\mathbf{x}} \sum_f \left\| \mathbf{J}_f(\mathbf{x}) - \mathbf{R}\left(\mathbf{J}_f(\mathbf{x})\right) \right\|_F^2$$

Iterate

Local $$\mathbf{R}^k = \mathbf{R}\left(\mathbf{J}_f(\mathbf{x}^k)\right)$$

Global $$\mathbf{x}^{k+1} = \min_{\mathbf{x}} \sum_f \left\| \mathbf{J}_f(\mathbf{x}) - \mathbf{R}^k \right\|_F^2$$

Thank You

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